Nonrepetitive colorings of lexicographic product of paths and other graphs

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A coloring c of the vertices of a graph G is nonrepetitive if there exists no path $v_1v_2\dots v_{2l}$ for which $c(v_i) = c(v_{l+i})$ for all $1 \le i \le l$. Given graphs G and H with $|V(H)| = k$, the lexicographic product $G[H]$ is the graph obtained by substituting every vertex of G by a copy of H, and every edge of G by a copy of $K_{k,k}$. We prove that for a sufficiently long path P, a nonrepetitive coloring of $P[K_k]$ needs at least $3k + \lfloor k/2 \rfloor$ colors. If $k > 2$ then we need exactly $2k + 1$ colors to nonrepetitively color $P[E_k]$, where E_k is the empty graph on k vertices. If we further require that every copy of E_k be rainbow-colored and the path P is sufficiently long, then the smallest number of colors needed for $P[E_k]$ is at least $3k + 1$ and at most $3k + \lceil k/2 \rceil$. Finally, we define fractional nonrepetitive colorings of graphs and consider the connections between this notion and the above results.

Keywords: nonrepetitive coloring, thue coloring, lexicographic product

1 Introduction

A sequence $x_1 \dots x_{2l}$ is a *repetition* if $x_i = x_{l+i}$ for all $1 \le i \le l$. A sequence is *nonrepetitive* if it does not contain a string of consecutive entries forming a repetition. In 1906, Thue [11] found an infinite nonrepetitive sequence using only three symbols.

Alon, Grytczuk, Hałuszczak, Riordan [2] generalized the notion of nonrepetitiveness to graph coloring: a coloring c of a graph G is *nonrepetitive* if there is no path $v_1, \dots, v_{2l}$ in G such that the string $c(v_1), \dots, c(v_{2l})$ is a repetition. Throughout the paper, for any vertex v and set A of vertices, $c(v)$ denotes the color of v, while $c[A] = \{c(v) : v \in A\}$ denotes the set of colors that appear on the set A. The *Thue chromatic number* of G is the least integer $\pi(G)$ such that there exists a nonrepetitive coloring c of G

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using $\pi(G)$ colors. With this notation, Thue's result says $\pi(P_\infty) = 3$ (the fact that 2 colors are not enough can be easily seen for a path of length at least 4). A survey and a good introduction to the topic is [7].

In this paper we are interested in nonrepetitive coloring of the lexicographic product of graphs.

Definition 1.1 *Let $G = (V_1, E_1)$ and $H = (V_2, E_2)$ be two graphs. The lexicographic product of G and H is the graph $G[H]$ with vertex set $V_1 \times V_2$ and (v_1, v_2) is joined to (v_1', v_2') if either $(v_1, v_1') \in E_1$ or $v_1 = v_1'$ and $(v_2, v_2') \in E_2$.*

For any vertex $v \in V_1$, the set $\{(v, v_2) : v_2 \in V_2\}$, denoted by $v[H]$, is called a *layer* of $G[H]$ and the subgraph induced by a layer is isomorphic to H. If all the vertices in $v[H]$ are colored by distinct colors, then we say $v[H]$ is *rainbow colored*. A *rainbow nonrepetitive coloring* of $G[H]$ is a nonrepetitive coloring c of $G[H]$ in which all the layers are rainbow colored. The *rainbow Thue chromatic number* of $G[H]$ is the least integer $\pi_R(G[H])$ such that there exists a rainbow nonrepetitive coloring c of $G[H]$ using $\pi_R(G[H])$ colors.

Denote by E_n, K_n, P_n the empty graph, the complete graph and the path on n vertices, respectively. It follows from the definition that $\pi(G[E_k]) \leq \pi_R(G[E_k]) \leq \pi_R(G[K_k]) = \pi(G[K_k])$ for any graph G (note that every nonrepetitive coloring of $G[K_n]$ is a rainbow nonrepetitive coloring).

Non-repetitive colorings of the lexicographic product of graphs has not been studied systematically before. However, a result of Barát and Wood [3] can be rephrased in our context: in Lemma 2 of their paper they showed that for any tree T and integer k, $\pi(T[K_k]) \leq 4k$. We shall prove that this bound is sharp, by constructing a tree T for which $\pi(T[E_k]) = 4k$ for every positive integer k (Lemma 2.4).

Our main results concentrate on the lexicographic product of paths with complete graphs or empty graphs.

Theorem 1.2 *For any $n \geq 4$ and $k \neq 2$, $\pi(P_n[E_k]) = 2k + 1$. For $k = 2$, $5 \leq \pi(P_n[E_2]) \leq 6$.*

Theorem 1.3 *For any pair of integers $n \geq 24$ and $k \geq 2$, $3k + 1 \leq \pi_R(P_n[E_k]) \leq 3k + \lceil k/2 \rceil$.*

Theorem 1.4 *For any integer $n \geq 28$, $3k + \lfloor k/2 \rfloor \leq \pi(P_n[K_k]) \leq 4k$.*

2 Proofs

We present the proofs of the lower and upper bounds in separate subsections. Most lower bounds rely on the same lemmas. The proofs for the upper bounds use earlier ideas and results by Kündgen and Pelsmajer [10].

2.1 Lower bounds

Lemma 2.1 *Let c be a nonrepetitive coloring of $G[E_k]$. If $v \in V(G)$ is a vertex of degree d and two vertices in $v[E_k]$ receive the same color, then c uses at least $dk + 1$ colors.*

Proof: Let $v_1, v_2, \ldots, v_d$ be the neighbors of v in G, and let $u_1, u_2 \in v[E_k]$ be vertices with $c(u_1) = c(u_2)$. For any pair of vertices $w_1, w_2 \in \cup_{i=1}^{d} v_i[E_k]$, we have $c(w_1) \neq c(w_2)$, for otherwise the coloring of the path $w_1 u_1 w_2 u_2$ would be a repetition. Also colors used for vertices in $\cup_{i=1}^{d} v_i[E_k]$ are different from that of u_1 and u_2. Hence c uses at least $dk + 1$ colors. □

Lemma 2.2 *Let $P = (v_1v_2v_3v_4)$ be a path of* 4 *vertices in G and c be a nonrepetitive coloring of $G[E_k]$. Then either the color sets of the first three layers are pairwise disjoint or the color sets of the last three layers are pairwise disjoint. In particular, if all the four layers are rainbow colored, then c uses at least $3k$ colors.*

Proof: Otherwise there exist $a \in c[v_1[E_k]] \cap c[v_3[E_k]]$ and $b \in c[v_2[E_k]] \cap c[v_4[E_k]]$, and hence a path with colors $abab$. □

We now construct a tree T with $\pi(T[E_k])$ matches the upper bound of Barát and Wood [3] mentioned in the introduction. Let $T_{3,6}$ denote the rooted tree in which all non-leaf vertices have degree three, and all leaves have distance 5 from root vertex, i.e. $T_{3,6}$ looks like the usual binary tree except that the root has three children. We will use the notions children and father in the standard way.

Lemma 2.3 *A rainbow nonrepetitive coloring c of $T_{3,6}[E_k]$ uses at least $4k$ colors.*

Proof: Assume c is a rainbow nonrepetitive coloring of $T_{3,6}[E_k]$ using at most $4k - 1$ colors. By pigeonhole principle, the root r has two children, say v_1, v_2 with $c[v_1[E_k] \cap c[v_2[E_k]] \neq \emptyset$. This implies that for the two children w_1, w_2 of v_1, $c[w_i[E_k]] \cap c[r[E_k]] = \emptyset$, for otherwise, if $a \in c[w_i[E_k]] \cap c[r[E_k]]$ and $b \in c[v_1[E_k] \cap c[v_2[E_k]]$, then there is a path coloured as $abab$.

By pigeonhole principle again, $c[w_1[E_k]] \cap c[w_2[E_k]] \neq \emptyset$. The same argument as above shows that the colour set of any child of w_1 is disjoint from the colour set of the father of w_1.

Repeat this argument, we find a path $u_0u_1u_2u_3u_4u_5$ in $T_{3,6}$ such that u_0 is the root of $T_{3,6}$ and $c[u_i[E_k]]$ is disjoint from $c[u_j[E_k]]$ for $j = i \pm 1, 2$. But then again as c uses at most $4k - 1$ colors we find vertices $w_i \in u_i[E_k]$ $i = 0, 1, \ldots, 5$ such that $c(w_0) = c(w_3), c(w_1) = c(w_4), c(w_2) = c(w_5)$ and thus $w_0w_1w_2w_3w_4w_5$ is a repetition of size six. □

Lemma 2.4 *There exists a tree T such that for any positive integer k, $\pi(T[E_k]) = 4k$.*

Proof: Let $T = T_{4,7}$ be the rooted tree in which all non-leaf vertices have degree four, and all leaves have distance 6 from the root vertex. As mentioned above, it was proved by Barát and Wood [3] that $\pi(T[E_k]) \leq 4k$. Let c be a nonrepetitive coloring of $T[E_k]$. We shall show that at least $4k$ colors are used. If a subgraph of $T_{4,7}[E_k]$ isomorphic to $T_{3,6}[E_k]$ is rainbow-colored, then we are done by Lemma 2.3. If not, then we are done by Lemma 2.1. □

To prove the lower bounds of Theorem 1.3 and Theorem 1.4 we need some preparations. Given a nonrepetitive sequence S over 3 letters A, B, C, by a *palindrome* we mean a subsequence of consecutive elements $x_1 \ldots x_{2l+1}$ of odd length $2l + 1 \geq 3$ such that $x_i = x_{2l+2-i}$ for $i = 1, 2, \ldots, l$. The middle letter x_{l+1} of a palindrome is called a *peak* of the sequence. When writing a sequence, we emphasize peaks by underlining them. The *gap* between two consecutive peaks is the number of letters between them in S. For technical reasons, the first and the last letter of a sequence is also regarded as a peak. In other words, a letter is not a peak if and only if its two neighbors exist and are different. Two sequences are *equivalent* if they are the same up to a permutation of the letters A, B and C.

Lemma 2.5 *In a sequence S over* 3 *letters that avoids repetitions of length at most* 6 *each gap is at most* 3 *and at least* 1*, except the first and the last gap that can be* 0*.*

Proof: If there is a 0 gap which is neither the first gap nor the last gap, then there would be a repetition of length 4 in S. To prove that a gap is at most 3, observe that between two peaks the letters are determined by the first peak-letter x and the letter after x. Indeed, without loss of generality, if these letters are $\underline{A}B$ then as B is not a peak, the third letter is C. In general the next letter is always the letter different from the previous two letters until we reach the next peak. Thus if there would be a gap of size 4 then there would be a sequence equivalent to $\underline{A}BCABC$ (the last letter may or may not be a peak), which includes a repetition. □

Lemma 2.6 *In a sequence over* 3 *letters, if* v *is a peak with gap* g_1 *on one side and* $g_2 \geq g_1$ *on the other side, then it is the center of a palindrome of length* $2g_1 + 3$*.*

Proof: This follows again from the fact that the peak and its neighbor determine all the letters until the next peak (on both sides). So going from v to each side, the $g_1 + 1$ letters are the same, and hence v is the center of a palindrome of length $2g_1 + 3$. □

Lemma 2.7 *Assume* S *is a sequence on* 3 *letters that avoids repetitions of length at most* 6*. If there are three consecutive gaps* $g_1 \geq g_2 \leq g_3$*, then there is a subsequence equivalent to one of the following*

1. $CB\underline{A}B\underline{C}BA$
2. $ACB\underline{A}BC\underline{A}CBA$
3. $BACB\underline{A}BCA\underline{B}ACBA$.

Proof: By Lemma 2.5, $g_2 = 1, 2$ or 3. By observing that letters between two peaks are determined by the peak-letters and the letter besides the peak letters, it is easy to verify that if $g_2 = 1$ (respectively, $g_2 = 2$ or $g_2 = 3$), then the resulting subsequence is as the first (respectively, the 2nd or the 3rd) listed above. Note that the first and last letters in these sequences might also be peaks. □

Lemma 2.8 *Given a sequence* S *of length* 22 *on* 3 *letters that avoids repetitions of length at most* 6*, there exist three consecutive gaps* $g_1 \geq g_2 \leq g_3$*.*

Proof: By Lemma 2.5, the series of gaps contains only the numbers $0, 1, 2, 3$. Suppose that the sequence S does not contain three consecutive gaps $g_1 \geq g_2 \leq g_3$. Then 0 can only be the length of the first or the last gap, a gap of length 1 must be adjacent to a gap of length 0, a gap of length 2 must be adjacent to a gap of length at most 1, and a gap of length 3 must be adjacent to a gap of length at most 2. The longest such sequence of gaps is the following: $0, 1, 2, 3, 3, 2, 1, 0$. Thus the sequence can have length at most $12 + 9 = 21$ (the number of letters in gaps plus the number of peak letters interceding them). □

Lemma 2.9 *For* $k \geq 2$*,* $\pi_R(P_{24}[E_k]) \geq 3k + 1$*.*

Proof: Let $P_{24} = p_1 p_2 \ldots p_{24}$ and $G = P_{24}[E_k]$. For simplicity we denote the layer corresponding to p_i by V_i. Suppose G has a nonrepetitive rainbow $3k$-coloring. By Lemma 2.2, all $3k$ colors are used. We distinguish two cases.

CASE A: There exists an index $2 \leq j \leq 21$ such that $c[V_j] \neq c[V_{j+2}]$ and $c[V_j] \cap c[V_{j+2}] \neq \emptyset$.

Suppose first that $2 \leq j \leq 19$. Let b be a color in $c[V_j] \cap c[V_{j+2}]$. By Lemma 2.2, $c[V_{j-1}] \cap c[V_{j+1}] = \emptyset$ and $c[V_{j+1}] \cap c[V_{j+3}] = \emptyset$.

As both $\{c[V_{j-1}], c[V_j], c[V_{j+1}]\}$ and $\{c[V_{j+1}], c[V_{j+2}], c[V_{j+3}]\}$ partition the colors into 3 parts of size k and $c[V_j] \neq c[V_{j+2}]$, there exist colors $d \in c[V_j] \cap c[V_{j+3}]$, $e \in c[V_{j-1}] \cap c[V_{j+2}]$ and $f \in c[V_{j-1}] \cap c[V_{j+3}]$. Now $c[V_{j+4}]$ must be disjoint from $c[V_{j+1}]$, as a color a appearing in both $c[V_{j+1}]$ and $c[V_{j+4}]$ would yield a repetition $edaeda$ of colors on $V_{j-1}, V_j, V_{j+1}, V_{j+2}, V_{j+3}, V_{j+4}$. As $c[V_{j+4}]$ is also disjoint from $c[V_{j+3}]$ we must have $c[V_{j+4}] = c[V_{j+2}]$. As $k \geq 2$, there are colors $b, h \in c[V_{j+2}] = c[V_{j+4}]$.

Now, $c[V_{j+5}]$ is disjoint from $c[V_{j+4}]$ and also disjoint from $c[V_{j+3}]$ (as otherwise there would be a repetition $hghg$, where $g \in c[V_{j+3}] \cap c[V_{j+5}]$). Thus $c[V_{j+5}] = c[V_{j+1}]$. Picking a color $a \in c[V_{j+5}] = c[V_{j+1}]$ we obtain a repetitively colored path $v_{-1}, v_0, v_1, v_2, v_3, v_4, v_5, v'_4$ ($v_i \in V_{j+i}$ and $v'_4 \in V_{j+4}$) with colors $fbahfbah$, a contradiction.

This proof works only if $2 \leq j \leq 19$, as we used the existence of $V_{j-1}, \dots, V_{j+5}$. Yet a symmetric reasoning works in case $6 \leq j \leq 23$, thus covering the whole range of possible values of j.

CASE B: For each $2 \leq j \leq 21$, either $c[V_j] = c[V_{j+2}]$ or $c[V_j] \cap c[V_{j+2}] = \emptyset$.

First we prove that there exists a partition $A \cup B \cup C$ of the $3k$ colors such that for every $2 \leq j \leq 21$, $c[V_j] = A$ or B or C. Indeed, write $A = c[V_2]$ and $B = c[V_3]$. We prove by induction that for every $4 \leq j \leq 21$, $c[V_j]$ equals to one of A, B, C. To avoid repetitions of size two $c[V_j]$ must be disjoint from $c[V_{j-1}]$ and if it is not the same as $c[V_{j-2}]$, then by the assumption of CASE B, $c[V_j] \cap c[V_{j-2}] = \emptyset$. As there are only $3k$ available colors, $c[V_j]$ must be equal to the third color set (the one different from $c[V_{j-1}]$ and $c[V_{j-2}]$, which by induction are two color sets from A, B, C).

Thus the coloring of the layers from $j = 2$ to $j = 23$ can be regarded as a sequence on the three letters A, B, C, which has length 22. Observe that this sequence is repetition-free, as otherwise there would be a repetitive path in the coloring of the original graph. By Lemma 2.7 and Lemma 2.8, there is a subsequence of the form $CBABCBA$ or $ACBABCACBA$ or $BACBABCABACBA$.

Each of A, B, C contains $k \geq 2$ colors. Let a_1, a_2 (respectively, b_1, b_2 and c_1, c_2) be two distinct colors in A (respectively, B and C). Then a path of color sequence $b_1c_1b_2a_1b_1c_1b_2a_1$ can be found from the parts with color sequence $CBABCBA$. Indeed, to find this start from the second part (which has color set B), go to the first part (which has color set A), then follow the original path to the end. Similarly, paths of color sequences $b_1c_1a_1c_2b_2a_1b_1c_1a_1c_2b_2a_1$ and $b_1c_1a_1b_1a_2c_2b_2a_1b_1c_1a_1b_1a_2c_2b_2a_1$ can be found from the parts with color sequence $ACBABCACBA$ and $BACBABCABACBA$, respectively. □

Theorem 2.10 *For any integer $k \geq 1$, $\pi(P_{28}[K_k]) \geq 3k + \lfloor k/2 \rfloor$.*

Proof: Assume to the contrary that there is a nonrepetitive coloring c of $G = P_{28}[K_k]$ with $3k + \lfloor k/2 \rfloor - 1$ colors. The vertices of P_{28} are $v_1, v_2, \dots, v_{28}$. Let $X_i = c(v_i[K_k])$. So each X_i is a k-subset of the $3k + \lfloor k/2 \rfloor - 1$ colors. For the remainder of this proof, a *set of colors* means a k-subset of the set of the $3k + \lfloor k/2 \rfloor - 1$ colors. For two sets of colors X and Y, we say X is *Y-rich* (and Y is X-rich) if $|X \cap Y| \geq \lceil k/2 + 1 \rceil$. We write $XYZ \in \mathcal{T}$ if X, Y, Z are three pairwise disjoint color sets, and write $XYZW \in \mathcal{Q}$ if $XYZ \in \mathcal{T}$ and $YZW \in \mathcal{T}$. We shall frequently use the following observation.

Proposition 2.11 *If Y is X-rich and Z is Y-rich then $|X \cap Z| \geq 2$. If $XYZW \in \mathcal{Q}$ then W is X-rich.*

Claim 2.12 *Assume $P_{11}[K_k]$ is nonrepetitively colored with $3k + \lfloor k/2 \rfloor - 1$ colors, and the color sets of the layers are $XYABCDEFGZW$ and $ABC \in \mathcal{T}$.*

(1) If $DEF \in \mathcal{T}$, then D, E, F are either B, A, C-rich respectively, or A, C, B-rich, respectively.

(2) If $D \cap F \neq \emptyset$, then $EFG \in \mathcal{T}$ and one of the following holds:

(i) F is D-rich and D, E, F, G are A, B, A, C-rich, respectively.

(ii) G is D-rich and D, E, F, G are B, A, C, B-rich, respectively.

The proof of this claim is postponed to the next subsection. Now we use this claim and continue with the proof of Theorem 2.10.

We (partially) label the sequence $X_3X_4 \dots X_{30}$ by three labels as follows: The first three consecutive pairwise disjoint color sets are labeled A, B, C, respectively. In other words, if $X_3X_4X_5 \in \mathcal{T}$, then X_3, X_4, X_5 are labeled A, B, C, respectively. Otherwise, $X_4X_5X_6 \in \mathcal{T}$, then X_4, X_5, X_6 are labeled A, B, C, respectively, and X_3 is unlabeled. Suppose we have already labeled $X_3X_4 \dots X_i$ (with X_3 possibly unlabeled). Let j be the largest index such that $j \leq i$ and X_{i+1} is X_j-rich. We label X_{i+1} the same label as X_j. By Claim 2.12, we can label three or four consecutive color sets simultaneously at each step. Note that by using Claim 2.12 to label three or four consecutive color sets, the last three consecutive color sets are always pairwise disjoint. So we can repeatedly apply Claim 2.12 to label the next three or four consecutive color sets. Thus the labeling is well-defined, except possibly the last three color sets are unlabeled.

Denote by S the label sequence constructed above, which has length at least 22 (the first two color sets were not labeled, further, the third color set and the last five color sets may not be labeled). The following observation follows from the definition.

Observation 2.13 *If two color sets X_i and X_j have the same label and there is at most one other color set between them that gets the same label, then $|X_i \cap X_j| \geq 2$.*

In particular, if $|i - j| \leq 3$ and X_i and X_j have the same label, then $|X_i \cap X_j| \geq 2$. Therefore, if S has a repetition of length at most 6, then it yields a repetitive path in G of length at most 6 along the corresponding layers. Thus S contains no repetition of length at most 6. By Lemma 2.7 and Lemma 2.8, there exists a subsequence S' that is equivalent to one of the following sequences:

CASE (I) $S' = CB\underline{A}B\underline{C}BA$

We write the sequence of color sets corresponding to S' as $CBAB_1C_1B_2A_1$. By Observation 2.13, there is a repetitive path in G with colors $cbab'cbab'$ where $c \in C, C_1; b \in B, B_2; a \in A, A_1, b' \in B_1, B_2$.

CASE (II) $S' = ACB\underline{A}BC\underline{A}CBA$

We write the sequence of color classes of the layers corresponding to S' as $ACBA_1B_1C_1A_2C_2B_2A_3$. Again it follows from Observation 2.13 that there is a repetitive path in G with colors $acba'b'c'acba'b'c'$ where $a \in A, A_2; c \in C, C_2; b \in B, B_2, a' \in A_1, A_3, b' \in B_1, B_2, c' \in C_1, C_2$.

CASE (III) $S' = BACB\underline{A}BCA\underline{B}ACBA$

We write the sequence of color sets corresponding to S' as $B_0A_0C_0B_1A_1B_2C_1A_2B_3A_3C_2B_4A_4$.

We claim that there is a repetitive path in G with colors $bacb'a'b''c'a''bacb'a'b''c'a''$ where $b \in B_0, B_3; a \in A_0, A_3; c \in C_0, C_2; b' \in B_1, B_4, a' \in A_1, A_4, b'' \in B_2, B_4, c' \in C_1, C_2; a'' \in A_2, A_3$. For this purpose, it suffices to show that in each pair of layers from which we need to pick vertices with the same color, we have at least two possible choices. This follows from Observation 2.13 if the two layers correspond to Y_i and Y_{i+1} or Y_i and Y_{i+2} for some letter $Y \in \{A, B, C\}$. There are some pairs of the form Y_i and Y_{i+3}

with $Y \in \{A, B\}$ for which we need to pick vertices with the same color. Hence we need to show that $|Y_i \cap Y_{i+3}| \geq 2$ for these pairs. For this purpose, by Proposition 2.11, it suffices to show that either Y_i is Y_{i+2}-rich or Y_{i+1} is Y_{i+3}-rich. The required properties follow from the following claim.

Claim 2.14 *B_1 is B_3-rich and A_1 is A_3-rich.*

Proof of Claim: Consider the reverse of the subsequence $C_0B_1A_1B_2C_1A_2B_3A_3$. Since A_2 is A_3-rich, by Lemma 2.2, C_1, A_2, B_3 are pairwise disjoint. Apply Claim 2.12 to the reverse of $C_0B_1A_1B_2C_1A_2B_3$, we conclude that B_1 is B_3-rich. Similarly, by Lemma 2.2, A_1, B_2, C_1 are pairwise disjoint, and apply Claim 2.12 to $A_1B_2C_1A_2B_3A_3C_2$, we know that A_1 is A_3-rich. □

This completes the proof of Theorem 2.10 (except that the proof of Claim 2.12 will be given in the next subsection). □

2.2 Proof of Claim 2.12

Claim 2.12 follows from the following three lemmas.

Lemma 2.15 *Assume $P_6[K_k]$ is nonrepetitively colored with $3k + \lfloor k/2 \rfloor - 1$ colors, and the color sets of the layers are $ABCDEF$. If $ABC \in \mathcal{T}$ and $DEF \in \mathcal{T}$, then D, E, F are either B, A, C-rich respectively, or A, C, B-rich respectively.*

Proof: We consider three cases.

CASE 1: $D \cap A = \emptyset$.

$BACD \in \mathcal{Q}$ implies that D is B-rich. As $D \cap B \neq \emptyset$, by Lemma 2.2, $E \cap C = \emptyset$. Now $ACDE \in \mathcal{Q}$, implies that E is A-rich, and $CDEF \in \mathcal{Q}$ implies that F is C-rich.

CASE 2: $D \cap B = \emptyset$.

$ABCD \in \mathcal{Q}$ implies that D is A-rich. If E intersects both B and C, then there is a repetitive path $abcabc$ where $a \in A, D$, $b \in B, E$ and $c \in C, E$, a contradiction. If E is disjoint from B, then $CBDE \in \mathcal{Q}$ implies that E is C-rich, and $BDEF \in \mathcal{Q}$ implies that F is B-rich. So D, E, F are A, C, B-rich, respectively, and we are done. If E is disjoint from C, then $BCDE \in \mathcal{Q}$ implies that E is B-rich, and $CDEF \in \mathcal{Q}$ implies that F is C-rich. But then there is a repetition $abcabc$, $a \in A, D; b \in B, E; c \in C, F$.

CASE 3: $D \cap A \neq \emptyset$ and $D \cap B \neq \emptyset$.

In this case, $E \cap C = \emptyset$, for otherwise there is a repetition $bcbc$, $b \in B, D; c \in C, E$. Now $CDEF \in \mathcal{Q}$ implies that F is C-rich. This implies that $E \cap B = \emptyset$, for otherwise there would be a repetition $abcabc$, $a \in A, D; b \in B, E; c \in C, F$. Then $ABCE \in \mathcal{Q}$ implies that E is A-rich, and $BCED \in \mathcal{Q}$ implies that D is B-rich. □

Lemma 2.16 *Assume $P_7[K_k]$ is nonrepetitively colored with $3k + \lfloor k/2 \rfloor - 1$ colors, and the color sets of the layers are $ABCDEFG$. If $ABC \in \mathcal{T}$ and $D \cap F \neq \emptyset$, then D, E, F, G are either F, B, A, C-rich respectively, or B, A, C, B-rich respectively.*

Proof: By Lemma 2.2 and $D \cap F \neq \emptyset$, we know that $EFG \in \mathcal{T}$ and $CDE \in \mathcal{T}$. We consider three cases

CASE 1: $D \cap B = \emptyset$.

As $BCD \in \mathcal{T}$, we can apply Lemma 2.15 to the color set sequence $BCDEFG$. Thus E, F, G are either C, B, D-rich respectively, or B, D, C-rich respectively. Also $ABCD \in \mathcal{Q}$ implies that D is A-rich, and $BCDE \in \mathcal{Q}$ implies that E is B-rich. Moreover, E cannot be C-rich, as $CDE \in \mathcal{T}$. So D, E, F, G are A, B, D, C-rich respectively. This implies that $F \cap B = \emptyset$, for otherwise there is a repetitive path with colors $abb'cabb'c$, $a \in A, D; b \in B, E; b' \in B, F; c \in C, G$. Also $F \cap C = \emptyset$, for otherwise there is a repetitive path with colors $abcabc$, $a \in A, D; b \in B, E; c \in C, F$. Now $ABCF \in \mathcal{Q}$ implies that F is A-rich. Thus we have proved that D, E, F, G are F, B, A, C-rich, respectively, and D is also A-rich.

CASE 2: $D \cap A = \emptyset$.

Then $BACD \in \mathcal{Q}$ implies that D is B-rich. As $CDE \in \mathcal{T}$, $E \cap C = \emptyset$. Thus $ACDE \in \mathcal{Q}$ and hence E is A-rich.

If $F \cap B \neq \emptyset$, then $(F \cup G) \cap C = \emptyset$, for otherwise there is a repetitive path with colors $bab'cbab'c$, $b \in B, D, ; a \in A, E; b' \in B, F; c \in C, F \cup G$. Then $(E \cup F \cup G) \cap C = \emptyset$, which is a contradiction as $EFG \in \mathcal{T}$. So $F \cap B = \emptyset$.

CASE 2(I): $E \cap B \neq \emptyset$.

Then $F \cap C = \emptyset$, for otherwise there is a repetitive path with colors $bab'cbab'c$, $b \in B, D; a \in A, E; b' \in B, E; c \in C, F$. Now $ABCF \in \mathcal{Q}$ implies that F is A-rich, which is a contradiction as E is A-rich and $E \cap F = \emptyset$.

CASE 2(II): $E \cap B = \emptyset$.

Now $BEFG \in \mathcal{Q}$ implies that G is B-rich, and $CBEF \in \mathcal{Q}$ implies that F is C-rich. So we have proved that D, E, F, G are B, A, C, B-rich, respectively.

CASE 3: $D \cap A \neq \emptyset$ and $D \cap B \neq \emptyset$.

CASE 3(I): $E \cap A = \emptyset$.

Now $BACE \in \mathcal{Q}$ implies that E is B-rich, and $ACED \in \mathcal{Q}$ implies that D is A-rich. This implies that F is disjoint from C, for otherwise there is a repetition $abcabc$, $a \in A, D; b \in B, E; c \in C, F$. Then $ACEF \in \mathcal{Q}$ implies that F is A-rich, and $DECF \in \mathcal{Q}$ implies that F is D-rich, and $CEFG \in \mathcal{Q}$ implies that G is C-rich. Thus D, E, F, G are F, B, A, C-rich respectively (and D is also A-rich), and we are done.

CASE 3(II): $E \cap B = \emptyset$.

Then $ABCE \in \mathcal{Q}$ implies that E is A-rich, and $BCED \in \mathcal{Q}$ implies that D is B-rich. If $F \cap B = \emptyset$, then $CBEF \in \mathcal{Q}$ implies that F is C-rich and $BEFG \in \mathcal{Q}$ implies that G is B-rich. So D, E, F, G are B, A, C, B-rich, respectively. Thus we assume $F \cap B \neq \emptyset$. Then $(F \cup G) \cap C = \emptyset$, for otherwise there is a repetitive path with colors $bab'cbab'c$, $b \in B, D; a \in A, E; b' \in B, F; c \in C, F \cup G$. Now $(D \cup E \cup F \cup G) \cap C = \emptyset$, which is a contradiction.

CASE 3(III): $E \cap A \neq \emptyset$ and $E \cap B \neq \emptyset$.

In this case, $F \cap C = \emptyset$, for otherwise there is a repetitive path with colors $abcabc$, $a \in A, D; b \in B, E; c \in C, F$. If $F \cap B \neq \emptyset$ then $G \cap C = \emptyset$, for otherwise there is a repetitive path with colors $abb'cabb'c$, $a \in A, D; b \in B, E; b' \in B, F; c \in C, G$. Then $(D \cup E \cup F \cup G) \cap C = \emptyset$, which is a contradiction. Thus $F \cap B = \emptyset$. Now $ABCF \in \mathcal{Q}$ implies that F is A-rich, and $BCFE \in \mathcal{Q}$ implies that E is B-rich, and $CEFG \in \mathcal{Q}$ implies that G is C-rich, and $DCEF \in \mathcal{Q}$ implies that F is D-rich. So D, E, F, G are F, B, A, C-rich, respectively. □

Lemma 2.17 *Assume $P_{11}[K_k]$ is nonrepetitively colored with $3k + \lfloor k/2 \rfloor - 1$ colors, and the color sets of the layers are $XYABCDEFGZW$, and $ABC \in \mathcal{T}$ and $D \cap F \neq \emptyset$.*

(i) If D, E, F, G are F, B, A, C-rich, respectively, then D is A-rich.

(ii) If D, E, F, G are B, A, C, B-rich, respectively, then G is D-rich.

Proof: Observe that in the proof of Theorem 2.10 we started the labeling process without using X_1, X_2 so that we can always find the color sets X, Y used in this lemma.

We assumed that $ABC \in \mathcal{T}$ and also by Lemma 2.2 and the assumption $D \cap F \neq \emptyset$, we know that $EFG \in \mathcal{T}$ and $CDE \in \mathcal{T}$.

First we prove (i), thus we assume that D, E, F, G are F, B, A, C-rich, respectively. Note that in this case we do not use the existence of Z, W.

CASE 1: $BAY \in \mathcal{T}$.

We can apply Lemma 2.15 to the color set sequence $EDCBAY$. We get that B, A, Y are either D, E, C-rich or E, C, D-rich, respectively. In the first case A is E-rich and E is B-rich by assumption, so $A \cap B \neq \emptyset$, a contradiction. In the second case, as $YAB \in \mathcal{T}$ and $ABC \in \mathcal{T}$, Proposition 2.11 implies that C is Y-rich. As D is also Y-rich, $C \cap D \neq \emptyset$, a contradiction.

CASE 2: $B \cap Y \neq \emptyset$.

We can apply Lemma 2.16 to the color set sequence $EDCBAYX$. We get that B, A, Y, X are either Y, D, E, C-rich or D, E, C, D-rich, respectively. In the first case, the only case not leading to contradiction, we get that A is D-rich, as we claimed. In the second case we get that A is E-rich and E is B-rich by assumption, so $A \cap B \neq \emptyset$, a contradiction.

Now we prove (ii), thus we assume that D, E, F, G are B, A, C, B-rich, respectively. Note that in the proof of this case we will already use the statement of (i) and also we need the existence of Z, W.

CASE 3: $DCB \in \mathcal{T}$.

We can apply Lemma 2.15 to the color set sequence $GFEDCB$. We get that D, C, B are either F, G, E-rich or G, E, F-tich, respectively. In the first case C is G-rich and by the assumption G is B-rich, thus $C \cap B \neq \emptyset$, a contradiction. In the second case similarly B is F-rich and by the assumption F is C-rich, thus $C \cap B \neq \emptyset$, a contradiction.

CASE 4: $B \cap D \neq \emptyset$.

We can apply Lemma 2.16 to the color set sequence $GFEDCBA$. We get that D, C, B, A are either B, F, G, E-rich or F, G, E, F-rich, respectively.

In the first case, as $GFE \in \mathcal{T}$ and $F \cap D \neq \emptyset$, we can apply Lemma 2.17(i) to the color set sequence $WZGFEDCBAYX$. We get that if D, C, B, A are B, F, G, E-rich , respectively (and this is exactly the case), then D is G-rich, as we claimed.

In the second case D is F-rich and by the assumption F is C-rich, thus $C \cap D \neq \emptyset$, a contradiction. □

2.3 Upper bounds

Before we start our proofs, we describe some tools from the paper of Kündgen and Pelsmajer [10].

Lemma 2.18 (Kündgen, Pelsmajer, [10, Lemma 3]) *If c is a nonrepetitive palindrome-free coloring of a path P, and P' is obtained from P by adding a loop at each vertex, then every repetitively colored walk W_1W_2 in P' satisfies $W_1 = W_2$.*

Let $V_1, \ldots, V_m$ be a partition of $V(G)$ and let G_k and $G_{>k}$ denote the subgraphs of G induced by V_k and $V_{k+1} \cup \ldots \cup V_m$, respectively. The *$k$-shadow* of a subgraph H of G is the set of vertices in V_k which have a neighbor in $V(H)$. We say that G is *shadow complete* (with respect to the partition) if the k-shadow of every component of $G_{>k}$ induces a complete graph.

Theorem 2.19 (Kündgen, Pelsmajer, [10, Theorem 6]) *If G is shadow complete and each G_k has a nonrepetitive coloring with b colors, then G has a nonrepetitive coloring with $4b$ colors.*

Proof of Theorem 1.2: Recall that we want to prove that for any $n \geq 4$ and $k \neq 2$, we have $\pi(P_n[E_k]) = 2k+1$ and for $k = 2$ we have $5 \leq \pi(P_n[E_2]) \leq 6$. The lower bounds of the theorem follow from Lemma 2.1 and Lemma 2.2.

To prove the upper bounds we need to define a nonrepetitive coloring c of $P_\infty[E_k]$. For $k \geq 3$ let Y denote the set $\{k+1, k+2, \ldots, 2k+1\}$ and X denote the set $\{1, 2, \ldots, k\}$. If $k = 2$, then let $X = \{1, 2\}, Y = \{3, 4, 5, 6\}$. Elements of Y will be denoted by lower case letters a, b, c, a_1, etc. Let $S = s_1s_2s_3s_4\ldots$. be an infinite palindrome-free nonrepetitive sequence. There exists such a sequence that uses only 4 symbols [2]. Thus we can pick all s_i's from Y. Let the vertex set of P_∞ be $\{v_1, v_2, \ldots\}$ and $E(P_\infty) = \{(v_i, v_{i+1}) : 1 \leq i\}$. If $j = 4(i-1)+1$, then define c on $v_j[E_k]$ such that $c[v_j[E_k]] = X$. If $j = 4(i-1)+2$ or $j = 4i$, then for any vertex $u \in v_j[E_k]$ let $c(u) = s_i$. Finally, if $j = 4(i-1)+3$, then define c on $v_j[E_k]$ such that $c[v_j[E_k]]$ is a k-subset of $Y \setminus s_i$ (note that if $k \geq 3$, then $|Y \setminus s_i| = k$ and if $k = 2$, then $|Y \setminus s_i| = 3$).

We claim that c is nonrepetitive. Assume to the contrary that there is a path Q_1Q_2 in $P_\infty[E_k]$ such that the sequence of colors on Q_1Q_2 is a repetition. Remove all vertices from Q_1Q_2 that have colors from the set X. The sequence of colors of the remaining vertices $Q'_1Q'_2 = (q'_{1,1}\ldots q'_{1,l}q'_{2,1}\ldots q'_{2,l})$ still form a repetition. Let P'_∞ be an infinite path with one loop added to each of its vertices. Furthermore, let c_S be the coloring of P'_∞ with $c_S(p'_j) = s_j$. Let us define the function $f : Q'_1Q'_2 \to P'_\infty$ with $f(q) = p'_i$ if and only if $q \in v_{4(i-1)+2}[E_k] \cup v_{4(i-1)+3}[E_k] \cup v_{4i}[E_k]$. Writing W_1 and W_2 for the images of Q'_1 and Q'_2, we obtain that W_1W_2 is a walk in P'_∞.

Claim 2.20 *The sequence of colors of vertices in W_1W_2 with respect to the coloring c_S is a repetition.*

Proof: Let $1 \leq m \leq l$. Consider the largest parts of Q_1 and Q_2 that contain $q'_{1,m}$ and $q'_{2,m}$ such that they form a subpath of Q'_1 and Q'_2, i.e. the subpaths of Q_1 and Q_2 that lie between consecutive X-colored vertices of Q_1 and Q_2. Clearly, the part in Q_1 lies entirely within $v_{4(i-1)+2}[E_k] \cup v_{4(i-1)+3}[E_k] \cup v_{4i}[E_k]$ for some i and the part in Q_2 lies entirely within $v_{4(j-1)+2}[E_k] \cup v_{4(j-1)+3}[E_k] \cup v_{4j}[E_k]$ for some j

and vertices of the former are mapped by f to p'_i and those of the latter are mapped by f to p'_j. If these paths are $(q'_{1,m_1} \dots q'_{1,m} \dots q'_{1,m_2})$ and $(q'_{2,m_1} \dots q'_{2,m} \dots q'_{2,m_2})$, then $c(q'_{1,m_1}) = c(q'_{2,m_1})$ and $c(q'_{1,m_2}) = c(q'_{2,m_2})$ and at least one of the pairs $(q'_{1,m_1}, q'_{2,m_1}), (q'_{1,m_2}, q'_{2,m_2})$, say the former one, lie next to an X-colored vertex and therefore their c-color is s_i and s_j. This shows that $c_S(f(q'_{1,m})) = s_i = c(q'_{1,m_1}) = c(q'_{2,m_1}) = s_j = c_S(f(q'_{2,m}))$. □ By Claim 2.20 and Lemma 2.18, $W_1 = W_2$. Suppose first that $W_1 = W_2$ contains at least two different vertices. This means that the original paths Q_1 and Q_2 had to cross from $v_{4(i-1)+2}[E_k] \cup v_{4(i-1)+3}[E_k] \cup v_{4i}[E_k]$ to $v_{4(i)+2}[E_k] \cup v_{4(i)+3}[E_k] \cup v_{4(i+1)}[E_k]$ or vice versa. But as the layer $v_{4i+1}[E_k]$ is rainbow colored with colors in X, the original color sequence of Q_1Q_2 could not be a repetition.

Suppose then that W_1W_2 is a walk repeating the same vertex p'_i. Then all vertices of Q_1Q_2 must lie in $v_{4(i-1)+1}[E_k] \cup v_{4(i-1)+2}[E_k] \cup v_{4(i-1)+3}[E_k] \cup v_{4i}[E_k] \cup v_{4i+1}[E_k]$. Therefore Q_1Q_2 cannot contain any vertex from $v_{4(i-1)+3}[E_k]$ as they have unique colors among vertices in these 5 layers, preventing the possibility of a repetition. By connectivity, we get that Q_1Q_2 must lie either in $v_{4(i-1)+1}[E_k] \cup v_{4(i-1)+2}[E_k]$ or in $v_{4i}[E_k] \cup v_{4i+1}[E_k]$, say the former. Observe that Q_1Q_2 must contain a vertex from $v_{4(i-1)+1}[E_k]$ which has a unique color among vertices in $v_{4(i-1)+1}[E_k] \cup v_{4(i-1)+2}[E_k]$. This contradicts the fact that the color sequence of Q_1Q_2 is a repetition. This finishes the proof of Theorem 1.2. □

Proof of Theorem 1.3: We will construct a nonrepetitive rainbow coloring c of $P_\infty[E_k]$ with $\lceil 7k/2 \rceil$ colors. Let us denote the vertices of P_∞ by p_i $i = 1, 2, 3, \dots$ with (p_i, p_j) forming an edge if and only if $|i - j| = 1$. We will write $V_i = p_i[E_k]$. Let X, A, B, C, D, E be pairwise disjoint sets with $|X| = k$, $|B| = |C| = |D| = \lceil k/2 \rceil$, $|A| = |E| = \lfloor k/2 \rfloor$. Let $S = s_1s_2s_3\dots$ be an infinite palindrome-free nonrepetitive sequence with $s_i \in \{1, 2, 3, 4\}$ for all positive integers i. We define a coloring of $P_\infty[E_k]$ using colors $X \cup A \cup B \cup C \cup D \cup E$ as follows:

- If $j = 4(i-1) + 1$ then $c[V_j] = X$.
- If $s_i = 1$, then $c[V_{4(i-1)+2}] = c[V_{4i}] = A \cup B$ and $c[V_{4(i-1)+3}]$ is a k-subset of $C \cup D$.
- If $s_i = 2$, then $c[V_{4(i-1)+2}] = c[V_{4i}] = A \cup C$ and $c[V_{4(i-1)+3}] = B \cup E$.
- If $s_i = 3$, then $c[V_{4(i-1)+2}] = c[V_{4i}] = C \cup E$ and $c[V_{4(i-1)+3}] = A \cup D$.
- If $s_i = 4$, then $c[V_{4(i-1)+2}] = c[V_{4i}] = D \cup E$ and $c[V_{4(i-1)+3}]$ is a k-subset of $B \cup C$.

It is easy to verify that for any index i, any two colors $c_1 \in c[V_{4(i-1)+2}] = c[V_{4i}]$ and $c_2 \in c[V_{4(i-1)+3}]$ uniquely determine s_i.

We shall show that c is a nonrepetitive coloring of $P_\infty[E_k]$. Assume to the contrary that there is a path Q_1Q_2 in $P_\infty[E_k]$ such that the sequence of colors on Q_1Q_2 form a repetition. Remove all vertices from Q_1Q_2 that have colors from the set X and also those vertices which on the path Q_1Q_2 have only neighbors that have colors from the set X. The sequence of colors of the remaining vertices $Q'_1Q'_2 = (q'_{1,1} \dots q'_{1,l}q'_{2,1} \dots q'_{2,l})$ still form a repetition. Let P'_∞ be an infinite path with one loop added to each of its vertices. Furthermore, let c_S be the coloring of P'_∞ with $c_S(p'_j) = s_j$. Let us define the function $f : Q'_1Q'_2 \to P'_\infty$ with $f(q) = p'_i$ if and only if $q \in v_{4(i-1)+2}[E_k] \cup v_{4(i-1)+3}[E_k] \cup v_{4i}[E_k]$. Writing W_1 and W_2 for the images of Q'_1 and Q'_2, we obtain that W_1W_2 is a walk in P'_∞. By the observation above, $c_1 \in c[V_{4(i-1)+2}] = c[V_{4i}]$ and $c_2 \in c[V_{4(i-1)+3}]$ uniquely determine s_i. This ensures that the color sequence of W_1W_2 with respect to c_S is a repetition. Therefore by Lemma 2.18 we obtain that $W_1 = W_2$.

The remainder of the proof is almost identical to that of Theorem 1.2. Suppose first that W_1 and thus W_2 contains at least two different vertices. This means that the original paths Q_1 and Q_2 had to cross from $v_{4(i-1)+2}[E_k] \cup v_{4(i-1)+3}[E_k] \cup v_{4i}[E_k]$ to $v_{4(i)+2}[E_k] \cup v_{4(i)+3}[E_k] \cup v_{4(i+1)}[E_k]$ or vice versa. But as the layer $v_{4i+1}[E_k]$ is rainbow colored with colors in X, the original color sequence of Q_1Q_2 could not be a repetition.

Suppose then that W_1W_2 is a walk repeating the same vertex p_i'. Then all vertices of Q_1Q_2 must lie in $v_{4(i-1)+1}[E_k] \cup v_{4(i-1)+2}[E_k] \cup v_{4(i-1)+3}[E_k] \cup v_{4i}[E_k] \cup v_{4i+1}[E_k]$. Therefore Q_1Q_2 cannot contain any vertex from $\cup v_{4(i-1)+3}[E_k]$ as they have unique colors among vertices in these 5 layers preventing the possibility of a repetition. By connectivity, we get that Q_1Q_2 must lie either in $v_{4(i-1)+1}[E_k] \cup v_{4(i-1)+2}[E_k]$ or in $v_{4i}[E_k] \cup v_{4i+1}[E_k]$, say the former. By connectivity, Q_1Q_2 must contain a vertex from $v_{4(i-1)+1}[E_k]$ which has a unique color among vertices in $v_{4(i-1)+1}[E_k] \cup v_{4(i-1)+2}[E_k]$. This contradicts the fact that the color sequence of Q_1Q_2 is a repetition.

Finally, if the walk W_1W_2 is empty, then all vetices of the path Q_1Q_2 are either X-colored or all their neighbors in their part of Q_1Q_2 are X-colored. By connectivity, this is only possible if all vertices of Q_1Q_2 lie with $v_{4i}[E_k] \cup v_{4i+1}[E_k] \cup v_{4i+2}[E_k]$ for some i. Then again by connectivity Q_1Q_2 must contain a vertex from $v_{4i+1}[E_k]$. This vertex has a unique c-color in $v_{4i}[E_k] \cup v_{4i+1}[E_k] \cup v_{4i+2}[E_k]$ thus the color sequence of Q_1Q_2 with respect to c cannot form a repetition. This contradiction completes the proof of Theorem 1.3. □

3 Some remarks and open problems

Kündgen and Pelsmajer [10] applied their method to outerplanar graphs. Their techniques can be used to prove the following theorem.

Theorem 3.1 *For every outerplanar graph G and integer $k \geq 2$, $\pi(G[K_k]) \leq 16k$. Furthermore, there exists an outerplanar graph G_0 such that $\pi(G_0[E_k]) > 6k$ for every positive integer k.*

Proof: Kündgen and Pelsmajer [10] proved that a maximal outerplanar graph has a shadow complete vertex-partition in which each G_k is a linear forest. Similarly, we can show that if G is a maximal outerplanar graph, then $G[K_n]$ has a shadow complete vertex-partition in which each G_k is of the form $P[K_n]$, where P is a linear forest. As $\pi(P[K_n]) \leq 4k$, it follows from Theorem 2.19 that $\pi(G[K_k]) \leq 16k$.

As for the lower bound, in [1, 5] an outerplanar graph is shown that has star-chromatic number at least 6 (a proper vertex coloring is a star-coloring if every path on four vertices uses at least three distinct colors), thus also nonrepetitive-chromatic number at least 6. We can modify this example so that it gives the desired lower bound. Start with a path P_{10} on 10 vertices. Add one vertex u connected to all vertices of P_{10}. Then, for each vertex p_i of P_{10} add a 24-vertex path Q_i whose 24 vertices are all connected to p_i. Let us call this the *core* of our future graph G_0. Finally, for every vertex v in the core, let us add 6 more leaves $\ell_{v,1}, ..., \ell_{v,6}$ connected to v. Suppose there is a coloring of $G_0[E_k]$ with less than $6k$ colors, we shall arrive to contradiction.

If on the vertices of a layer corresponding to a vertex of the core there is a repeated color, then by Lemma 2.1 we need at least $6k + 1$ colors. Thus we can suppose that the layers corresponding to the vertices of the core are rainbow colored. The k colors $1, 2, \dots k$ used for coloring $u[E_k]$ do not appear on $P_{10}[E_k]$. We call a color *redundant* if it appears at least on two vertices of $P_{10}[E_k]$. As non-redundant colors are all different, there are at most $5k$ non-redundant colors. Thus by the pigeon-hole principle there exist two neighboring layers $p_i[E_k]$ and $p_{i+1}[E_k]$ whose coloring contains at least one redundant color each. Observe that on $Q_i[E_k]$ the colors $1, 2, \dots k$ cannot appear, as otherwise we would have a repetitive path of length 4 (through $u[E_k]$ and using the vertices of the redundant color). Also, either on $Q_i[E_k]$ or on $Q_{i+1}[E_k]$ none of the $2k$ colors of $p_i[E_k]$ and $p_{i+1}[E_k]$ appear, as otherwise there would be a repetitive path of length 4 with its endpoints in $Q_i[E_k]$ and $Q_{i+1}[E_k]$. Suppose that they do not appear on $Q_i[E_k]$. Thus we can use at most $6k - k - 2k = 3k$ colors to color $Q_i[E_k]$, but Theorem 1.3 implies that we would need at least $3k + 1$ colors for this, a contradiction. □

Tightening the gap between lower and upper bounds in Theorem 1.3, Theorem 1.4 and Theorem 3.1 are natural open problems related to results in this paper.

Fractional versions of graph parameters have attracted the attention of researchers. We now introduce a fractional version of nonrepetitive coloring. For a pair of positive integers $p < q$, a *p-tuple nonrepetitive q-coloring* of G is a mapping $c : V(G) \to \binom{[q]}{p}$ such that for any path $v_1 \dots v_{2l}$ in G the sequence $c_1 \dots c_{2l}$ of colors is not a repetition for any choice of $c_i \in c(v_i)$. The *fractional Thue chromatic number* $\pi_f(G)$ of a graph G is defined as

$$\pi_f(G) = \inf \left\{ \frac{q}{p} : \exists\, p\text{-tuple nonrepetitive } q\text{-coloring } c \text{ of } G \right\}.$$

By definition, for any graph G, $\pi_f(G) \le \pi(G)$. It is easy to see that $\pi_f(P_n) = \pi(P_n)$ for all n. On the other hand, already for the cycle of length 7, the ordinary Thue chromatic number and the fractional Thue chromatic number do not coincide as $\pi(C_7) = 4$ and $\pi_f(C_7) = 3.5$. For the upper bound take the following $(7, 2)$-nonrepetitive coloring of C_7: $v_1 \to \{1, 2\}; v_2 \to \{3, 4\}; v_3 \to \{1, 7\}; v_4 \to \{5, 6\}; v_5 \to \{3, 4\}; v_6 \to \{2, 6\}; v_7 \to \{5, 7\}$. The lower bound is an elementary case analysis.

Problem 3.2 *How big can the be the difference $\pi(G) - \pi_f(G)$? Is $\pi(G)$ bounded from above by a function of $\pi_f(G)$?*

For arbitrary graphs, it was proved [2, 4] that if the maximum degree of G is Δ then $\pi(G) \le c\Delta^2$ (c is a constant independent of G and Δ). This immediately gives that $\pi(G[K_k]) \le ck^2\Delta^2$, as the maximum degree of $G[K_k]$ is $k(\Delta + 1) - 1$. As the graphs $G[K_k]$ have special structure, one may expect that the upper bound to be improved. Barát and Wood investigated nonrepetitive colorings of walks [3]. Following their definitions, a walk $\{v_1, v_2, \dots, v_{2t}\}$ is *boring* if $v_i = v_{t+i}$ for all $1 \le i \le t$. Clearly, a boring walk is repetitively colored by every coloring. A coloring f is *walk-nonrepetitive* if only boring walks are repetitively colored by f. Let $\pi^W(G)$ denote the least integer such that G has a walk-nonrepetitive coloring with $\pi^W(G)$ colors. Barát and Wood pose the following problem: is there a function f such that $\pi^W(G) \le f(\Delta)$? If this is true, then a rainbow blow-up of such a coloring would immediately imply that $\pi_R(G[E_k]) \le k\pi^W(G) \le kf(\Delta)$. Indeed a repetitive path in $G[E_k]$ would be a 'lift' of a repetitive walk in the original coloring, thus boring, which is a contradiction (as such a path in $G[E_k]$ cannot be repetitive). It is also easy to see that the same coloring would actually show that $\pi(G[K_k]) \le k\pi^W(G) \le kf(\Delta)$.

Problem 3.3 *Is there a function f such that for every graph G of maximum degree Δ, $\pi(G[K_k]) \le kf(\Delta)$? Perhaps $\pi(G[K_k]) \le ck\Delta^2$ for some constant c?*

A natural marriage of the above two notions is the *fractional walk-nonrepetitive chromatic number*, where in the definition of p-tuple nonrepetitive q-coloring of G, the path $v_1v_2 \dots v_{2l}$ in G is replaced by a walk. We denote by $\pi_f^W(G)$ the fractional walk-nonrepetitive chromatic number of G. It is obvious that for path P of length at least 4, $\pi_f^W(P_n) \geq \pi_f(P_n) = \pi(P_n) = 3$ and $\pi_f^W(P_n) \leq \pi^W(P_n) \leq 4$. It is also easy to see that $\inf(\pi_R(P_n[E_k])/k) \leq \pi_f^W(P_n)$. A natural question is to determine $\pi_f^W(P_n)$ and also to see whether equality holds in the previous inequality.

Given a list assignment L with $L(v) \subset \mathbb{N}$ for all vertices v of a graph G, we say that G is L-nonrepetitively colorable if there exists a nonrepetitive coloring C of G with $c(v) \in L(v)$ for all $v \in V(G)$. The *Thue choice number* $\pi_L(G)$ of a graph G is the minimum integer m such that G is L-nonrepetitive colorable for every list assignment L provided $|L(v)| = m$ for all $v \in V(G)$. It is known [8] that the Thue choice number of a path is at most 4. However, the Thue choice number of trees is unbounded [6].

Problem 3.4 *Is there a constant c such that $\pi_L(P_\infty[K_k]) \leq ck$?*

In the first draft of this paper, we posed the following conjecture, which has recently been confirmed by Kozik [9].

Conjecture 3.5 *There exists an infinite sequence on four letters, A, B, C and D such that the sequence is nonrepetitive, palindrome-free and avoids the subsequences CD and DC.*

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