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► **To cite this version:**

Moshe Rosenfeld, Ziqing Xiang. Hamiltonian decomposition of prisms over cubic graphs. *Discrete Mathematics and Theoretical Computer Science, DMTCS*, 2015, Vol. 16 no. 2 (in progress) (2), pp.111–124. <hal-01185619>

**HAL Id: hal-01185619**

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Submitted on 20 Aug 2015

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# Hamiltonian decomposition of prisms over cubic graphs

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received 28<sup>th</sup> Nov. 2013, revised 22<sup>nd</sup> Nov. 2014, accepted 28<sup>th</sup> Dec. 2014.

The prisms over cubic graphs are 4-regular graphs. The prisms over 3-connected cubic graphs are Hamiltonian. In 1986 Brian Alspach and Moshe Rosenfeld conjectured that these prisms are Hamiltonian decomposable.

In this paper we present a short survey of the status of this conjecture and various constructions proving that certain families of prisms over 3-connected cubic graphs are Hamiltonian decomposable. Among others, we prove that the prisms over cubic Halin graphs, cubic generalized Halin graphs of order  $4k + 2$  and other infinite sequences of cubic graphs are Hamiltonian decomposable.

**Keywords:** Cubic graph, planar cubic graph, Hamiltonian cycle, prism

## 1 Introduction

**Definition 1.** The *prism* over a graph  $G$  is the Cartesian product  $G \square K_2$ . In other words, we take two copies of  $G$ , *upper* copy and *lower* copy, and join each vertex to its clone in the other copy by a *vertical* edge (see Figure 1).

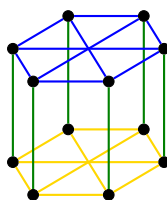


Fig. 1: The prism over  $K_{3,3}$ .

\*Email: moische@uw.edu. This research was done while visiting Shanghai Jiao Tong University. Their support and hospitality is gratefully acknowledged. Partial support from NSF grant DMS-1244294 is acknowledged.

†Email: ziqingxiang@gmail.com. This research is supported by the National Natural Science Foundation of China (No. 11271255) and the Chun-Tsung program of Shanghai Jiao Tong University.

*Remark 2.* It is easy to see that the prism over a 2-connected cubic graph is a 4-connected 4-regular graph.

**Definition 3.** A *Hamiltonian decomposition* of a graph is a partition of its edges such that each part induces a Hamiltonian cycle. A graph is *Hamiltonian decomposable* if it admits a Hamiltonian decomposition. A graph is *prism-decomposable* if the prism over it is Hamiltonian decomposable.

Throughout this paper, we use standard notation and definitions for graphs as in [2] or any other book on graph theory. Our study of prisms over graphs was motivated by D. Barnette's still open conjecture (1970) (see [5, page 1145]) that all simple 4-polytopes, which are all 4-connected 4-regular graphs, are Hamiltonian. This conjecture was probably motivated by Tutte's remarkable and surprising theorem (see [11]) that all 4-connected 3-polytopes are Hamiltonian. It is a remarkable result as these graphs are sparse, at most  $3n - 6$  edges in a graph of order  $n$ ; the prisms over sparse graphs are also sparse. The simplicity requirement in Barnette's conjecture is essential as it is easy to construct non-Hamiltonian 4-polyhedral graphs.

In 1973, D. Barnette and M. Rosenfeld tested this conjecture on prisms over simple 3-polytopes which are simple 4-polytopes (see [9]). It was observed that the 4-color conjecture (which became a theorem in 1976) implies that these prisms are Hamiltonian. This paper introduced the "*B-Y spanning subgraph*" which was later used to prove that prisms over 3-connected cubic graphs (even non-planar) are Hamiltonian (see [3, 8]). In 1986, Brian Alspach and M. Rosenfeld observed that the prisms over all 3-connected cubic graphs they tested actually were Hamiltonian decomposable. In [1] it was conjectured that:

**Conjecture 4.** The prisms over 3-connected cubic graphs are Hamiltonian decomposable.

In 2008, A. Bondy and U. S. R. Murty wrote in their book [2]: "*We present here an updated selection of interesting unsolved problems and conjectures.*" This conjecture is listed as problem #85 in this book.

The conjecture has been verified for 3-connected cubic bipartite planar graphs, for the duals of kleetopes (cubic graphs that are obtained by starting from  $K_4$  and repeatedly "inflating" vertices to triangles) (see [3]), for prisms ( $C_k \square K_2$ ) and for 3-edge-colorable cubic graphs such that every two colored 1-factors form a Hamiltonian cycle (such as  $K_4$  or the Dodecahedron see [1]).

The 3-connectivity is essential as it is possible to find 2-connected cubic graphs which are not prism-decomposable (see [3]). As an aside, prisms over 2-connected planar graphs are Hamiltonian (see [4]). Probably there are 2-connected cubic graphs whose prisms are not Hamiltonian, but so far they have eluded us.

In this note, we exhibit a variety of constructions of Hamiltonian decompositions of prisms over cubic graphs. Some of these constructions apply to certain cubic graphs while they do not apply to others. We wonder whether one will be able to find a single proof for Conjecture 4, assuming of course that it is true, which we believe.

## 1.1 Preliminaries

Our main tool for constructing Hamiltonian cycle in the prisms over cubic graphs is the B-Y graph.

Given a cubic graph  $G$ , if the prism over  $G$  has a Hamiltonian cycle  $C$ , the edges of  $C$  can be partitioned into four sets:

- The edges of  $C$  that appear only in the upper part of  $G$ ;
- The edges of  $C$  that appear only in the lower part of  $G$ ;

- The edges of  $C$  that appear in both parts;
- The edges of  $C$  that connect upper part and lower part.

Motivated by this observation, we get the following definition.

**Definition 5.** A *B-Y graph* is a connected, sub-cubic graph  $G$  with the following properties:

- Its edges are colored by three colors: Blue, Yellow and Green.
- The graph  $\bar{G}$  with  $V(\bar{G}) = \{v, v' \mid v \in V(G)\}$  and the following edges is a cycle:
  - $(v, w)$  if  $(v, w)$  is a blue edge;
  - $(v', w')$  if  $(v, w)$  is a yellow edge;
  - $(v, w), (v', w')$  if  $(v, w)$  is a green edge;
  - $(v, v')$  if  $\deg(v) = 1$  or  $\deg(v) = 2$  and the edges incident with  $v$  are blue and yellow.

It is clear from the definition that, the prism over a B-Y graph  $G$  has a Hamiltonian cycle  $\bar{C}$ . It was observed in [9] that in a B-Y graph, the blue-yellow edges induce a vertex-disjoint union of blue-yellow colored even cycles, and the green edges induce a vertex-disjoint union paths.

A *cactus* is a connected graph in which every two circuits intersects in at most one vertex. And an *even-cactus* is a cactus in which every circle is of even length. Every even-cactus is a B-Y graph (see Figure 2).

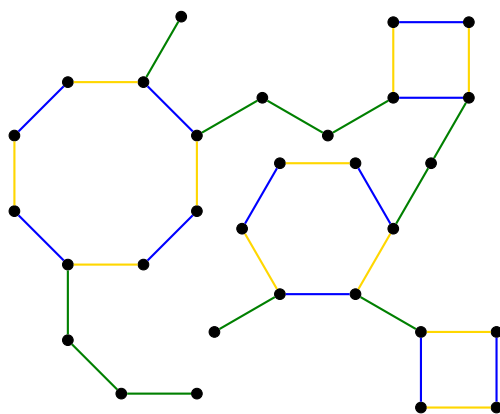


Fig. 2: An even cactus viewed as a B-Y graph. The edges in a circuit are colored B-Y and the edges that are not in any circuits are colored G.

The proofs that the prisms over 3-connected cubic graphs are Hamiltonian were accomplished in two steps: first, they have a spanning 2-connected bipartite sub-cubic subgraph, and second, every such sub-graph has a spanning even-cactus subgraph, which is a spanning B-Y subgraph. In general, for the prism over a graph  $G$  to be Hamiltonian, it is not necessary to have a spanning 2-connected bipartite sub-cubic

subgraph. Every kleeptope have a Hamiltonian prism but no spanning 2-connected bipartite sub-cubic subgraphs.

The following theorem was, and still is the main tool for proving that a cubic graph  $G$  is prism-decomposable.

**Theorem 6** ([1, Theorem 3]). *A cubic graph is prism-decomposable if and only if there exists two spanning B-Y subgraphs of  $G$  such that:*

- (i) *The two B-Y spanning subgraphs share the same B-Y cycles;*
- (ii) *Each edge in  $G$  other than the ones in the common B-Y cycles belongs to exactly one B-Y spanning subgraph.*

We call the two spanning B-Y subgraphs that satisfy the two conditions in Theorem 6 a *prism Hamiltonian decomposition*. From now on, in order to prove that a cubic graph is prism-decomposable, instead of showing two concrete edge-disjoint Hamiltonian cycles, we construct a prism Hamiltonian decomposition.

*Remark 7.* Throughout the paper, the vertices in the upper copy of a prism are colored blue and the vertices in the lower copy of a prism are colored yellow. When we construct two B-Y subgraphs which share the same B-Y cycles, in order to distinguish two sets of green edges in the two subgraphs, we color green the edges in one copy and red in the other. See Figure 3(a) for an example.

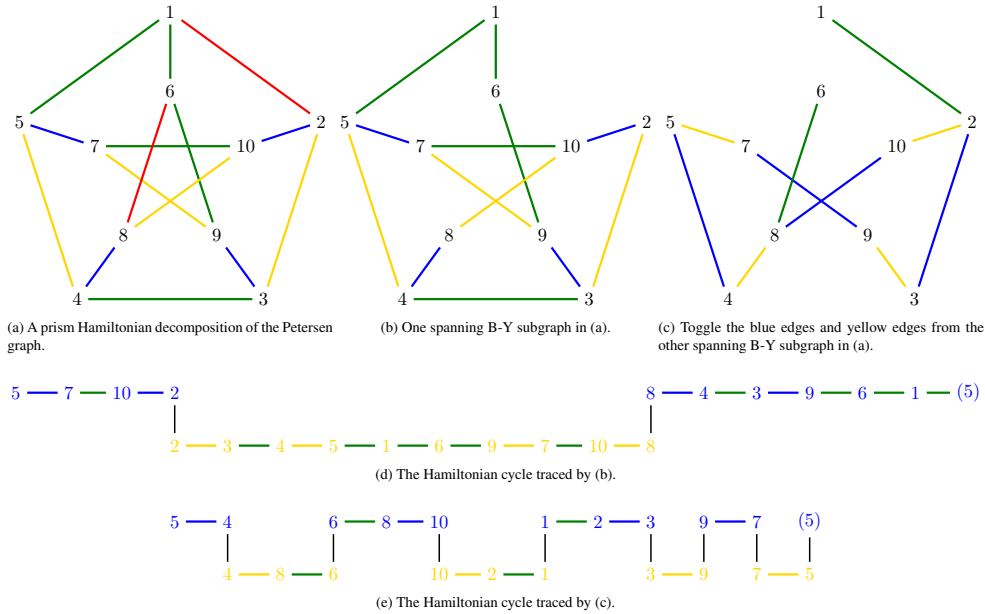


Fig. 3: Figures (d), (e) are two edge-disjoint Hamiltonian cycles of the prism over the Petersen graph traced by the two spanning B-Y subgraphs shown in (a). The color usage is described in Remark 7.

Figure 3 shows that the Petersen graph is prism-decomposable and also demonstrates the use of prism Hamiltonian decomposition.

## 1.2 Gadgets

In this section we introduce gadgets, a collection of B-Y graphs that will help us find the prism Hamiltonian decomposition.

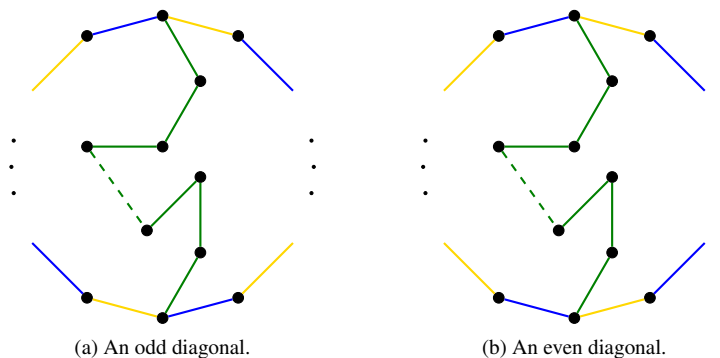


Fig. 4: Two kinds of diagonals in a B-Y cycle.

**Definition 8.** Let  $u, v$  be two vertices on an even cycle  $C$ . A *diagonal* connecting  $u$  and  $v$  in  $C$ , denoted by  $u-v$ , is a path connecting  $u$  and  $v$  which contains no other vertices of the cycle  $C$ . A diagonal  $u-v$  is an *odd* diagonal if  $u$  and  $v$  have even distance along  $C$ . See Figure 4(a). Similarly, a diagonal  $u-v$  is an *even* diagonal if  $u$  and  $v$  have odd distance along  $C$ . See Figure 4(b). Two diagonals  $a-b$  and  $c-d$  are *intersecting* if they are disjoint and  $a, c, b, d$  appear in this order on the cycle  $C$ . See Figure 5.

The following is a list of a few gadgets and methods to combine B-Y graphs in order to form new B-Y graph.

- G1: An even cycle colored B-Y alternatively.
- G2: Given a B-Y graph, adding disjoint dangling green paths hanging from degree 2 vertices incident with blue and yellow edges. See Figure 5(a).
- G3: An even cycle with disjoint non-intersecting odd diagonals. See Figure 5(b).
- G4: An even cycle with an even number of disjoint non-intersecting odd diagonals and one additional odd diagonal that intersects all of them. See Figure 5(c).
- G5: An even cycle with an odd number of disjoint non-intersecting odd diagonals and one additional even diagonal that intersects all of them. See Figure 5(d).
- G6: An even cycle with two intersecting even diagonals. See Figure 5(e).
- G7: Given a B-Y graph, adding a  $C_4$  by splitting a green edge as shown in Figure 6.

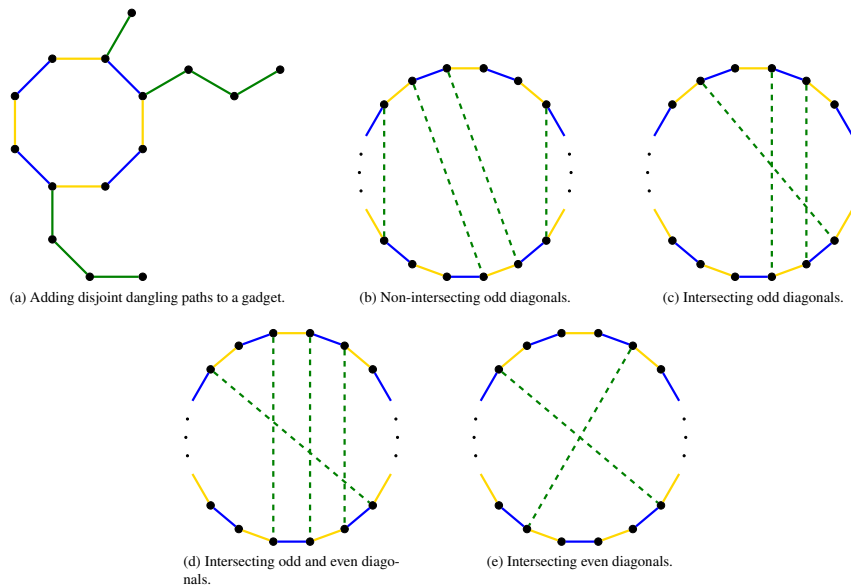


Fig. 5: Dashed lines are disjoint diagonals.

G8: Given two vertex-disjoint B-Y graph, joining two vertices, each on a different B-Y graph, each of degree 2 and each the end vertex of a blue edge, by a green path. For instance, an even-cactus with at least two circuits can be obtained in this way. See Figure 2.

All listed gadgets are B-Y graphs. We leave this task to the reader.

## 2 Prism decomposable cubic graphs

In this section we demonstrate various techniques to prove that certain 3-connected cubic graphs are prism-decomposable. We begin with an example of 2-connected cubic graphs (see Figure 7) that are not prism-decomposable. It is noteworthy that they demonstrate another subtle point. Nash-Williams conjectured that all 4-connected 4-regular graphs are Hamiltonian decomposable. Meredith [7] showed us how to construct infinitely many counterexamples. Prisms over the graphs in Figure 7 are Hamiltonian 4-connected 4-regular but not Hamiltonian decomposable (see [3]) yielding examples of Hamiltonian 4-regular, 4-connected graphs that do not have a Hamiltonian decomposition.

The following proposition was proved in [1]. We include it here in order to demonstrate the use of the gadgets.

**Proposition 9.** *The prisms  $Pr_n := C_n \square K_2$  are prism-decomposable.*

**Proof:** When  $n$  is even,  $C_n \square K_2$  is planar and bipartite. We can quickly demonstrate a direct construction of the two Y-B spanning subgraphs. Figure 8(a) shows a decomposition into two gadgets of type 2 over gadgets of type 1.

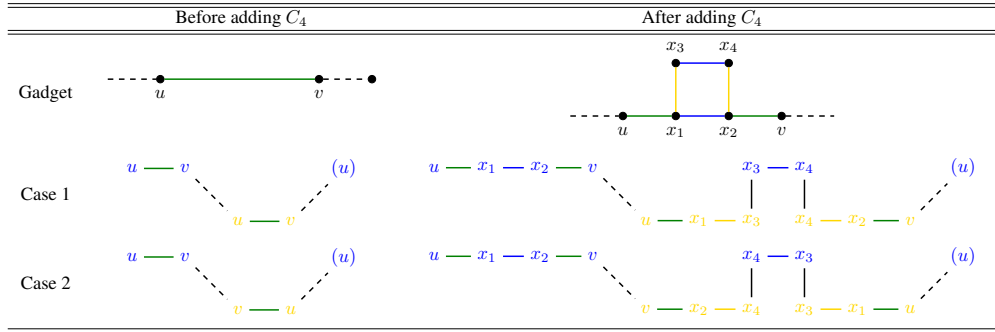


Fig. 6: Suppose we have a gadget in which  $uv$  is a green edge. By adding a  $C_4$  between  $u$  and  $v$ , we obtain a new gadget.

Explanation: assume that we orient the Hamiltonian cycle so that the green edge  $uv$  is traversed in the upper level (blue) from  $u \rightarrow v$ . Case 1 describes the new Hamiltonian cycle in case the edge  $uv$  on the bottom level (yellow) is also traversed from  $u \rightarrow v$ , while case 2 describes the Hamiltonian cycle in case it is traversed from  $v \rightarrow u$ .

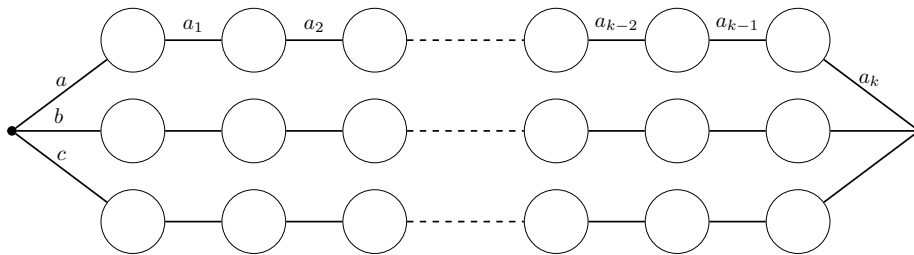


Fig. 7: Each circle stands for a 2-connected cubic graph with one edge deleted. The two resulting vertices of degree 2 are connected along the path  $a, a_1, \dots, a_k$  and similarly the paths starting at  $b$  and  $c$ . Prisms over such graphs are 4-connected 4-regular (see [3]). They may be Hamiltonian but definitely not prism-decomposable.

When  $n$  is odd we resort to another approach. Figure 8(b) shows a prism Hamiltonian decomposition of the prism  $Pr_3$ . We can now use the gadget of type 7 to add  $k - 1$  copies of  $C_4$  to obtain a prism Hamiltonian decomposition of the prism  $Pr_{2k+1}$  as shown in Figure 8(c).  $\square$

We next apply our gadgets to prove that the generalized Petersen graphs are prism-decomposable.

**Definition 10.** A *generalized Petersen graph* is a cubic graph  $G$  that has an induced cycle  $C$  (called *generalized Petersen cycle*) and an induced 2-regular graph  $H$  such that  $V(C)$  and  $V(H)$  give a partition of  $V(G)$ . Note that, the edges between  $V(C)$  and  $V(H)$  form a perfect matching.

**Proposition 11.** *Every generalized Petersen graph  $G$  of order  $4k$  is prism-decomposable.*

**Proof:** Let  $C$  be a generalized Petersen cycle of order  $2k$  in  $G$ . The graph  $G \setminus V(C)$  is a union of disjoint cycles  $A_1, \dots, A_m$  where  $A_i = a_{i,1}a_{i,2} \dots a_{i,|A_i|}$ . We color the edges of  $C$  by blue and yellow



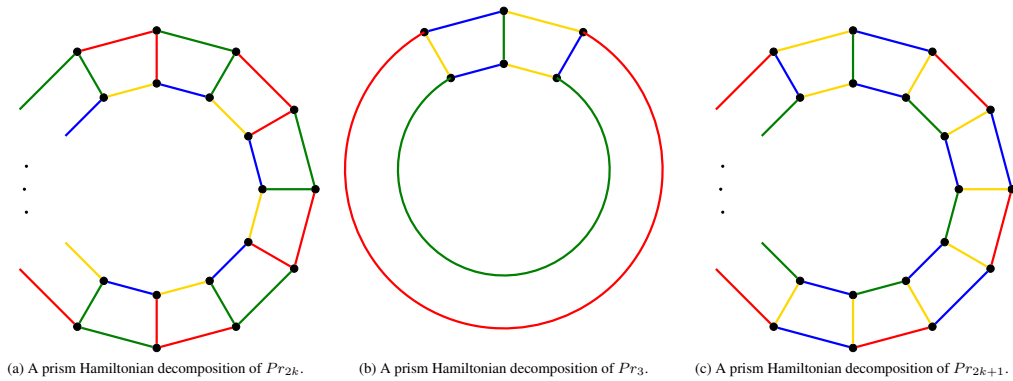


Fig. 8: The prisms are prism-decomposable.

alternatingly. Let  $\pi(a_{i,t})$  be the vertex of the cycle  $C$  matched to  $a_{i,t}$  in  $G$ . We are now ready to construct two spanning B-Y subgraphs of  $G$ .

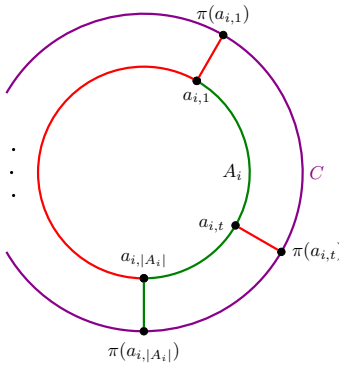


Fig. 9: Part of a prism Hamiltonian decomposition of a generalized Petersen graph of order  $4k$ . The purple cycle  $C$  is a B-Y cycle.

For each cycle  $A_i$  we color the edges along the path  $a_{i,1}a_{i,2} \dots a_{i,|A_i|}\pi(a_{i,|A_i|})$  green. Clearly, we get a single B-Y cycle plus dangling green paths, which are a type 2 gadget over a type 1 gadget, hence a spanning B-Y subgraph of  $G$ .

To form the second spanning B-Y subgraph, we color red the edges  $a_{i,1}a_{i,|A_i|}$  and  $\pi(a_{i,t})a_{i,t}$ ,  $1 \leq t < |A_i|$ . It is easy to see that the cycle  $C$  plus the dangling red paths are again a type 2 gadget over a type 1 gadget, hence a spanning B-Y subgraph of  $G$ .  $\square$

**Definition 12.** A *Halin graph* is a planar graph consisting of a tree with no vertices of degree 2, embedded in the plane, plus a cycle (called the *Halin cycle*) through its leaves connecting them in the order they are drawn in the plane.

**Definition 13.** A *generalized Halin graph* is a tree, with no vertices of degree 2, plus a cycle (called the *generalized Halin cycle*) through the leaves.

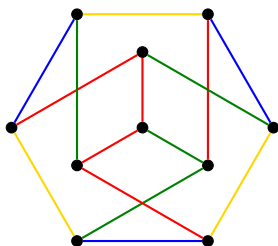


Fig. 10: The Petersen graph is a generalized Halin graph. A prism Hamiltonian decomposition using a single B-Y cycle (the generalized Halin cycle) is indicated.

In [6] it was proved that the prisms over generalized Halin graphs are Hamiltonian. A cubic generalized Halin graph of order  $2k$  consists of a cycle  $C_{k+1}$  plus a tree with  $k - 1$  non-leaves (vertices of degree 3 in the tree). We believe that the prisms over cubic generalized Halin graphs are prism-decomposable. We can only prove it for a bit more than half of them.

Before embarking on the proof we need the following Lemma:

**Lemma 14.** Let  $T$  be a cubic tree (all non-leaves have degree 3). Given any two distinct leaves  $u, v \in V(T)$ , the edges of the tree can be partitioned into two disjoint sets  $A_{u,v}^T, B_{u,v}^T$  satisfying:

- The set  $A_{u,v}^T$  contains the edges of the unique path connecting  $u$  and  $v$ .
- Each set induces vertex disjoint paths that cover all non-leaves.
- For each set, every path covers exactly one leaf, except for the one path containing  $u$  and  $v$ .

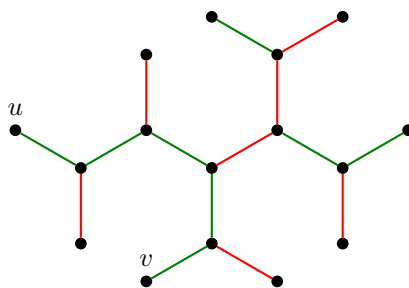


Fig. 11: A partition of the edges of the tree  $T$  in Lemma 14. The paths in  $A_{u,v}^T$  are colored green, and those in  $B_{u,v}^T$  are colored red.

**Proof:** By induction on the size of  $T$ . The result holds for  $K_1, K_2$ . Let  $T$  be a cubic tree of size at least 4. Pick a cherry, namely two different leaves  $x, y$  with a common neighbor  $z$ . Delete  $x, y$  to obtain a cubic

tree  $T'$  in which  $z$  is a leaf. By the induction hypothesis,  $E(T')$  has the required partition for every two distinct leaves in  $T'$ .

Case	$A_{u,v}^T$	$B_{u,v}^T$
$\{u, v\} \cap \{x, y\} = \emptyset$	$A_{u,v}^{T'} \cup \{xz\}$	$B_{u,v}^{T'} \cup \{yz\}$
$u = x, v \neq y$	$A_{v,z}^{T'} \cup \{xz\}$	$B_{v,z}^{T'} \cup \{yz\}$
$\{u, v\} = \{x, y\}$ ( $z' \neq z$ is a leaf in $T'$ )	$B_{z,z'}^{T'} \cup \{xz, zy\}$	$A_{z,z'}^{T'}$

There are essentially three cases, and we list the decomposition for each case in the above table.  $\square$

**Proposition 15.** *The generalized Halin graph  $G$  of order  $4k + 2$  is prism-decomposable.*

**Proof:** The generalized Halin cycle  $C$  is of order  $2k + 2$ , hence an even cycle. Choose two different vertices  $u, v$  on the cycle  $C$  whose distance along the cycle is even. Let  $T = G \setminus E(C)$ , and  $A_{u,v}^T, B_{u,v}^T$  be the decomposition obtained by Lemma 14. The set  $E(C) \cup A_{u,v}^T$  is a type 2 gadget over a type 3 gadget. The set  $E(C) \cup B_{u,v}^T$  is a type 2 gadget over a type 1 gadget.  $\square$

**Proposition 16.** *Every cubic generalized Halin graph  $G$  containing a vertex-induced  $C_3$  is prism-decomposable.*

**Proof:** If  $G$  is of order  $4k + 2$ ,  $G$  is prism-decomposable by Proposition 15. Now, assume that the graph  $G$  is of order  $4k$ . Let  $C$  be a generalized Halin cycle, which is of order  $2k + 1$ . A  $C_3 = \{a, b, c\}$  in  $G$  must contain two adjacent vertices in  $C$ , say  $a, b$ . Construct an even cycle  $C'$  by setting  $E(C') = E(C) \Delta E(C_3)$ , where  $\Delta$  stands for the symmetric difference. Note that, in  $C'$ , the diagonal  $a-b$  is an odd diagonal. Let  $T = G \setminus E(C') \setminus E(C_3)$ , which is a tree. From  $T$ , pick a leaf  $d$  which is not  $c$ . Applying Lemma 14, we obtain a decomposition  $A_{c,d}^T, B_{c,d}^T$ .

If the diagonal in  $C'$ ,  $c-d$ , is an odd diagonal, then  $E(C') \cup A_{c,d}^T$  and  $E(C') \cup B_{c,d}^T \cup \{ab\}$  are type 2 gadgets over type 3 gadgets. If the diagonal  $c-d$  is an even diagonal, then  $E(C') \cup A_{c,d}^T \cup \{ab\}$  is a type 2 gadget over a type 5 gadget, and  $E(C') \cup B_{c,d}^T$  is a type 2 gadget over a type 1 gadget.  $\square$

**Corollary 17.** *Cubic Halin graphs are prism-decomposable.*

**Proof:** Given a Halin graph  $G$ , for every odd number  $i$  such that  $3 \leq i \leq |V(G)|$ , there exists a cycle of length  $i$  in  $G$  (see [10]). In particular, every Halin graph contains a  $C_3$ . The result follows from Proposition 16.  $\square$

The difficulty of finding prism Hamiltonian decompositions arises when we are short of “good” even cycles, that is, even cycles of length near  $\frac{|V(G)|}{2}$  that have very few single edge diagonals. For cubic generalized Halin graphs with an odd length generalized Halin cycle, we know that every two adjacent vertices on the cycle are connected by a unique path on the tree, which, together with the edge connecting the two vertices, form a cycle. We also know that at least one of these cycles is of odd length. When we combine it with the generalized Halin cycle, we obtain an even cycle. Sometimes, based on this single even cycle, we can construct the prism Hamiltonian decomposition.

But in general, a single even cycle is not enough. Actually, we can prove that if a cubic graph  $G$  admits a prism Hamiltonian decomposition based on a single B-Y cycle  $C$ , the subgraph  $G \setminus E(C)$  must be a

union of trees and connected unicyclic graphs. Two extreme cases are the odd prisms  $Pr_{2k+1}$  (see Figure 8) and the Möbius ladders of order  $4k$  (see Figure 12). While they are prism-decomposable, it is not difficult to prove that their prism Hamiltonian decompositions must involve multiple B-Y cycles.

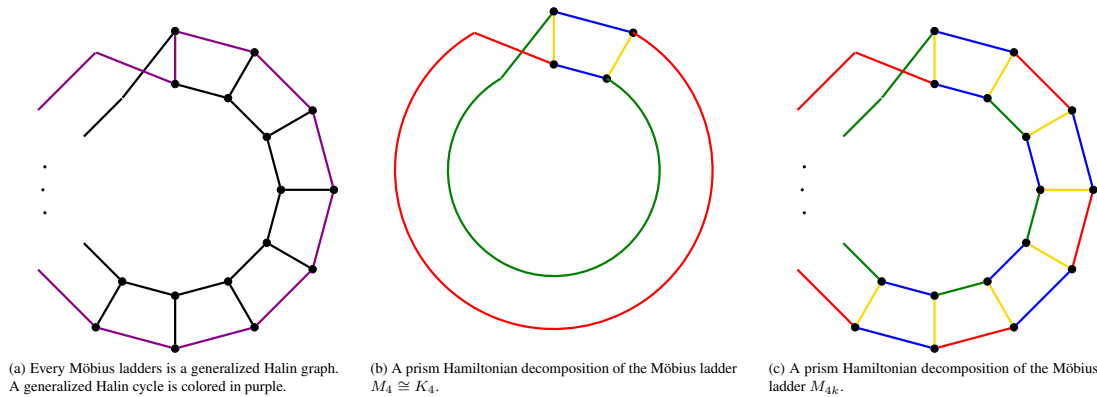


Fig. 12: Möbius ladders are prism-decomposable.

**Proposition 18.** *The Möbius ladders  $M_n$  (see Figure 12) are prism Hamiltonian decomposable.*

**Proof:** Möbius ladders are generalized Halin graphs (see Figure 12(a)). Proposition 15 takes care of the graphs  $M_{4k+2}$ . In Figures 12(b) and 12(c), we employ the same strategy we used in Proposition 9 to prove that  $M_{4k}$  is prism-decomposable.  $\square$

We note that the Möbius graph is a graph based on a caterpillar and a cycle through its leaves (see Figure 12(a)). Such graphs may present the biggest challenge when we try to increase the size of the generalized Halin cycle to obtain an even cycle, as it will generate many single edge diagonals in the cycle. It is conceivable that proving that all cubic generalized Halin graphs based on a caterpillar are prism-decomposable may lead to a general proof.

We conclude this section with an example of such a graph where we have many different prism Hamiltonian decompositions.

### 3 Concluding remarks

Is there a single method that will resolve the conjecture that all 3-connected cubic graphs are prism-decomposable? For instance, does the existence of a Hamiltonian cycle in  $G$  help us prove that  $G$  is Hamiltonian decomposable? The Möbius Ladder casts doubt whether the Hamilton cycle can always be used. In general, we feel that adding one additional property could help proving that the given family of cubic graphs are prism-decomposable.

We list some families of cubic graphs. We do not know whether the following graphs have prisms that are Hamiltonian decomposable.

- (i) Cubic bipartite graphs.  
In [6] it was proved that the prisms over planar, cubic, 3-connected graphs are Hamiltonian decomposable. The proof used an inductive generation of these graphs. It did not extend to general cubic 3-connected bipartite graphs. There are planar cubic 2-connected graphs whose prisms are not Hamiltonian decomposable.
- (ii) Cubic 3-connected planar graphs.  
These graphs are 3-edge colorable, thus they admit an even 2-factor (all cycles are even). This might help in finding a proof that their prisms are Hamiltonian decomposable.
- (iii) Cubic 3-connected Hamiltonian graphs. In our explorations, we found that in most cases the Hamiltonian cycle could be used as the B-Y subgraph. The Möbius Ladder is an exception two regions. All other edges are split into two sets: the edges contained in one region. These edges form a set of non-intersecting diagonals.
- (iv) Cubic 3-connected Hamiltonian planar graphs.  
When these graphs are drawn in the plane, the Hamiltonian cycles gives us natural partitions of edges that are not in the Hamiltonian cycles. This partition may help us construct a Hamiltonian decomposition of the prism over the graph.
- (v) Cubic generalized Halin graphs of order  $4k$ .  
It seems like a big gap of difficulty between the generalized cubic Halin graphs of order  $4k + 2$  and order  $4k$ .
- (vi) Cubic generalized Halin graphs whose trees are caterpillars.  
The Möbius Ladders of order  $4k$ , viewed as a generalized Halin graph, had consisted of an odd cycle  $C_{2k+1}$  and a tree which is a caterpillar. This meant that the even cycles created by diagonals were large, unlike the planar case.  
We wondered whether solving this case can help prove the conjecture for all cubic generalized Halin graphs.

We conclude with three decision problems.

- (i) Given a cubic graph of order  $2n$ , is it a generalized Halin graph? That is, does the graph contain a cycle  $C_{n+1}$  such that when its edges are deleted we are left with a tree? Note that, there is a linear time algorithm to determine whether a graph is Halin or not.
- (ii) Given a cubic graph and a Hamiltonian cycle in it, can the diagonals be partitioned into two sets such that each set, in which edges are colored Green, plus the Hamiltonian cycle, colored Blue-Yellow, is a spanning B-Y graph?
- (iii) Same as in the previous problem except we in addition assume that the graph is planar. Does the embedding of the planar graph into the plane help us find a desired partition of diagonals?

## Acknowledgments

We would like to thank the referees for their helpful remarks and suggested corrections.

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