Discrete Mathematics and Theoretical Computer Science DMTCS vol. **16:2**, 2014, 21–34

Uniquely monopolar-partitionable block graphs

Xuegang Chen[1,2*] and Jing Huang[3†]

[1]*Department of Mathematics, North China Electric Power University, Beijing 102206, China*
[2]*Department of Applied Mathematics, Beijing University of Technology, Beijing 100124, China*
[3]*Department of Mathematics and Statistics, University of Victoria, B.C., Canada*

received 27th Dec. 2013, *revised 18th Apr. 2014*, *accepted 20th Apr. 2014*.

As a common generalization of bipartite and split graphs, monopolar graphs are defined in terms of the existence of certain vertex partitions. It has been shown that to determine whether a graph has such a partition is an NP-complete problem for general graphs, and is polynomial time solvable for several classes of graphs. In this paper, we investigate graphs that admit a unique such partition and call them uniquely monopolar-partitionable graphs. By employing a tree trimming technique, we obtain a characterization of uniquely monopolar-partitionable block graphs. Our characterization implies a polynomial time algorithm for determining whether a block graph is uniquely monopolar-partitionable.

Keywords: Monopolar graph, monopolar partition, uniquely monopolar-partitionable grap, block graph, characterization, polynomial time algorithm

1 Introduction

Given a graph G, a *monopolar partition* of G is a partition (A, B) of its vertex set where A is an independent set and B induces a disjoint union of cliques in G. A graph which admits a monopolar partition is called *monopolar* or *monopolar-partitionable*.

Monopolar graphs were introduced in [17] as a common generalization of bipartite graphs and split graphs. Every bipartition of a bipartite graph is a monopolar partition. Graphs which admit monopolar partitions (A, B) where B induces a single clique are precisely split graphs [12, 14].

A monopolar graph is called *uniquely monopolar-partitionable* if it has exactly one monopolar partition, that is, if (A, B) and (A', B') are both monopolar partitions of G then $A = A'$ (and $B = B'$). Since each complete graph has two monopolar partitions (A, B) and (A', B') where A and A' are the empty set and singleton set respectively, no complete graph is uniquely monopolar-partitionable. On the other hand, the graph obtained from two complete graphs of order at least three by identifying two vertices, one from each, is uniquely monopolar-partitionable.

*Email: gxcxdm@163.com. Research supported by Beijing Postdoctoral Research Foundation.
†Email: huangj@uvic.ca. Research supported by NSERC.

Unlike bipartite graphs and split graphs which are easy to recognize, recognizing monopolar graphs in general is an NP-complete problem (cf. [2] and [11]). It is currently unknown whether uniquely monopolar-partitionable graphs are recognizable in polynomial time.

In this paper, we consider the uniqueness of monopolar partitionability of block graphs. A *block* of a graph G is a maximal subgraph of G without cut-vertices. A graph is a *block* graph if every block is a clique (cf. [1, 15]). We shall give a structural characterzation of uniquely monopolar-partitionable block graphs by using a tree trimming technique. As a by-product, we obtain a polynomial time algorithm for determining whether a block graph is uniquely monopolar-partitionable. We note that such an algorithm can be extracted from [8].

2 Basic definitions

We follow the standard definition and terminology from [18] and consider only simple graphs. Let $G = (V, E)$ be a graph. The *neighbourhood* $N(v)$ of a vertex v in G consists of all vertices adjacent to v. The size of $N(v)$ is the *degree* of v and denoted by $d(v)$. If $d(v) = 1$, then vertex v is called a *leaf* of G. The *closed neighhood* $N[v]$ of v is $N(v) \cup \{v\}$. For any $S \subseteq V$, the subgraph of G induced by S is denoted by $G[S]$. For convenience we write $G - S = G[V - S]$. For any two vertices $u, v \in V$, the *distance* $d_G(u, v)$ between u and v in G is the length of a shortest u, v-path in G, and the *diameter* $diam(G)$ of G is the maximum distance of any two vertices in G. We shall use P_n, C_n and K_n to denote the path, cycle and clique with order n, respectively.

Let H be a subgraph of a graph G. By *contracting* H in G we mean to obtain a new graph G' from $G - V(H)$ by adding a new vertex w adjacent to all vertices which have at least one neighbour in H.

Let $G_1, G_2, \cdots, G_t$ be the components of $G - v$. For any $1 \leq i_1 < i_2 < \cdots < i_s \leq t$, $R_v^s = G[V(\bigcup_{1\leq j\leq s} G_{i_j}) \cup \{v\}]$ is called a *tangent subgraph* of G. If a star with order at least two is a component of $G - v$ and only one center of the star is adjacent to v, then the star is called an *end star* of G, and v is said to be *adjacent to* the end star. The vertex of the star that is adjacent to v is taken as the center of the end star.

Let G be a block graph with at least two blocks and let Q be a block of G. A vertex v of G is said to be *adjacent to* Q if v is not in Q but adjacent to a vertex of Q. If the order of Q is at least 3, then Q is called a *big block* of G. If a big block Q contains a unique cut vertex and $G - Q$ is connected, then Q is called a *terminal block*; if the big block Q contains a unique cut vertex and $G - Q$ is disconnected, then Q is called a *suspending block* of G. If two big blocks have no common vertex but contain adjacent vertices, then the two blocks are called *adjacent*. If two big blocks Q_1 and Q_2 have a common vertex z. then we call the subgraph $G[V(Q_1 \cup Q_2)]$ a *bowtie* of G and the common vertex z is the *center* of the bowtie. The bowtie is called a *terminal bowtie* if $G - Q_1 - Q_2$ is connected and its center is adjacent to exactly one vertex w of $V(G) - V(Q_1 \cup Q_2)$. Vertex w is said to be *adjacent to* the terminal bowtie. If the center of an end star or a terminal bowtie is adjacent to a big block, then the end star or the terminal bowtie is said to be *adjacent* to the big block. Two terminal bowties are *adjacent* if their centers are adjacent. A big block is said to be *adhered to a vertex* v if v is identified with a vertex from the block. Adhering a bowtie to a vertex v means adhering two big blocks to the vertex. If a block graph G is induced by $t \geq 3$ big blocks with a common vertex, then G is called a *flower*.

Let G' be an induced subgraph of G. Suppose that (A', B') is a monopolar partition of G'. If there is a monopolar partition (A, B) of G such that $A \supseteq A'$ and $B \supseteq B'$, then we say that the monopolar partition (A', B') can be *extended to* a monopolar partition of G. If there is a unique monopolar partition (A, B)

of G such that $A \supseteq A'$ and $B \supseteq B'$, then we say that the monopolar partition (A', B') can be *extended to exactly* one monopolar partition of G.

3 Basic properties

Suppose that G is a uniquely monopolar-partitionable graph and (A, B) is the unique monopolar partition of G. For any vertex $v \in V(G)$, if $G[N(v)]$ has an induced P_3, then $v \in B$. If $G[N(v)]$ has no induced P_3, then $G[N(v)]$ is a disjoint union of cliques. If $G[N(v)]$ contains two cliques Q_1 and Q_2 such that each has at least two vertices, then $v \in A$.

Proposition 3.1 *Let T be a tree with order $n \geq 2$. For any edge $uv \in E(T)$, there exists a monopolar partition (A, B) such that $u, v \in B$.*

Proof: Let T' be obtained from T by contracting edge uv and let w denote the new vertex of T'. Then T' is a tree. Let (A', B') be the bipartition of T'. Say $w \in B'$. Let $(A, B) = (A', (B' \setminus \{w\}) \cup \{u, v\})$. Then (A, B) is a monopolar partition of T such that $u, v \in B$. □

Corollary 3.2 *No tree is uniquely monopolar-partitionable.*

Proposition 3.3 *Let G be a uniquely monopolar-partitionable graph and v be a vertex of G. Suppose that C is a component of $G - v$ and $V(C) \cup \{v\}$ induces a tree in G. Then $V(C) \cup \{v\}$ induces a star in G.*

Proof: Since $V(C) \cup \{v\}$ induces a tree, v has a unique neighbour in C which we denoted by x. We show that x is adjacent to every other vertex in C. Suppose not. Then there exist vertices y, z such that xyz is a path in C. Since G is a monopolar graph and $G - C$ is a subgraph of G, $G - C$ is a monopolar graph. Let (A', B') be a monopolar partition of $G - C$.

Assume first that $v \in A'$. By Proposition 3.1, C has a monopolar partition (A_1, B_1) such that $x, y \in B_1$. Let (A_2, B_2) be a bipartition of C where $x \in B_2$ and $y \in A_2$. Then $(A' \cup A_1, B' \cup B_1)$ and $(A' \cup A_2, B' \cup B_2)$ are different monopolar partitions of G, which is a contradiction.

Assume now that $v \in B'$. By Proposition 3.1, C has a monopolar partition (A_1, B_1) such that $y, z \in B_1$. Let (A_2, B_2) be a bipartition of C where $x, z \in A_2$ and $y \in B_2$. Then $(A' \cup A_1, B' \cup B_1)$ and $(A' \cup A_2, B' \cup B_2)$ are different monopolar partitions of G, which is a contradiction. □

Suppose that C is a component of $G - v$ of order at least two and $G[V(C) \cup \{v\}]$ is a tree. By Proposition 3.3, if G is a uniquely monopolar-partitionable graph, then C is an end star of G.

For any monopolar partition (A, B) of G, the center of a bowtie must belong to A. Hence, we have the following.

Proposition 3.4 *Let Q_i be a big block of block graph G for $i = 1, 2, 3$. If $G[V(Q_1 \cup Q_2)]$ and $G[V(Q_2 \cup Q_3)]$ are two bowties of G with different centers, then G has no monopolar partition.* □

Proposition 3.5 *Let $Q_1, \cdots, Q_t$ be big blocks of block graph G containing vertex u and $t \geq 2$. Let $\widehat{G} = G - V(\bigcup_{1 \leq j \leq t} Q_j)$, $S_1 = N(u) \cap V(\widehat{G})$, and $S_2 = N(V(\bigcup_{1 \leq j \leq t} Q_j) - u) \cap V(\widehat{G})$. Assume $S_1 \cup S_2 \neq \emptyset$. Let G' be obtained from $\widehat{G}$ by the following two operations:*

- *For every $w \in S_1$, adding a bowtie and joining its center to w;*

- *For every $w \in S_2$, adhering a bowtie to w.*

Then G is uniquely monopolar-partitionable if and only if G' is.

Proof: Suppose that G is uniquely monopolar-partitionable. Since $\widehat{G}$ is a subgraph of G, $\widehat{G}$ is a monopolar graph. The monopolar partition of G, when restricted to $\widehat{G}$, can be extended to a monopolar partition of G'. Hence G' is a monopolar graph. Assume that G' has at least two different monopolar partitions. For any monopolar partition (A', B') of G', it is obvious that $S_1 \subseteq B'$ and $S_2 \subseteq A'$. So, $\widehat{G}$ has at least two different monopolar partitions. Furthermore, each monopolar partition of $\widehat{G}$ can be extended to a monopolar partition of G. Hence G has at least two different monopolar partitions, which is a contradiction. Hence G' is uniquely monopolar-partitionable.

Suppose that G' is uniquely monopolar-partitionable. It is obvious that G is a monopolar graph. For any monopolar partition (A, B) of G, $S_1 \subseteq B$ and $S_2 \subseteq A$. If G has at least two different monopolar partitions, then $\widehat{G}$ has at least two different monopolar partitions. Since each monopolar partition of G, when restricted to $\widehat{G}$, can be extended to a monopolar partition of G', G' has at least two different monopolar partitions, which is a contradiction. Hence G is uniquely monopolar-partitionable. □

By Proposition 3.5, we can assume that block graph G has no three big blocks with a common vertex. Moreover, each bowtie of G is a terminal bowtie. A proof similar to that of Proposition 3.5 yields the following.

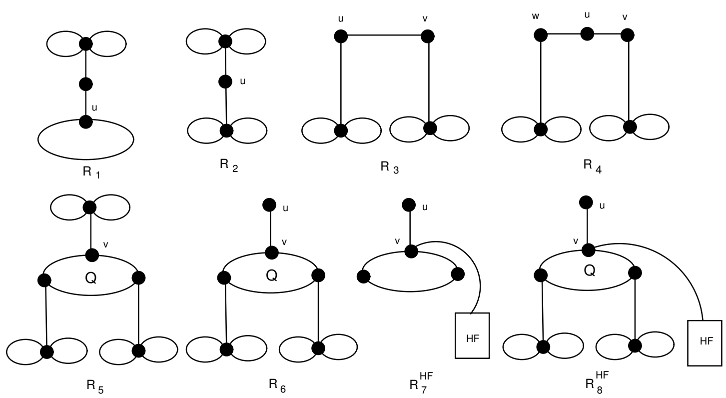

Fig. 1. Each ellipse is a big block, each vertex of $Q \setminus \{v\}$ in $R_5 \cup R_6$ is adjacent to exactly one terminal bowtie, and HF is a subgraph of G.

Proposition 3.6 *Let G be a block graph and let G' be defined as follows:*

- *Suppose that G contains induced subgraph R_1. Let G' be obtained from G by deleting the terminal bowtie of R_1 and adhering a big block to vertex u;*

- *Suppose that G contains induced subgraph R_2. Let G' be obtained from G by deleting a terminal bowtie of R_2;*

- *Suppose that G contains induced subgraph R_3, where $V(G) - V(R_3) \neq \emptyset$ and $N(u) \cap N(v) = \emptyset$. Let G' be obtained from G by deleting R_3 and adhering a bowtie to each vertex $w \in (N(u) \cup N(v)) \setminus V(R_3)$;*
- *Suppose that G contains induced subgraph R_4. Let G' be obtained from G by deleting the two terminal bowties of R_4 and adhering a bowtie to vertex u;*
- *Suppose that G contains induced subgraph R_5 and $V(G) - V(R_5) \neq \emptyset$. Let G' be obtained from G by deleting R_5 and adhering a bowtie to each vertex of $N(Q) \setminus V(R_5)$.*

Then G is uniquely monopolar-partitionable if and only if G' is. □

Proposition 3.7 *Suppose that (A, B) is the unique monopolar partition of block graph G.*

(1) If Q is either a terminal block or a suspending block, then the cut vertex v of Q belongs to A;

(2) The center x of each end star belongs to A.

Proof: (1) Suppose that $v \in B$. If $A \cap V(Q) = \emptyset$, say $u \in V(Q) \setminus \{v\}$, then $(A \cup \{u\}, B \setminus \{u\})$ is a monopolar partition of G. If $A \cap V(Q) \neq \emptyset$, say $u \in A \cap V(Q)$, then $(A \setminus \{u\}, B \cup \{u\})$ is a monopolar partition of G. Hence, if $v \in B$, then G has a monopolar partition different from (A, B), which is a contradiction. So, $v \in A$.

(2) Assume that $vx \in E(G)$ and v does not belong to the end star. Suppose that $x \in B$. If $v \in B$, then $((A - N(x)) \cup \{x\}, (B \setminus \{x\}) \cup N(x))$ is a monopolar partition of G. Suppose that $v \in A$. If $B \cap N(x) = \emptyset$, say $w \in N(x) \setminus \{u\}$, then $(A \setminus \{w\}, B \cup \{w\})$ is a monopolar partition of G. If $B \cap N(x) \neq \emptyset$, say $w \in N(x) \cap B$, then $(A \cup \{w\}, B \setminus \{w\})$ is a monopolar partition of G. Hence, if $x \in B$, then G has a monopolar partition different from (A, B), which is a contradiction. So, $x \in A$. □

Corollary 3.8 *Let G be a uniquely monopolar-partitionable block graph. Then no suspending block of G is adjacent to a terminal block, an end star, or a terminal bowtie.* □

Proposition 3.9 *Let G' be an induced subgraph of block graph G. Suppose that each monopolar partition of G' can be extended to at least one monopolar partition of G. Moreover, suppose that if G' has a unique monopolar partition, then it can be extended to exactly one monopolar partition of G. Then G is uniquely monopolar-partitionable if and only if G' is.*

Proof: Suppose that G is uniquely monopolar-partitionable. Since G' is an induced subgraph of G, it follows that G' is a monopolar graph. If G' has two different monopolar partitions, then these monopolar partitions can be extended to two different monopolar partitions of G, which is a contradiction. Hence, G' is uniquely monopolar-partitionable.

Suppose that G' is uniquely monopolar-partitionable. It is obvious that G is uniquely monopolar-partitionable.

□

By Proposition 3.7 and Proposition 3.9, we have the following corollary.

Corollary 3.10 *Let G be a block graph and let G' be defined as follows:*

- *if G contains a leaf w adjacent to a block, then $G' = G - w$;*
- *if a vertex v is adjacent to two terminal blocks Q_1 and Q_2, then $G' = G - Q_2$;*
- *if G contains the tangent subgraph $R_6 = R_u^1$, then G' is obtained from G by deleting $V(R_6) \setminus \{u\}$.*

Then G is uniquely monopolar-partitionable if and only if G' is. □

4 Reductions of block graphs

Let G be a block graph. In view of Propositions 3.4, 3.5 and 3.6, we may assume that G satisfies the following two conditions:

- Each bowtie of G is a terminal bowtie and G has no two adjacent terminal bowties.
- G has no induced subgraph R_i for $i = 1, 2, 3, 4, 5$, where u and v in R_3 do not belong to the same big block of G.

Let T be a tree obtained from G by contracting each terminal bowtie, end star, and big block, respectively. Let $v_0v_1 \cdots v_d$ be a longest path of T. Let $V_i = \{u \in V(T) | d(u, v_0) = i\}$ for $i = 0, 1, 2, \cdots, d$. Note that $(V_0, V_1, \cdots, V_d)$ is a vertex partition of T. From the vertex partition of T, we obtain a vertex partition $(V_G^0, V_G^1, \cdots, V_G^d)$ of G as follows: $u \in V_i$ if and only if $u \in V_G^i$ or all the vertices of the corresponding terminal bowtie, end star or big block belong to V_G^i.

For each big block Q of G, if the block belongs to V_G^i of G, then Q is called an i^{th} *level big block*. Suppose that Q is the i^{th} level big block. If $v \in V(Q)$ is adjacent to a vertex in V_G^{i-1}, then v is called the *upper vertex* of Q, the other vertices are called *down vertices* of Q. For any vertex $v \in V_G^i$, if there exists a vertex $u \in V_G^{i-1}$ such that $vu \in E(G)$, then u is called the *parent* of v. Both Q and v are called *children* of u.

In the section, a family of some special graphs $\{H_i, F_j, Y_k | 1 \le i \le 4, 1 \le j \le 5, k = 2, 5\}$ is given in Fig. 2, Fig. 3 and Fig. 4. Say $d \ge 4$. The basic idea in the section is as follows: By employing a tree trimming techniques, we delete all the vertices of V_G^d. Firstly, we consider each component C in the induced subgraph $G[V_G^{d-1} \cup V_G^d]$ having nonempty intersection with V_G^d. By local structure of C, if C is not isomorphic to H_i for $i \in \{1, 2, 3, 4\}$, either we can determine that G is not uniquely monopolar-partitionable or some blocks of C are deleted. Secondly, we consider each component C' in the induced subgraph $G[V_G^{d-2} \cup V_G^{d-1} \cup V_G^d]$ containing H_i as a tangent subgraph, where $i \in \{1, 2, 3, 4\}$. By local structure of C', if C' is not isomorphic to F_i for $i \in \{1, 2, 3, 4, 5\}$, either we can determine that G is not uniquely monopolar-partitionable or some blocks of C' are deleted. Thirdly, we consider each component C'' in the induced subgraph $G[V_G^{d-3} \cup V_G^{d-2} \cup V_G^{d-1} \cup V_G^d]$ containing F_j as a tangent subgraph, where $j \in \{1, 2, 3, 4, 5\}$. By local structure of C'', if C'' is not isomorphic to Y_k for $k \in \{2, 5\}$, either we can determine that G is not uniquely monopolar-partitionable or some blocks of C'' are deleted. Finally, we consider each component C''' in the induced subgraph $G[V_G^{d-4} \cup V_G^{d-3} \cup V_G^{d-2} \cup V_G^{d-1} \cup V_G^d]$ containing Y_j as a tangent subgraph, where $j \in \{2, 5\}$. By local structure of C''', either we can determine that G is not uniquely monopolar-partitionable or all blocks of C''' belonging to V_G^d are deleted. Then we obtain a new block graph whose associated tree has diameter less than d.

Proposition 4.1 *Let Q be a big block of G, and let $G_1, G_2, \cdots, G_t$ be the components of $G - Q$. Assume that the upper vertex v of Q is adjacent to $G_1, \cdots, G_s$. Suppose that G_j is a terminal bowtie, a terminal block, an end star or an isolated vertex for $j = s + 1, \cdots, t$.*

(1) Suppose that there exists a down vertex w of Q such that w is not adjacent to a terminal bowtie. Let $G' = G[V(Q) \cup V(\bigcup_{1\leq i\leq s} G_i)]$. Then G is uniquely monopolar-partitionable if and only if G' is.

(2) Suppose that each down vertex of Q is adjacent to a terminal bowtie. Let G' be obtained from G by deleting all the G_k except exactly one terminal bowtie for each down vertex, where $k \in \{s+1, \cdots, t\}$. Then G is uniquely monopolar-partitionable if and only if G' is.

Proof: (1) Suppose that G is uniquely monopolar-partitionable. Since G' is an induced subgraph of G, G' is a monopolar graph. For any monopolar partition (A', B') of G', $v \in A'$. Otherwise, G' has at least two monopolar partitions. One has $V(Q) \subseteq B'$, the other has $V(Q) \setminus \{w\} \subseteq B'$ and $w \in A'$. Both of them can be extended to a monopolar partition of G, which is a contradiction. Since $v \in A'$, any monopolar partition (A', B') of $G[V(Q) \cup V(\bigcup_{1\leq i\leq s} G_i)]$ can be extended to exactly one monopolar partition of G. So G' is uniquely monopolar-partitionable.

Suppose that G' is uniquely monopolar-partitionable. Let (A', B') be its monopolar partition. By Proposition 3.7, $v \in A'$. Since $v \in A'$, (A', B') can be extended to exactly one monopolar partition of G. By Proposition 3.9, G is uniquely monopolar-partitionable.

(2) By Proposition 3.9, G is uniquely monopolar-partitionable if and only if G' is. □

Proposition 4.2 *Let v be a vertex of G not belonging to any big block, and let t denote the parent of v. Suppose v is not adjacent to a terminal bowtie and every child of v is a leaf, a terminal block or end star.*

(1) If G is uniquely monopolar-partitionable, then v is not adjacent to an end star.

(2) If G is uniquely monopolar-partitionable, then v is not adjacent to both a leaf and a terminal block.

Proof: (1) Suppose v is adjacent to an end star. Let u and w be the center and a leaf of the end star, respectively. Let (A, B) be the unique monopolar partition of G. By Proposition 3.7, $u \in A$ and $v \in B$. If $t \in A$, then let $u \in B$ and $N(u) \cup N(v) \setminus \{u, v\} \subseteq A$. So there exists a monopolar partition such that $u \in B$, which is a contradiction. If $t \in B$, let $N(v) \subseteq B$ and $N(u) \subseteq A$, then there exists a monopolar partition such that $u \in B$, which is a contradiction.

(2) Suppose v is adjacent to both a leaf s and a terminal block. Let u and w be the upper vertex and a down vertex of the terminal block, respectively. Let (A, B) be the unique monopolar partition of G. By Proposition 3.7, $u \in A$ and $v \in B$. If $t \in A$, then $s \in A$ or $s \in B$. Then G has two different monopolar partitions, which is a contradiction. If $t \in B$, let $v \in A$ and $N[u] \setminus \{v\} \subseteq B$, then there exists a monopolar partition such that $u \in B$, which is a contradiction. □

By Proposition 4.1, Proposition 4.2, Corollary 3.8, Corollary 3.10 and the fact that G has no induced subgraph R_1 and R_5, if $d \geq 2$, then we can assume that each component of $G[V_G^{d-1} \cup V_G^d]$, having nonempty intersection with V_G^d, is isomorphic to H_i for $i \in \{1, 2, 3, 4\}$.

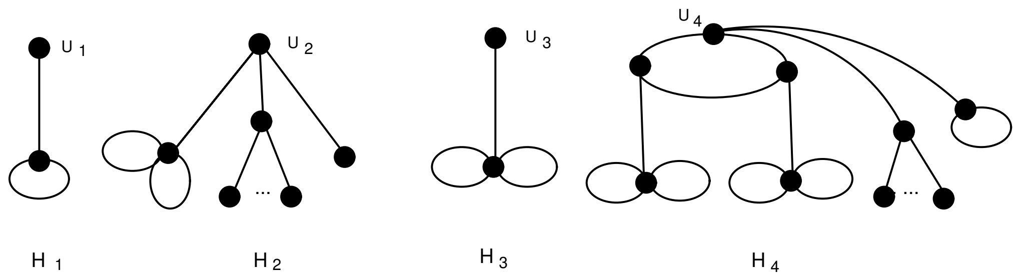

Fig. 2. Each ellipse is a big block, u_2 is adjacent to at least one leaf or end star,

u_4 is adjacent to at least one end star or terminal block.

For each H_i, let w_i denote the parent of u_i. For $i = 1, \cdots, 4$, let $F_i = G[V(H_i) \cup \{w_i\}]$. If $w_i = w_j$, then let $F_{ij} = G[V(F_i) \cup V(F_j)]$. That is $F_{ij} = G[V(H_i) \cup V(H_j) \cup \{w_i\}]$. For $k = 1, 2, 3, 4$, let F_j^k denote the graph obtained from F_j by joining w_j to a terminal bowtie, an end star, a terminal block and a leaf, respectively.

By a proof similar to that of Proposition 3.7, we have the following.

Proposition 4.3 *Suppose that (A, B) is the unique monopolar partition of block graph G. If G contains the tangent subgraph $F_i = R^1_{w_i}$ for $i = 1, 4$, then $w_i \in A$. If G contains the tangent subgraph $F_2 = R^1_{w_2}$, then $w_2 \in B$.* □

By Proposition 4.3 and Proposition 3.7, we have the following corollary.

Corollary 4.4 *Let G be uniquely monopolar-partitionable. Then G does not contain tangent subgraph F_{12}, F_{24}, F_1^1, F_1^2, F_1^3, F_4^1, F_4^2 and F_4^3.* □

Proposition 4.5 *If G contains the tangent subgraph F_{ij} for $i, j \in \{1, 4\}$, then G is uniquely monopolar-partitionable if and only if $G - H_j$ is.*

Proof: It is obvious that $G - H_j$ is an induced subgraph of G. For any monopolar partition of $G - H_j$, it can be extended to at least a monopolar partition of G. Suppose that $G - H_j$ is uniquely monopolar-partitionable. Let (A', B') be the unique monopolar partition of $G - H_j$. By Proposition 4.3, $w_i \in A'$. Then (A', B') can be extended to exactly one monopolar partition of G. By Proposition 3.9, G is uniquely monopolar-partitionable if and only if $G - H_j$ is. □

By a proof similar to that of Proposition 4.5, we have the following.

Proposition 4.6 *(1) For any $i \in \{1, 2, 3, 4\}$, if G contains the tangent subgraph F_i^4, then G is uniquely monopolar-partitionable if and only if $G - t$ is, where $t \in V(F_i^4)$ is the leaf and is adjacent to w_i.*

(2) Suppose that G contains the tangent subgraph F_2^i for $i = 2, 3$. Let G' be the graph obtained from G by deleting the end star and the terminal block that are adjacent to w_2. Then G is uniquely monopolar-partitionable if and only if G' is. □

Proposition 4.7 *Suppose that G contains the tangent subgraph F_{3j} for $j \in \{1,4\}$. Let G' be obtained from G by deleting $F_{3j} \setminus \{w_3\}$ and adhering a big block to w_3. Then G is uniquely monopolar-partitionable if and only if $G^{'}$ is.*

Proof: Let $G'' = G - F_{3j} \setminus \{w_3\}$. Suppose that G is uniquely monopolar-partitionable. Let (A,B) be the unique monopolar partition of G. By Proposition 4.3, $w_3 \in A$. Then $(A \cap V(G''), B \cap V(G''))$ can be extended to a monopolar partition of G'. Hence, G' is a monopolar graph. For any monopolar partition (A', B') of G', $w_3 \in A'$. Otherwise, $(A' \cap V(G''), B' \cap V(G''))$ can be extended to two different monopolar partitions of G, which is a contradiction. Since $w_3 \in A'$, $(A' \cap V(G''), B' \cap V(G''))$ can be extended to exactly one monopolar partition of G. Hence, $G^{'}$ is uniquely monopolar-partitionable.

Suppose that G' is uniquely monopolar-partitionable. Let (A', B') be the unique monopolar partition of G'. By Proposition 3.7, $w_3 \in A'$. Then $(A' \cap V(G''), B' \cap V(G''))$ can be extended to a monopolar partition of G. Hence, G is a monopolar graph. For any monopolar partition (A,B) of G, $w_3 \in A$. Otherwise, G' has two different monopolar partitions, which is a contradiction. If G has two different monopolar partitions, then G'' has two different monopolar partitions. So G' has two different monopolar partitions, which is a contradiction. Hence, G is uniquely monopolar-partitionable. □

By Corollary 4.4, Proposition 4.5, Proposition 4.6, Proposition 4.7 and the fact that G has no induced subgraph R_3 and R_4, we can assume that the subgraph induced by w_i and its descendant, having nonempty intersection with V_G^d, is isomorphic to F_i in Fig 3 for $i \in \{1, \cdots, 5\}$.

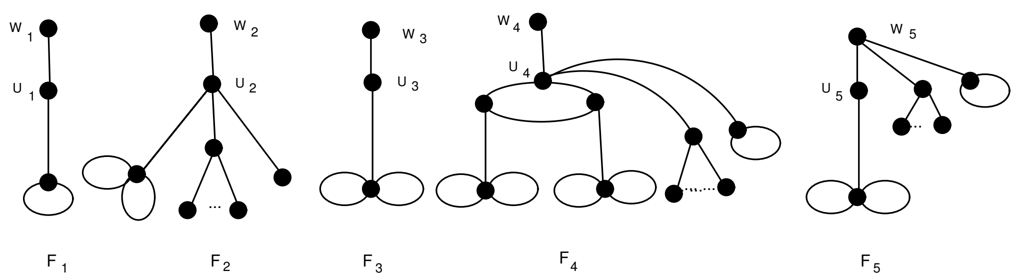

Fig. 3. w_5 is adjacent to at least one terminal block or an end star

Since G has no induced subgraph R_1, w_2, w_3, w_5 do not belong to a big block. Let $R_7^{H_i}$ and $R_8^{H_i}$ be defined in Fig. 1 for $i \in \{1,4\}$. For each F_i, $i = 1,4$, if w_i belongs to a big block of G, then we have the following:

Proposition 4.8 *Let G be a block graph.*

(1) Suppose that G contains the tangent subgraph $F_i = R_{w_i}^1$, where $i \in \{1,4\}$. If G is uniquely monopolar-partitionable, then w_i is not a down vertex of a big block .

(2)Suppose that G contains the tangent subgraph $R_7^{H_i} = R_u^1$, where $G[V(H_i) \cup \{v\}] = F_i$ and $i \in \{1,4\}$. Then G is uniquely monopolar-partitionable if and only if $G - H_i$ is.

(3)Suppose that G contains the tangent subgraph $R_8^{H_i} = R_u^1$, where $G[V(H_i) \cup \{v\}] = F_i$ and $i \in \{1,4\}$. Let G' be obtained from G by deleting H_i and all the children of the big block Q. Then G is uniquely monopolar-partitionable if and only if G' is.

Proof: (1) Suppose that w_i is a down vertex of a big block. Let (A, B) be the unique monopolar partition of G. By Proposition 4.3, $w_i \in A$. Let $G' = G - H_i$. Then $(A \cap V(G'), B \cap V(G'))$ is a monopolar partition of G'. So $(A \cap (V(G') - w_i), (B \cap V(G')) \cup \{w_i\})$ is also a monopolar partition of G' and it can be extended to a monopolar partition of G. Hence, G has two different monopolar partitions, which is a contradiction.

(2) It is obvious that $G - H_i$ is an induced subgraph of G. For any monopolar partition (A', B') of $G - H_i$, it can be extended to at least a monopolar partitions of G. Suppose that $G - H_i$ is uniquely monopolar-partitionable. Let (A', B') be the unique monopolar partition. By Proposition 3.7, $v \in A'$. Then (A', B') can extend to exactly one monopolar partition of G. By Proposition 3.9, G is uniquely monopolar-partitionable if and only if $G - H_i$ is.

(3) A proof similar to that of Case 1 in Proposition 4.1 shows that G is uniquely monopolar-partitionable if and only if G' is. □

Let m_i denote the parent of w_i and $Y_i = G[V(F_i) \cup \{m_i\}]$ for $i = 1, 2, \cdots, 5$. If $m_i = w_j$, let $YF_{ij} = G[V(Y_i) \cup V(F_j)]$ for $i = 2, 5$ and $j = 1, 2, \cdots, 5$. If $m_i = m_j$, let $Y_{ij} = G[V(Y_i) \cup V(Y_j)]$ for $i, j = 2, 5$. Let Y_i^j be the graph obtained from Y_i by joining m_i to a terminal bowtie, an end star, a terminal block or an isolated vertex, respectively, for $i = 2, 5$ and $j = 1, 2, 3, 4$. By Proposition 4.8, we can assume that $Y_i = R^1_{m_i}$ is a tangent subgraph of G for $i \in \{1, \cdots, 5\}$.

Proposition 4.9 *Suppose that G contains the tangent subgraph Y_i, where $i \in \{1, 4\}$. Let H_i be the tangent subgraph of Y_i. Then G is uniquely monopolar-partitionable if and only if $G - H_i \setminus \{u_i\}$ is.*

Proof: It is obvious that $G - H_i \setminus \{u_i\}$ is an induced subgraph of G. For any monopolar partition of $G - H_i \setminus \{u_i\}$ can be extended to at least a monopolar partition of G. Suppose that $G - H_i \setminus \{u_i\}$ is uniquely monopolar-partitionable. Let (A', B') be the unique monopolar partition. By Proposition 3.7, $w_i \in A'$ and $u_i \in B'$. Then (A', B') can extend to exactly one monopolar partition of G. By Proposition 3.9, G is uniquely monopolar-partitionable if and only if $G - H_i \setminus \{u_i\}$ is. □

Proposition 4.10 *Suppose that G contains the tangent subgraph Y_3. Let F_3 be the tangent subgraph of Y_3. Then G is uniquely monopolar-partitionable if and only if $G - F_3$ is.*

Proof: By Proposition 3.9, G is uniquely monopolar-partitionable if and only if $G - F_3$ is. □

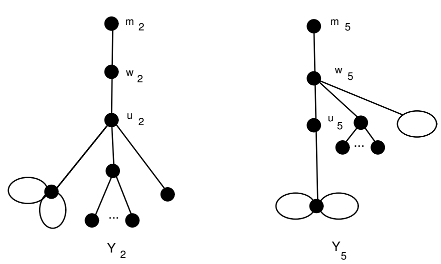

Fig. 4. Subgraphs Y_2 and Y_5

By Proposition 3.7 and Proposition 4.3, we have the following corollary.

Corollary 4.11 *Suppose that block graph G has unique monopolar partition (A, B). If G contains the tangent subgraph Y_i for $i = 2, 5$, then $m_i \in A$.* □

By Propositions 4.3 and Corollary 4.11, we have the following corollary.

Corollary 4.12 *Let G be uniquely monopolar-partitionable block graph. Then G does not contain tangent subgraph YF_{22}, YF_{25}, YF_{52}, YF_{55}, Y_2^j and Y_5^j for $j = 1, 2, 3$.* □

By a proof similar to that of Proposition 4.5, we have the following.

Proposition 4.13 *(1) For $i = 2, 5$ and $j = 1, 4$, suppose that G contains the tangent subgraph YF_{ij}. Let H_j be the tangent subgraph of F_j. Then G is uniquely monopolar-partitionable if and only if $G - H_j$ is.*

(2) For $i, j = 2, 5$, suppose that G contains the tangent subgraph Y_{ij}. Let F_j be the tangent subgraph of Y_i. Then G is uniquely monopolar-partitionable if and only if $G - F_j$ is.

(3) For $i = 2, 5$, if G contains the tangent subgraph Y_i^4, then G is uniquely monopolar-partitionable if and only if $G - t$ is, where $t \in V(Y_i^4)$ is a leaf and is adjacent to m_i.

By a proof similar to that of Proposition 4.7, we have the following.

Proposition 4.14 *Suppose that G contains the tangent subgraph YF_{i3} for $i \in \{2, 5\}$. Let G' be obtained from G by deleting $YF_{i3} \setminus \{w_3\}$ and adhering a big block to w_3. Then G is uniquely monopolar-partitionable if and only if G' is.*

Let $R_7^{F_i}$ and $R_8^{F_i}$ be defined in Fig. 1 for $i \in \{2, 5\}$. By a proof similar to that of Proposition 4.8, we have the following.

Proposition 4.15 *Let G be a block graph.*

(1) Suppose that G contains the tangent subgraph Y_i. If G is uniquely monopolar-partitionable block graph, then m_i is not a down vertex of a big block, where $i \in \{2, 5\}$.

(2)Suppose that G contains the tangent subgraph $R_7^{F_i} = R_u^1$, where $G[V(F_i) \cup \{v\}] = Y_i$ and $i \in \{2, 5\}$. Then G is uniquely monopolar-partitionable if and only if $G - F_i$ is.

(3)Suppose that G contains the tangent subgraph $R_8^{F_i} = R_u^1$, where $G[V(F_i) \cup \{v\}] = Y_i$ and $i \in \{2, 5\}$. Let G' be obtained from G by deleting F_i and all the children of the big block Q. Then G is uniquely monopolar-partitionable if and only if G' is.

Let t_i denote the parent of m_i and $X_i = G[V(Y_i) \cup \{t_i\}]$ for $i = 2, 5$. By Corollary 4.12, Proposition 4.13, Proposition 4.14 and Proposition 4.15, we can assume that $X_i = R_{t_i}^1$ is a tangent subgraph of G for $i \in \{2, 5\}$.

By a proof similar to that of Proposition 4.1, we have the following.

Proposition 4.16 *For $i = 2, 5$, suppose that G contains the tangent subgraph X_i. Let F_i be the tangent subgraph of X_i. Then G is uniquely monopolar-partitionable if and only if $G - F_i \setminus \{w_i\}$ is.*

Remark. By above Propositions, we have deleted all of vertices of V_G^d. So we obtain a new block graph whose associated tree has diameter less than d.

5 Uniquely monopolar-partitionable block graphs

In order to determine whether a given block graph has a unique monopolar partition, we first define a family of block graphs. Let Φ be the family of block graphs G satisfying the following conditions:

(1) Either G is a bowtie or each bowtie of G is a terminal bowtie.

(2) G has no induced subgraph R_i for $i = 1, 2, 3, 4, 5$, where u and v in R_3 do not belong to the same big block of G.

(3) G has no two adjacent terminal bowties.

Let G be a block graph. By Proposition 3.4, if G has two bowties $G[V(Q_1 \cup Q_2)]$ and $G[V(Q_2 \cup Q_3)]$ with different centers, then G is not uniquely monopolar-partitionable. Obviously, if G has two adjacent terminal bowties, then G is not uniquely monopolar-partitionable. Without loss of generality, we can assume that G has neither two bowties $G[V(Q_1 \cup Q_2)]$ and $G[V(Q_2 \cup Q_3)]$ with different centers nor two adjacent terminal bowties. By repeatedly applying Proposition 3.5 and Proposition 3.6, we obtain block graph $G_1, \cdots, G_t$ such that $G_i \in \Phi$ or $G_i \in \{$ a flower, R_3, R_5 $\}$ for $1 \leq i \leq t$. We have the following.

Theorem 5.1 *Let G be a block graph. Suppose that G has neither two bowties $G[V(Q_1 \cup Q_2)]$ and $G[V(Q_2 \cup Q_3)]$ with different centers nor two adjacent terminal bowties. Then G is uniquely monopolar-partitionable if and only if G_i is uniquely monopolar-partitionable for $1 \leq i \leq t$, where G_i is defined as above.* □

If G_i is a flower or $G_i \in \{R_3, R_5\}$, then G_i is uniquely monopolar-partitionable. Now we determine whether or not a given block graph in Φ is uniquely monopolar-partitionable. Suppose that G is a block graph and $G \in \Phi$. Let T denote the tree structure of G. Now we define some operations on G as follows:

Operation τ_1: If a block is adjacent to a leaf, delete the leaf; If a vertex v is adjacent to two terminal blocks, delete one terminal block; If G contains the tangent subgraph $R_6 = R_u^1$, delete $V(R_6) \setminus \{u\}$.

Operation τ_2: Suppose that each down vertex of a big block is only adjacent to a leaf, a terminal bowtie, a terminal block or an end star. If there exists a down vertex such that it is not adjacent to a terminal bowtie, then delete all the children of each down vertex of the block; otherwise, delete all the children of each down vertex except one terminal bowtie.

Operation τ_3: Suppose that $i = 1, 4$. If G contains the tangent subgraph F_{ij} for $j = 1, 4$, delete H_j; If G contains tangent subgraph F_j^4 for $j = 1, 2, 3, 4$, delete the leaf; If G contains tangent subgraph F_{3i}, delete $H_i \cup H_3$ and adhere a big block to w_3; If G contains tangent subgraph F_2^j for $j = 2, 3$, delete the end star and the terminal block; If G contains the tangent subgraph Y_i, delete $V(H_i) \setminus \{u_i\}$; If G contains the tangent subgraph Y_3, delete F_3.

Operation τ_4: Suppose that $i = 2, 5$. If G contains tangent the subgraph YF_{ij} for $j = 1, 4$, delete H_j; If G contains tangent subgraph YF_{3i}, delete $F_i \cup H_3$ and adhere a big block to m_3; If G contains the tangent subgraph Y_{ij} for $j = 2, 5$, delete F_j; If G contains the tangent subgraph Y_i^4, delete the leaf; If G contains the tangent subgraph X_i, delete $V(F_i) \setminus \{w_i\}$.

Operation τ_5: For $i \in \{1, 4\}$, if G contains the tangent subgraph $R_7^{H_i} = R_u^1$, delete H_i; if G contains the tangent subgraph $R_8^{H_i} = R_u^1$, delete all the children of the big block of $R_8^{H_i}$. For $i \in \{2, 5\}$, if G contains the tangent subgraph $R_7^{F_i} = R_u^1$, delete F_i; if G contains the tangent subgraph $R_8^{F_i} = R_u^1$, delete all the children of the big block of $R_8^{F_i}$.

By Propositions in Section 3 and Section 4, we have the following.

Theorem 5.2 *Let G' be the graph obtained from $G \in \Phi$ by some operation τ_i for $i \in \{1, \cdots, 5\}$. Then G is uniquely monopolar-partitionable if and only if G' is.* □

Let G^* be the graph obtained from $G \in \Phi$ by a series of operations τ_i, where $i \in \{1, \cdots, 5\}$. It is obvious that $G^* \in \Phi$.

Theorem 5.3 *Let $G \in \Phi$ with $diam(T) \leq 1$, where T denotes the tree structure of G. Then G is uniquely monopolar-partitionable if and only if G is isomorphic to a bowtie or H_3.* □

To present our algorithm for recognizing uniquely monopolar-partitionable block graphs, we introduce the following five properties P_i and two sets $\mathcal{G}_i$ of graphs:

P_1 : there exists a vertex v and a component C of $G - v$ such that $G[V(C) \cup \{v\}]$ is a tree and is not a star;

P_2: some suspending block is adjacent to a terminal block, an end star or a terminal bowtie;

P_3: there exists a vertex v such that v neither belongs to a big block nor adjacent to a terminal bowtie, but v is adjacent to either an end star or adjacent to both a leaf and a terminal block;

P_4: for $i \in \{1, 4\}$, F_i is a tangent subgraph of G and w_i is a down vertex of a big block;

P_5: for $j \in \{2, 5\}$, Y_j is a tangent subgraph of G and m_j is a down vertex of a big block.

$\mathcal{G}_1 = \{F_{12}, F_{24}, YF_{22}, YF_{25}, YF_{52}, YF_{55}, F_i^j, Y_k^j \,|\, i = 1, 4\; k = 2, 5,\; j = 1, 2, 3\}$

$\mathcal{G}_2 = \{F_1, F_2, F_3, H_4, Y_2, Y_5\}$

Algorithm

Input: A connected block graph $G \in \Phi$. Let T denote the tree structure of G, and let $v_0v_1 \cdots v_d$ be a longest path of T and $(V_G^0, V_G^1, \cdots, V_G^d)$ be a vertex partition of G according to T.

Output: Determine whether or not G is uniquely monopolar-partitionable.

Repeatedly apply operation τ_i for $i = 1, \cdots, 5$, until one of the following occurs

- G has property P_i for $i \in \{1, \cdots, 5\}$ (// G is not uniquely monopolar-partitionable);
- G contains a graph in $\mathcal{G}_1$ as a tangent subgraph (// G is not uniquely monopolar-partitionable);
- the reduced graph is in $\mathcal{G}_2$ (// G is not uniquely monopolar-partitionable);
- $diam(T) \leq 1$ (// If $diam(T) \leq 1$ and G is isomorphic to a bowtie or to H_3, then G is uniquely monopolar-partitionable; otherwise G is not uniquely monopolar-partitionable).

We now discuss the correctness of the algorithm. In applying operations τ_i, if any property P_i occurs, then G is not uniquely monopolar-partitionable according to Proposition 3.3, Corollary 3.8, and Propositions 4.2, 4.8 and 4.15; if some graph in $\mathcal{G}_1$ is a tangent subgraph of G, then by Corollaries 4.4 and 4.12, G is not uniquely monopolar-partitionable. Suppose that none of properties P_i occurs and G does not contain any graph of $\mathcal{G}_1$ as a tangent subgraph. Then the operations applied to G yield either a graph in $\mathcal{G}_2$ or a graph whose associated T has diameter at most one. If the reduced graph is in $\mathcal{G}_2$, then it is obvious that G is not uniquely monopolar-partitionable. When $diam(T) \leq 1$, by Theorem 5.3, G is uniquely monopolar-partitionable if it is isomorphic to a bowtie or to H_3; otherwise G is not uniquely monopolar-partitionable. Moreover all these steps can be implemented in polynomial time. Therefore we have the following:

Theorem 5.4 *There is a polynomial time algorithm to decide if an input block graph is uniquely monopolar-partitionable.* □

References

[1] H-J Bandelt and H.M. Mulder, Distance-hereditary graphs, J. Combinatorial Theory B 41 (1986) 182 - 208.

[2] Z.A. Chernyak and A.A. Chernyak, About recognizing (α, β)-classes of polar graphs, Discrete Math. 62 (1986) 133 - 138.

[3] R. Churchley and J. Huang, Line-polar graphs: characterization and recognition, SIAM Journal on Discrete Mathematics 25 (2011) 1269 - 1284.

[4] R. Churchley and J. Huang, List monopolar partitions of claw-free graphs, Discrete Mathematics (2011) 2545 - 2549.

[5] R. Churchley and J. Huang, Solving partition problems with colour-bipartitions, Graphs and Combinatorics (2012) DOI 10.1007/s00373-012-1270-z.

[6] R. Churchley and J. Huang, On the polarity and monopolarity of graphs, Journal of Graph Theory (2013) DOI 10.1002/jgt.21755.

[7] T. Ekim, P. Heggernes, and D. Meister, Polar permutation graphs, J. Fiala, J. Kratochvil, and M. Miller (eds.): IWOCA 2009, LNCS 5874, pp 218 - 229, 2009.

[8] T. Ekim, P. Hell, J. Stacho, and, D. de Werra, Polarity of chordal graphs, Discrete Applied Math. 156 (2008) 2469 - 2479.

[9] T. Ekim and J. Huang, Recognizing line-polar bipartite graphs in time $O(n)$, Discrete Applied Math. 158 (2010) 1593 - 1598.

[10] T. Ekim, N.V.R. Mahadev and D. de Werra, Polar cographs, Discrete Applied Math. 156 (2008) 1652 - 1660.

[11] A. Farrugia, Vertex-partitioning into fixed additive induced-hereditary properties is NP-hard, Electronic J. Combinatorics 11 (2004).

[12] S. Foldes and P.L. Hammer, Split graphs, Proc. 8th South-Eastern Conf. on Combinatorics, Graph Theory and Computing (1977) 311 - 315.

[13] M.R. Garey and D.S. Johnson, **Computers and Intractability**, W.H. Freeman and Company, San Francisco (1979)., Graph Theory and Computing (1977) 311 - 315.

[14] M.C. Golumbic, **Algorithmic Graph Theory and Perfect Graphs**, Academic Press, New York, 1980.

[15] F. Harary, A characterization of block graphs, Canadian Mathematical Bulletin 6 (1963) 1 - 6.

[16] J. Huang and B. Xu, A forbidden subgraph characterization of line-polar bipartite graphs, Discrete Applied Math. 158 (2010) 666 - 680.

[17] R.I. Tyshkevich and A.A. Chernyak, Decompositions of graphs, Cybernetics and System Analysis 21 (1985) 231 - 242.

[18] D. West, **Introduction to Graph Theory**, Second Edition, Prentice Hall, 2001.