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Packing and covering the balanced complete bipartite multigraph with cycles and stars

Hung-Chih Lee

Department of Information Technology, Ling Tung University, Taichung, Taiwan

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Let $C_k$ denote a cycle of length $k$ and let $S_k$ denote a star with $k$ edges. For multigraphs $F$, $G$ and $H$, an $(F,G)$-decomposition of $H$ is an edge decomposition of $H$ into copies of $F$ and $G$ using at least one of each. For $L \subseteq H$ and $R \subset rH$, an $(F,G)$-packing (resp. $(F,G)$-covering) of $H$ with leave $L$ (resp. padding $R$) is an $(F,G)$-decomposition of $H - E(L)$ (resp. $H + E(R)$). An $(F,G)$-packing (resp. $(F,G)$-covering) of $H$ with the largest (resp. smallest) cardinality is a maximum $(F,G)$-packing (resp. minimum $(F,G)$-covering), and its cardinality is referred to as the $(F,G)$-packing number (resp. $(F,G)$-covering number) of $H$. In this paper, we determine the packing number and the covering number of $\lambda K_{n,n}$ with $C_k$’s and $S_k$’s for any $\lambda$, $n$ and $k$, and give the complete solution of the maximum packing and the minimum covering of $\lambda K_{n,n}$ with 4-cycles and 4-stars for any $\lambda$ and $n$ with all possible leaves and paddings.

Keywords: complete bipartite multigraph, cycle, star, packing, covering

1 Introduction

For positive integers $m$ and $n$, $K_{m,n}$ denotes the complete bipartite graph with parts of sizes $m$ and $n$. If $m = n$, the complete bipartite graph is referred to as balanced. A $k$-cycle, denoted by $C_k$, is a cycle of length $k$. A $k$-star, denoted by $S_k$, is the complete bipartite graph $K_{1,k}$. A $k$-path, denoted by $P_k$, is a path with $k$ vertices. For a graph $H$ and a positive integer $\lambda$, we use $\lambda H$ to denote the multigraph obtained from $H$ by replacing each edge $e$ by $\lambda$ edges each having the same endpoints as $e$. When $\lambda = 1$, $1H$ is simply written as $H$.

Let $F$, $G$, and $H$ be multigraphs. A decomposition of $H$ is a set of edge-disjoint subgraphs of $H$ whose union is $H$. An $(F,G)$-decomposition of $H$ is a decomposition of $H$ into copies of $F$ and $G$ using at least one of each. If $H$ has an $(F,G)$-decomposition, we say that $H$ is $(F,G)$-decomposable and write $(F,G)\mid H$. If $H$ does not admit an $(F,G)$-decomposition, two natural questions arise:

1. What is the minimum number of edges needed to be removed from the edge set of $H$ so that the resulting graph is $(F,G)$-decomposable, and what does the collection of removed edges look like?
These questions are respectively called the maximum packing problem and the minimum covering problem of $H$ with $F$ and $G$.

Let $F$, $G$, and $H$ be multigraphs. For $L \subseteq H$ and $R \subseteq rH$, an $(F,G)$-packing of $H$ with leave $L$ is an $(F,G)$-decomposition of $H - E(L)$, and an $(F,G)$-covering with padding $R$ is an $(F,G)$-decomposition of $H + E(R)$. For an $(F,G)$-packing $\mathcal{P}$ of $H$ with leave $L$, if $|\mathcal{P}|$ is as large as possible (so that $|L|$ is as small as possible), then $\mathcal{P}$ and $L$ are referred to as a maximum $(F,G)$-packing and a minimum leave, respectively. Moreover, the cardinality of the maximum $(F,G)$-packing of $H$ is called the $(F,G)$-packing number of $H$, denoted by $p(H;F,G)$. For an $(F,G)$-covering $\mathcal{C}$ of $H$ with padding $R$, if $|\mathcal{C}|$ is as small as possible (so that $|R|$ is as small as possible), then $\mathcal{C}$ and $R$ are referred to as a minimum covering and a minimum padding, respectively. Moreover, the cardinality of the minimum $(F,G)$-covering of $H$ is called the $(F,G)$-covering number of $H$, denoted by $c(H;F,G)$. Clearly, an $(F,G)$-decomposition of $H$ is a maximum $(F,G)$-packing with leave the empty graph, and also a minimum $(F,G)$-covering with padding the empty graph.

Recently, decomposition into a pair of graphs has attracted a fair share of interest. Abueida and Daven [3] investigated the problem of $(K_k, S_k)$-decomposition of the complete graph $K_n$. Abueida and Daven [4] investigated the problem of the $(C_4, E_2)$-decomposition of several graph products where $E_2$ denotes two vertex disjoint edges. Abueida and O’Neil [7] settled the existence problem for $(C_k, S_{k-1})$-decomposition of the complete multigraph $\lambda K_n$ for $k \in \{3, 4, 5\}$. Priyadharsini and Muthusamy [12, 13] gave necessary and sufficient conditions for the existence of $(G_n, H_n)$-decompositions of $\lambda K_n$ and $\lambda K_{n,n}$ where $G_n, H_n \in \{C_n, P_n, S_{n-1}\}$. A graph-pair $(G, H)$ of order $m$ is a pair of non-isomorphic graphs $G$ and $H$ on $m$ non-isolated vertices such that $G \cup H$ is isomorphic to $K_m$. Abueida and Daven [2] and Abueida, Daven and Roblee [5] completely determined the values of $n$ for which $\lambda K_n$ admits a $(G, H)$-decomposition where $(G, H)$ is a graph-pair of order 4 or 5. Abueida, Clark and Leach [1] and Abueida and Hampson [6] considered the existence of decompositions of $K_n - F$ for the graph-pair of order 4 and 5, respectively, where $F$ is a Hamiltonian cycle, a 1-factor, or almost 1-factor. Furthermore, Shyu [14] investigated the problem of decomposing $K_n$ into paths and stars with $k$ edges, giving a necessary and sufficient condition for $k = 3$. In [15, 16], Shyu considered the existence of a decomposition of $K_n$ into paths and cycles with $k$ edges, giving a necessary and sufficient condition for $k \in \{3, 4\}$. Shyu [17] investigated the problem of decomposing $K_n$ into cycles and stars with $k$ edges, settling the case $k = 4$. In [18], Shyu considered the existence of a decomposition of $K_{m,n}$ into paths and stars with $k$ edges, giving a necessary and sufficient condition for $k = 3$. Recently, Lee [9] and Lee and Lin [10] established necessary and sufficient conditions for the existence of $(C_k, S_k)$-decompositions of the complete bipartite graph and the complete bipartite graph with a 1-factor removed, respectively. However, much less work has been done on the problem of packing and covering graphs with a pair of graphs. Abueida and Daven [3] obtained the maximum packing and the minimum covering of the complete graph $K_n$ with $(K_k, S_k)$. Abueida and Daven [2] and Abueida, Daven and Roblee [5] gave the maximum packing and the minimum covering of $K_n$ and $\lambda K_n$ with $G$ and $H$, respectively, where $(G, H)$ is a graph-pair of order 4 or 5. In this paper, we determine the packing number and the covering number of $\lambda K_{n,n}$ with $k$-cycles and $k$-stars for any $\lambda$, $n$ and $k$, and give the complete solution of the maximum packing and the minimum covering of $\lambda K_{n,n}$ with 4-cycles and 4-stars for any $\lambda$ and $n$ with all possible leaves and paddings.
2 Preliminaries

In this section we first collect some needed terminology and notation, and then present a result which is useful for our discussions to follow.

Let \( G \) be a multigraph. The degree of a vertex \( x \) of \( G \), denoted by \( \deg_G x \), is the number of edges incident with \( x \). The vertex of degree \( k \) in \( S_k \) is the center of \( S_k \) and any vertex of degree 1 is an endvertex of \( S_k \). For \( W \subseteq V(G) \), we use \( G[W] \) to denote the subgraph of \( G \) induced by \( W \). Furthermore, \( \mu(uv) \) denotes the number of edges of \( G \) joining \( u \) and \( v \), \( (v_1, \ldots, v_k) \) and \( v_1 \ldots v_k \) denote the \( k \)-cycle and the \( k \)-path through vertices \( v_1, \ldots, v_k \) in order, respectively, and \( (x; y_1, \ldots, y_k) \) denotes the \( k \)-star with center \( x \) and endvertices \( y_1, \ldots, y_k \). When \( G_1, G_2, \ldots, G_t \) are multigraphs, not necessarily disjoint, we write \( G_1 \cup G_2 \cup \cdots \cup G_t \) or \( \bigcup_{i=1}^{t} G_i \) for the graph with vertex set \( \bigcup_{i=1}^{t} V(G_i) \) and edge set \( \bigcup_{i=1}^{t} E(G_i) \).

When the edge sets are disjoint, \( G = \bigcup_{i=1}^{t} G_i \) expresses the decomposition of \( G \) into \( G_1, G_2, \ldots, G_t \). Given an \( S_k \)-decomposition of \( G \), a central function \( c \) from \( V(G) \) to the set of non-negative integers is defined as follows. For each \( v \in V(G) \), \( c(v) \) is the number of \( k \)-stars in the decomposition whose center is \( v \).

The following result is essential to our proof.

Proposition 2.1 (Hoffman [8]) For a positive integer \( k \), a multigraph \( H \) has an \( S_k \)-decomposition with central function \( c \) if and only if

(i) \[ k \sum_{v \in V(H)} c(v) = |E(H)|, \]

(ii) \[ \text{for all } x, y \in V(H), \mu(xy) \leq c(x) + c(y), \]

(iii) \[ \text{for all } S \subseteq V(H), k \sum_{v \in S} c(v) \leq \varepsilon(S) + \sum_{x \in S, y \in V(H) - S} \min\{c(x), \mu(xy)\}. \]

where \( \varepsilon(S) \) denotes the number of edges of \( H \) with both ends in \( S \).

In the sequel of the paper, \((A, B)\) denotes the bipartition of \( \lambda K_{n,n} \), where \( A = \{a_0, a_1, \ldots, a_{n-1}\} \) and \( B = \{b_0, b_1, \ldots, b_{n-1}\} \).

3 Packing numbers and covering numbers

In this section the packing number and the covering number of the balanced complete bipartite multigraph with \( k \)-cycles and \( k \)-stars are determined. We begin with a criterion for decomposing the complete bipartite graph into \( k \)-cycles.

Proposition 3.1 (Sotteau [19]) For positive integers \( m, n, \) and \( k \), the graph \( K_{m,n} \) is \( C_k \)-decomposable if and only if \( m, n, \) and \( k \) are even, \( k \geq 4, \) \( \min\{m, n\} \geq k/2, \) and \( k \) divides \( mn \).

Let \( K^*_{m,n} \) denote the symmetric complete bipartite digraph with parts of size \( m \) and \( n \), and let \( C^*_k \) denote the directed \( k \)-cycle.

Proposition 3.2 (Sotteau [19]) For positive integers \( m, n, \) and \( k \), the digraph \( K^*_{m,n} \) is \( C^*_k \)-decomposable if and only if \( k \) is even, \( k \geq 4, \) \( \min\{m, n\} \geq k/2, \) and \( k \) divides \( 2mn \).

Removing the directions from the arcs of directed cycles in a \( C^*_k \)-decomposition of \( K^*_{m,n} \), we obtain the following result by Proposition 3.2.
Lemma 3.3  For positive integers $m$, $n$, and $k$, the multigraph $2K_{m,n}$ is $C_k$-decomposable if $k$ is even, $k \geq 4$, $\min\{m,n\} \geq k/2$, and $k$ divides $2mn$.

Lemma 3.4  Let $\lambda$, $k$, $m$, and $n$ be positive integers with $\lambda m \equiv \lambda n \equiv k \equiv 0 \pmod{2}$ and $\min\{m,n\} \geq k/2 \geq 2$. If $m$ or $n$ is divisible by $k$, then $\lambda K_{m,n}$ is $C_k$-decomposable.

Proof: Since $\lambda K_{m,n}$ is isomorphic to $\lambda K_{n,m}$, it suffices to show that the result holds for $k \mid m$. If $\lambda$ is odd, then $m$ and $n$ are even from the assumption $\lambda m \equiv \lambda n \equiv 0 \pmod{2}$. Since $k$ divides $mn$, Proposition 3.1 implies that $K_{m,n}$ is $C_k$-decomposable. If $\lambda$ is even, then $2K_{m,n} \mid \lambda K_{m,n}$. Since $k$ divides $2mn$, $2K_{m,n}$ is $C_k$-decomposable by Lemma 3.3. Hence $\lambda K_{m,n}$ is $C_k$-decomposable. □

Lemma 3.5  If $k$ is a positive even integer with $k \geq 4$, then $\lambda K_{k,k}$ is $(C_k, S_k)$-decomposable.

Proof: Note that $\lambda K_{k,k} = \lambda K_{k,k-2} \cup \lambda K_{k,2}$. By Lemma 3.4, $\lambda K_{k,k-2}$ is $C_k$-decomposable. Trivially, $\lambda K_{k,2}$ is $C_k$-decomposable. Therefore, $\lambda K_{k,k}$ is $(C_k, S_k)$-decomposable. □

Lemma 3.6  Let $k$ be a positive even integer and let $n$ be a positive integer with $4 \leq k < n < 2k$. If $\lambda(n-k)^2 < k$, then $\lambda K_{n,n}$ has a $(C_k, S_k)$-packing with leave $\lambda K_{n-k,n-k}$ and a $(C_k, S_k)$-covering with padding $P_{k-\lambda(n-k)^2+1}$.

Proof: Let $n = k + r$. The assumption $k < n < 2k$ implies $0 < r < k$. We first give the required packing. Note that $\lambda K_{n,n} = \lambda K_{k,k} \cup \lambda K_{k,r} \cup \lambda K_{r,k} \cup \lambda K_{r,r}$.

By Lemma 3.4, $\lambda K_{k,k}$ has a $C_k$-decomposition $\mathcal{D}_1$. Trivially, $\lambda K_{k,r}$ and $\lambda K_{r,k}$ have $S_k$-decompositions $\mathcal{D}_2$ and $\mathcal{D}_3$, respectively. Thus $\bigcup_{i=1}^3 \mathcal{D}_i$ is a $(C_k, S_k)$-packing of $\lambda K_{n,n}$ with leave $\lambda K_{r,r}$, as desired.

Now we give the required covering. Let $s = \lambda r^2$. Let $A_0 = \{a_0, a_1, \ldots, a_{\lfloor (s-1)/2 \rfloor}\}$, $A_1 = A - A_0$, $B_0 = \{b_0, b_1, \ldots, b_{k-1}\}$ and $B_1 = B - B_0$. Define a $k$-cycle $C$ and a $(k-s+1)$-path $P$ as follows:

$$C = (b_0, a_0, b_1, a_1, \ldots, b_{k/2-1}, a_{k/2-1})$$

$$P = \begin{cases} 
  b_0 a_{k/2-1} b_{k/2-1} a_{k/2-2} \cdots b_{(s-1)/2} a_{(s-1)/2} & \text{if } s \text{ is odd,} \\
  b_0 a_{k/2-1} b_{k/2-1} a_{k/2-2} \cdots a_{s/2} b_{s/2} & \text{if } s \text{ is even.}
\end{cases}$$

Let

$$H = \lambda K_{n,n} - E(C) + E(P).$$

Note that $V(H) = V(\lambda K_{n,n})$, $|E(H)| = \lambda n^2 - k + (k-s) = \lambda n^2 - \lambda r^2 = \lambda k(k+2r)$, and $\mu(uv) \leq \lambda$ for all $u, v \in V(H)$. Furthermore, for $H' = H[A \cup B_0]$, we have

$$\deg_{H'} v = \begin{cases} 
  \lambda k - 2 & \text{if } v \in A_0 - \{a_{\lfloor (s-1)/2 \rfloor}\}, \\
  \lambda k - \rho & \text{if } v = a_{\lfloor (s-1)/2 \rfloor}, \\
  \lambda k & \text{if } v \in A_1,
\end{cases}$$

where $\rho = 1$ if $s$ is odd, and $\rho = 2$ if $s$ is even. Define a function $c : V(H) \to \mathbb{N}$ as follows:

$$c(v) = \begin{cases} 
  0 & \text{if } v \in B_0, \\
  \lambda & \text{otherwise.}
\end{cases}$$
Now we show that there exists an $S_k$-decomposition of $H$ with central function $c$ by Proposition 2.1.

First, $k \sum_{v \in V(H)} c(v) = k\lambda(k + 2r) = |E(H)|$. This proves (i). Next, if $u, v \in B_0$, then $c(u) + c(v) = 0 = \mu(uv)$; otherwise, $c(u) + c(v) \geq \lambda \geq \mu(uv)$. This proves (ii). Finally, for $S \subseteq V(H)$ and $i \in \{0, 1\}$, let $S \cap A_i = X_i$ and $S \cap B_i = Y_i$. Moreover, let $X = X_0 \cup X_1$ and $Y = Y_0 \cup Y_1$. Define a set $T$ of ordered pairs of vertices as follows:

$$T = \{(u, v) | u \in X, v \in B_1 - Y_1 \text{ or } u \in X_1, v \in B_0 - Y_0 \text{ or } u \in Y_1, v \in A - X\}.$$  

Note that

$$k \sum_{w \in S} c(w) = k\lambda(|X| + |Y_1|), \quad (1)$$

$$\varepsilon(S) = \lambda(|X||Y_1| + |X_1||Y_0|) + \sum_{u \in X_0, v \in Y_0} \mu(uv), \quad (2)$$

and for $u \in S$ and $v \in V(H) - S$

$$\min\{c(u), \mu(uv)\} = \begin{cases} 
\lambda & \text{if } (u, v) \in T, \\
\mu(uv) & \text{if } u \in X_0, v \in B_0 - Y_0, \\
0 & \text{otherwise}.
\end{cases} \quad (3)$$

For $S \subseteq V(H)$, let

$$g(S) = \varepsilon(S) + \sum_{u \in S, v \in V(H) - S} \min\{c(u), \mu(uv)\} - k \sum_{w \in S} c(w).$$

Note that

$$\begin{aligned}
&\sum_{u \in X_0, v \in Y_0} \mu(uv) + \sum_{u \in X_0, v \in B_0 - Y_0} \mu(uv) \\
&= \sum_{u \in X_0, v \in B_0} \mu(uv) \\
&= \begin{cases} 
|X_0|(|\lambda k - 2| & \text{if } a_{(s-1)/2} \notin X_0, \\
|X_0|(|\lambda k - 2| + 2 - \rho & \text{if } a_{(s-1)/2} \in X_0.
\end{cases}
\end{aligned}$$

By (1)–(3) and $|X_0| + |X_1| = |X|$, we have

$$g(S) = \lambda(|X||Y_1| + |X_1||Y_0|) + \sum_{u \in X_0, v \in Y_0} \mu(uv)$$

$$+ \lambda(|X|(r - |Y_1|) + |X_1|(k - |Y_0|) + |Y_1|(k + r - |X|))$$

$$+ \sum_{u \in X_0, v \in B_0 - Y_0} \mu(uv) - k\lambda(|X| + |Y_1|)$$

$$= \begin{cases} 
\lambda(r|X| + |Y_1|(r - |X|) - 2|X_0| & \text{if } a_{(s-1)/2} \notin X_0, \\
\lambda(r|X| + |Y_1|(r - |X|)) - 2|X_0| + 2 - \rho & \text{if } a_{(s-1)/2} \in X_0.
\end{cases}$$
If $a_{(s-1)/2} \notin X_0$, then $|X_0| \leq [(s-1)/2]$, which implies $-2|X_0| \geq -s$. If $a_{(s-1)/2} \in X_0$, then $|X_0| \leq [(s-1)/2] + 1$, which implies $-2|X_0| + 2 - \rho \geq -2[(s-1)/2] - \rho = -2(s-\rho)/2 - \rho = -s$. Thus for $|X| \geq r$, we have

$$g(S) \geq \lambda(r|X| - |Y_1|(|X| - r)) - s$$

$$= \lambda(r|X| - |Y_1|(|X| - r)) - \lambda r^2$$

$$= \lambda(|X| - r)(r - |Y_1|)$$

$$\geq 0.$$

If $\lambda r = 1$ and $|X| < r$, then $|X_0| = |X| = 0$, which implies $-2|X_0| = -\lambda r|X_0|$. If $\lambda r \geq 2$, then $-2|X_0| \geq -\lambda r|X_0|$. Hence for $|X| < r$, we have

$$g(S) \geq \lambda(r|X| + |Y_1|(|X| - r)) - \lambda r|X_0|$$

$$= \lambda(r|X_1| + |Y_1|(|X| - |X_1|))$$

$$\geq 0.$$

This settles (iii) and completes the proof. □

Before going on, the following results are needed.

**Proposition 3.7** (Ma et al. [11]) For positive integers $k$ and $n$, the graph obtained by deleting a 1-factor from $K_{n,n}$ is $C_k$-decomposable if and only if $n$ is odd, $k$ is even, $4 \leq k \leq 2n$, and $n(n-1)$ is divisible by $k$.

**Lemma 3.8** If $\lambda$ and $p$ are positive integers and $k$ is a positive even integer with $k \geq 4$, then there exist $\lambda pk/2 - p$ edge-disjoint $k$-cycles in $\lambda K_{k/2, pk}$ (also in $\lambda K_{pk, k/2}$).

**Proof:** It suffices to show that the result holds for $\lambda K_{k/2, pk}$. If $\lambda$ or $k/2$ is even, then by Lemma 3.4 there exists a $C_k$-decomposition $D$ of $\lambda K_{k/2, pk}$ with $|D| = \lambda pk/2$, in which $k$-cycles are edge-disjoint. If $k/2$ is odd, then by Proposition 3.7 there exists a $C_k$-decomposition $D'$ of $K_{k/2, k/2} - I$ with $|D'| = (k-2)/4$, where $I$ is a 1-factor of $K_{k/2, k/2}$. Since $K_{k/2, pk}$ can be decomposed into $2p$ copies of $K_{k/2, k/2}$, there exist $2p|D'| = pk/2 - p$ edge-disjoint $k$-cycles in $K_{k/2, pk}$. For odd $\lambda$ with $\lambda \geq 3$, $\lambda K_{k/2, k} = (\lambda - 1)K_{k/2, k} \cup K_{k/2, k}$. By Lemma 3.4 there exists a $C_k$-decomposition $D''$ of $(\lambda - 1)K_{k/2, pk}$ with $|D''| = (\lambda - 1)pk/2$. Hence there exist $(\lambda - 1)pk/2 + pk/2 - p = \lambda pk/2 - p$ edge-disjoint $k$-cycles in $\lambda K_{k/2, pk}$. □

**Lemma 3.9** Let $\lambda$ and $r$ be positive integers and let $k$ be a positive even integer with $k \geq 4$ and $r < k$. If $t = \lfloor \lambda r^2/k \rfloor$, then there exist $[t/2]$ edge-disjoint $k$-cycles in $\lambda K_{k/2, k}$. Moreover, if $\lambda \geq 2$ or $r \leq k - 2$ and $\lambda r^2 \geq k$, then $[t/2] + 1 \leq \lambda r/2$ and there exist $[t/2] + 1$ edge-disjoint $k$-cycles in $\lambda K_{k/2, k}$.

**Proof:** Since $r < k$, we have $t < \lambda r$. Thus $t + 1 \leq \lambda r$; in turn, $[t/2] \leq (t + 1)/2 \leq \lambda r/2 < \lambda k/2$, which implies $[t/2] \leq \lambda k/2 - 1$. By Lemma 3.8 there exist $[t/2]$ edge-disjoint $k$-cycles in $\lambda K_{k/2, k}$. When $\lambda r^2 = k$, the result is trivial. When $\lambda r^2 > k$, we have $r > 2/\sqrt{\lambda}$ since $k \geq 4$. For $\lambda \geq 2$,

$$\frac{\lambda r^2}{k} \leq \frac{\lambda r^2}{r + 1} = \lambda r - \frac{\lambda}{1 + 1/r} < \lambda r - \frac{2\lambda}{2 + \sqrt{\lambda}} < \lambda r - \frac{4}{2 + \sqrt{2}}.$$
For $r \leq k - 2$, 
\[
\frac{\lambda r^2}{k} = \frac{\lambda r^2}{r + 2} = \lambda r - \frac{2\lambda}{1 + 2/r} < \lambda r - \frac{2\lambda}{1 + \sqrt{\lambda}} < \lambda r - 1.
\]
Therefore, $t = \lfloor \lambda r^2/k \rfloor \leq \lambda r - 2$. In turn, $\lfloor t/2 \rfloor + 1 \leq t/2 + 1 \leq \lambda r/2$ for $\lambda \geq 2$ or $r \leq k - 2$. It implies $\lfloor t/2 \rfloor + 1 < \lambda k/2$. Hence $\lfloor t/2 \rfloor + 1 \leq \lambda k/2 - 1$ for $\lambda \geq 2$ or $r \leq k - 2$. This assures us that there exist $\lfloor t/2 \rfloor + 1$ edge-disjoint $k$-cycles in $\lambda K_{k/2,k}$ by Lemma 3.8.

**Lemma 3.10** Let $k$ be a positive even integer and let $n$ be a positive integer with $4 \leq k < n < 2k$. If $\lambda(n-k)^2 \geq k$, then $\lambda K_{n,n}$ has a $(C_k, S_k)$-packing $\mathcal{P}$ with $|\mathcal{P}| = \lfloor \lambda n^2/k \rfloor$ and a $(C_k, S_k)$-covering $\mathcal{C}$ with $|\mathcal{C}| = \lfloor \lambda n^2/k \rfloor$.

**Proof:** Let $n = k + r$. From the assumption $k < n < 2k$, we have $0 < r < k$. Let $\lambda r^2 = tk + s$ such that $s$ and $t$ are integers with $0 \leq s < k$. Note that $t = \lfloor \lambda r^2/k \rfloor$. Hence $\lfloor \lambda n^2/k \rfloor = \lfloor \lambda (k + r)^2/k \rfloor = \lambda(k + 2r) + t$ and
\[
\left\lfloor \frac{\lambda n^2}{k} \right\rfloor = \left\lfloor \frac{\lambda(k + r)^2}{k} \right\rfloor = \begin{cases} 
\lambda(k + 2r) + t & \text{if } s = 0 \\
\lambda(k + 2r) + t + 1 & \text{if } s > 0.
\end{cases}
\]
Since $\lambda(n-k)^2 \geq k$, $t \geq 1$. Let $p_0 = \lfloor t/2 \rfloor$ and $p_1 = \lfloor t/2 \rfloor$. We have $p_0 = 1$ and $p_1 = 0$ for $t = 1$, and $p_0 = 0$ and $p_1 = 1$ for $t \geq 2$. In the sequel, we will show that $\lambda K_{n,n}$ has a packing $\mathcal{P}$ consisting of $t$ copies of $k$-cycles and $\lambda(k+2r)$ copies of $k$-stars with leave $P_{s+1}$ (except in the case $s = 0$, in which the leave is the empty graph), and a covering $\mathcal{C}$ with $|\mathcal{C}| = \lfloor \lambda n^2/k \rfloor$.

Let $A_0 = \{a_0, a_1, \ldots, a_{k/2-1}\}$, $A_1 = \{a_{k/2}, a_{k/2+1}, \ldots, a_{k-1}\}$, $A_2 = A - (A_0 \cup A_1)$, $B_0 = \{b_0, b_1, \ldots, b_{k-1}\}$ and $B_1 = B - B_0$. In addition, letting $A_i' = \{a_{k/2}, a_{k/2+1}, \ldots, a_{[(k+s)/2]-1}\}$ for $s > 0$ and $G_i = \lambda K_{n,n}[A_i \cup B_i]$ for $i = 0, 1$. Clearly, $G_0$ and $G_1$ are isomorphic to $\lambda K_{k/2,k}$. By Lemma 3.9, there exist $p_i$ edge-disjoint $k$-cycles in $G_i$ for $i \in \{0, 1\}$, and there exist $p_1 + 1$ edge-disjoint $k$-cycles in $G_1$ for $\lambda \geq 2$ or $r \leq k - 2$. Let $\delta = 0$ for $p_1 = 0$ and $\delta = 1$ for $p_1 \geq 1$. Suppose that $Q_{i,0}, Q_{i,1}, \ldots, Q_{i,pi-1}$ are edge-disjoint $k$-cycles in $G_i$ for $0 \leq i \leq \delta$. Moreover, for $\lambda \geq 2$ or $r \leq k - 2$, let $Q$ be a $k$-cycle in $G_1$ which is edge-disjoint with $Q_{1,j}$ for $0 \leq j \leq p_1 - 1$. Without loss of generality, we assume that 
\[
Q = (b_{j_1}, a_{k/2}, b_{j_2}, a_{k/2+1}, \ldots, b_{j_{k/2}}, a_{k-1}).
\]
Note, for $\lambda = 1$ and $r = k - 1$, that $\lambda r^2 = (k-1)^2 = k(k-2) + 1$, which implies $t = k - 2$ and $s = 1$. For $s > 0$, define an $(s+1)$-path $P$ as follows:
\[
P = \begin{cases} 
 a_{k/2}b^s & \text{if } \lambda = 1, r = k - 1, \\
 b_{j_1}, a_{k/2}b_{j_2}a_{k/2+1} \ldots b_{j_s}, a_{(k+s)/2-1}b_{j_{s+1}} & \text{if } \lambda \geq 2 \text{ or } r \leq k - 2, s \text{ is even}, \\
 b_{j_1}, a_{k/2}b_{j_2}a_{k/2+1} \ldots b_{j_{(\lambda s+1)/2}}, a_{k+s+1}/2-1 & \text{if } \lambda \geq 2 \text{ or } r \leq k - 2, s \text{ is odd},
\end{cases}
\]
where $a_{k/2}b^s$ is any edge (incident with $a_{k/2}$) not in $Q_{i,0}, Q_{i,1}, \ldots, Q_{i,pi-1}$. Let 
\[
H = \lambda K_{n,n} - E(\bigcup_{i=0}^{p_i-1} (\bigcup_{h=0}^{\delta} Q_{i,h})) \cup P).
\]
Note that $V(H) = V(\lambda K_{n,n})$, $|E(H)| = \lambda n^2 - \frac{(tk + s)}{2} = \lambda n^2 - \frac{\lambda r^2}{2} = \lambda k + 2r$, and $\mu(uv) \leq \lambda$ for all $u, v \in V(H)$. Moreover, for $H' = H[A \cup B_0]$, we have

$$\deg_{H', v} = \begin{cases} 
\lambda k - 2\lfloor t/2 \rfloor & \text{if } v \in A_0, \\
\lambda k - 2\lfloor t/2 \rfloor + 1 & \text{if } s > 0 \text{ and } v \in A'_1 - \{a_{2(k+s)}/2 - 1\}, \\
\lambda k - 2\lfloor t/2 \rfloor - \rho & \text{if } s > 0 \text{ and } v = a_{2(k+s)}/2 - 1, \\
\lambda k - 2\lfloor t/2 \rfloor & \text{if } s > 0 \text{ and } v \in A_1 - A'_1, \text{ or } s = 0 \text{ and } v \in A_1, \\
\lambda k & \text{if } v \in A_2,
\end{cases}$$

where $\rho = 1$ if $s$ is odd, and $\rho = 2$ if $s$ is even. Define a function $c : V(H) \to \mathbb{N}$ as follows:

$$c(v) = \begin{cases} 
0 & \text{if } v \in B_0, \\
\lambda & \text{otherwise.}
\end{cases}$$

Now we show that there exists an $S_t$-decomposition $\varnothing$ of $H$ with central function $c$ by Proposition 2.1.

First, $k \sum_{v \in V(H)} c(v) = k\lambda(k + 2r) = |E(H)|$. This proves (i). Next, if $u, v \in B_0$, then $c(u) + c(v) = 0 = \mu(uv)$; otherwise, $c(u) + c(v) \geq \lambda \geq \mu(uv)$. This proves (ii). Finally, for $S \subseteq V(H)$, $i \in \{0, 1, 2\}$, and $j \in \{0, 1\}$, let $S \cap A_i = X_i$ and $S \cap B_j = Y_j$. Moreover, letting $S \cap A_0 = X_0$, $X = X_0 \cup X_1 \cup X_2$, and $Y = Y_0 \cup Y_1$. Define a set $T$ of ordered pairs of vertices as follows:

$$T = \{(u, v) | u \in X, v \in B_1 - Y_1 \text{ or } u \in X_2, v \in B_0 - Y_0 \text{ or } u \in Y_1, v \in A - X\}.$$  

Note that

$$k \sum_{w \in S} c(w) = k\lambda(|X| + |Y|), \tag{4}$$

$$\varepsilon(S) = \lambda(|X||Y_1| + |X_2||Y_0|) + \sum_{u \in X_0 \cup X_1, v \in Y_0} \mu(uv), \tag{5}$$

and for $u \in S$ and $v \in V(H) - S$

$$\min\{c(u), \mu(uv)\} = \begin{cases} 
\lambda & \text{if } (u, v) \in T, \\
\mu(uv) & \text{if } u \in X_0 \cup X_1, v \in B_0 - Y_0, \\
0 & \text{otherwise.}
\end{cases} \tag{6}$$

For $S \subseteq V(H)$, let

$$g(S) = \varepsilon(S) + \sum_{u \in S, v \in V(H) - S} \min\{c(u), \mu(uv)\} - k \sum_{w \in S} c(w).$$

Note that

$$\sum_{u \in X_0 \cup X_1, v \in Y_0} \mu(uv) + \sum_{u \in X_0 \cup X_1, v \in B_0 - Y_0} \mu(uv)$$

$$= \sum_{u \in X_0 \cup X_1, v \in B_0} \mu(uv)$$

$$= \begin{cases} 
|X_0|(|\lambda k - 2\lfloor t/2 \rfloor| + |X_1|(|\lambda k - 2\lfloor t/2 \rfloor|) & \text{if } s = 0, \\
|X_0|(|\lambda k - 2\lfloor t/2 \rfloor| + |X_1|(|\lambda k - 2\lfloor t/2 \rfloor|) - 2|X_1|) & \text{if } s > 0, a_{2(k+s)}/2 - 1 \notin X'_1, \\
|X_0|(|\lambda k - 2\lfloor t/2 \rfloor| + |X_1|(|\lambda k - 2\lfloor t/2 \rfloor|) - 2|X_1| + 2 - \rho) & \text{if } s > 0, a_{2(k+s)}/2 - 1 \in X'_1.
\end{cases}$$
By (4)–(6) and $|X_0| + |X_1| + |X_2| = |X|$, we have

$$g(S) = \lambda(|X| |Y_1| + |X_2| |Y_0|) + \sum_{v \in X_0 \cup X_1, u \in Y_0} \mu(uv)$$

$$+ \lambda(|X| (r - |Y_1|) + |X_2|(k - |Y_0|) + |Y_1|(k + r - |X|))$$

$$+ \sum_{u \in X_0 \cup X_1, v \in B_0 - Y_0} \mu(uv) - k\lambda(|X| + |Y_1|)$$

$$= \lambda(|X| + |Y_1|(r - |X|)) + m,$$

where

$$m = \begin{cases} -2(|X_0| \lceil t/2 \rceil + |X_1| \lceil t/2 \rceil) & \text{if } s = 0, \\ -2(|X_0| \lceil t/2 \rceil + |X_1| \lceil t/2 \rceil) - 2|X_1| & \text{if } s > 0, a_{(k+s)/2} - 1 \notin X_1', \\ -2(|X_0| \lceil t/2 \rceil + |X_1| \lceil t/2 \rceil) - 2|X_1| + 2 - \rho & \text{if } s > 0, a_{(k+s)/2} - 1 \in X_1'. \end{cases}$$

If $a_{(k+s)/2} - 1 \notin X_1'$, then $|X_1| \leq |A_1'| - 1 = \lceil s/2 \rceil - 1$. Hence $-2|X_1| \geq -2(\lceil s/2 \rceil - 1) \geq -s$. If $a_{(k+s)/2} - 1 \in X_1$, then $|X_1| \leq |A_1'| = \lceil s/2 \rceil$. In addition, $\rho = 1$ for odd $s$ and $\rho = 2$ for even $s$.

Therefore, $-2|X_1| + 2 - \rho \geq -2\lceil s/2 \rceil + 2 - \rho = -s$. Together with the fact $\max\{|X_0|, |X_1|\} \leq k/2$, we have

$$m \geq -2(k/2 \lceil t/2 \rceil + k/2 \lceil t/2 \rceil) - s = -(kt + s) = -\lambda r^2.$$

Thus for $|X| \geq r$, we have

$$g(S) \geq \lambda(|X| - |Y_1|(|X| - r)) - \lambda r^2 = \lambda(|X| - r)(r - |Y_1|) \geq 0.$$

So it remains to consider the case $|X| < r$. Recall that $t = k - 2$ and $s = 1$ for $(\lambda, r) = (1, k - 1)$.

Thus $\lceil t/2 \rceil = \lceil t/2 \rceil = (\lambda - 1)/2$. In addition, $|X_1'| = 0$ for $a_{(k+s)/2} - 1 \notin X_1'$, and $\rho = 1$ as well as $|X_1| \geq 1$ (which implies $|X_1| \geq 1$) for $a_{(k+s)/2} - 1 \in X_1'$. Hence for $a_{(k+s)/2} - 1 \notin X_1'$,

$$m = -2(|X_0| + |X_1|)(\lambda r - 1)/2 \geq -\lambda r(|X_0| + |X_1|),$$

and for $a_{(k+s)/2} - 1 \in X_1'$,

$$m = -2(|X_0| + |X_1|)(\lambda r - 1)/2 - 1 = -\lambda r(|X_0| + |X_1|) + |X_0| + |X_1| - 1 \geq -\lambda r(|X_0| + |X_1|).$$

On the other hand, for $\lambda \geq 2$ or $r \leq k - 2$, we have $|t/2| + 1 \leq \lambda r/2$ by Lemma 3.9 this implies

$$m \geq -2(|X_0| \lceil t/2 \rceil + |X_1| \lceil (t/2) + 1 \rceil + (|X_1| - |X_1|) \lceil t/2 \rceil) \geq -\lambda r(|X_0| + |X_1|) = \lambda r(|X_2| + |Y_1|(r - |X|)) \geq 0.$$

Therefore, for $|X| < r$, we have

$$g(S) \geq \lambda(|X| - |Y_1|(|X| - r)) - \lambda r(|X_0| + |X_1|) = \lambda(|X| - |X|) \geq 0.$$

This settles (iii).
Let $\mathcal{P} = \mathcal{D} \cup \{Q_{i,0}, Q_{i,1}, \ldots, Q_{i,n-1}\}$. Clearly, $\mathcal{P}$ is the required packing. Let
$$C = \begin{cases} 
\mathcal{P} & \text{if } s = 0, \\
\mathcal{P} \cup \{Q\} & \text{if } s \geq 1.
\end{cases}$$

It is easy to check that $C$ is the covering as required.

Now, we are ready for the main result of this section.

**Theorem 3.11** If $\lambda$ and $n$ are positive integers and $k$ is a positive even integer with $4 \leq k \leq n$, then
$$p(\lambda K_{n,n}; C_k, S_k) = \left\lfloor \frac{\lambda n^2}{k} \right\rfloor$$
and
$$c(\lambda K_{n,n}; C_k, S_k) = \left\lceil \frac{\lambda n^2}{k} \right\rceil.$$

**Proof:** Obviously,
$$p(\lambda K_{n,n}; C_k, S_k) \leq \left\lfloor \frac{\lambda n^2}{k} \right\rfloor \leq \left\lceil \frac{\lambda n^2}{k} \right\rceil \leq c(\lambda K_{n,n}; C_k, S_k),$$

Let $n = qk + r$ where $q$ and $r$ are integers with $0 \leq r < k$. For $q = 1$, the result follows from Lemmas 3.5, 3.6, and 8.10. If $q \geq 2$, then $\lambda K_{n,n} = \lambda K_{k+r,k+r} \cup \lambda K_{k+r,q-1} \cup \lambda K_{q-1,k+n}$. Note that $\lambda K_{k+r,q-1}$ has a $(C_k, S_k)$-packing $\mathcal{P}$ with $|\mathcal{P}| = \lfloor \lambda (k + r)^2/k \rfloor$ and a $(C_k, S_k)$-covering $C'$ with $|C'| = \lfloor \lambda (k + r)^2/k \rfloor$. Trivially, $\lambda K_{k+r,q-1}$ and $\lambda K_{q-1,k+n}$ have $S_k$-decompositions $\mathcal{D}$ and $\mathcal{D}'$ with $|\mathcal{D}| = \lambda (k + r)(q - 1)$ and $|\mathcal{D}'| = \lambda (q - 1)n$, respectively. Since $\lambda (k + r)^2/k + \lambda (k + r)(q - 1) + \lambda (q - 1)n = \lambda (k + r)^2/k + \lambda n^2/k$, $\mathcal{P} \cup \mathcal{D} \cup \mathcal{D}'$ is a $(C_k, S_k)$-packing of $\lambda K_{n,n}$ with cardinality $\lceil \lambda n^2/k \rceil$ and $C \cup \mathcal{D} \cup \mathcal{D}'$ is a $(C_k, S_k)$-covering of $\lambda K_{n,n}$ with cardinality $\lceil \lambda n^2/k \rceil$. This completes the proof.

Clearly, if $\lambda K_{n,n}$ admits a $(C_k, S_k)$-decomposition, then $4 \leq k \leq n$ and $k$ is even and $\lambda n^2$ is divisible by $k$. When $k$ divides $\lambda n^2$, a $(C_k, S_k)$-packing $\mathcal{P}$ with $|\mathcal{P}| = \lfloor \lambda n^2/k \rfloor$ is a $(C_k, S_k)$-decomposition. Therefore, with the aid of Theorem 3.11 we have the following.

**Corollary 3.12** For positive integers $\lambda$, $k$, and $n$, the balanced complete bipartite multigraph $\lambda K_{n,n}$ is $(C_k, S_k)$-decomposable if and only if $4 \leq k \leq n$, $k$ is even, and $\lambda n^2$ is divisible by $k$.

## 4 Packing and covering with 4-cycles and 4-stars

In this section a complete solution to the minimum packing and minimum covering problem of $\lambda K_{n,n}$ with $C_4$ and $S_4$ is given. Before that, we need more notations. For multigraphs $G$ and $H$, $G \oplus H$ denotes the disjoint union of $G$ and $H$, $G \circ H$ denotes the union of $G$ and $H$ with a common vertex. For a set $\mathcal{R}$ and a positive integer $t$, $t\mathcal{R}$ denotes the multiset in which each element in $\mathcal{R}$ appears $t$ times. In addition, $M_t$ denotes the graph induced by $t$ nonadjacent edges. We begin with the discussion for the possible minimum leaves and paddings of $\lambda K_{n,n}$ with $C_4$ and $S_4$.

Note that $|E(\lambda K_{n,n})| = \lambda n^2$. If $\lambda \equiv 0 \pmod{4}$ or $n \equiv 0 \pmod{2}$, then $|E(\lambda K_{n,n})| \equiv 0 \pmod{4}$. By Corollary 3.12 both of the possible minimum leave and the possible minimum padding are the empty graph. If $\lambda \equiv 1 \pmod{4}$ and $n \equiv 1 \pmod{2}$, then $|E(\lambda K_{n,n})| \equiv 1 \pmod{4}$. This implies that the possible minimum leave is only $P_2$, and the possible minimum paddings are $S_3$, $P_3$, $P_3 \oplus P_2$, $M_1$, $2P_2 \oplus P_2$, $2P_2 \circ P_2$, and $3P_2$. If $\lambda \equiv 2 \pmod{4}$ and $n \equiv 1 \pmod{2}$, then $|E(\lambda K_{n,n})| \equiv 2 \pmod{4}$. This implies that the possible minimum leaves are $P_3$, $M_2$, and $2P_2$, so are the possible minimum paddings. If $\lambda \equiv 3 \pmod{4}$ and $n \equiv 1 \pmod{2}$, then $|E(\lambda K_{n,n})| \equiv 3 \pmod{4}$. This implies that the possible minimum leaves are $S_3$, $P_3$, $P_3 \oplus P_2$, $M_3$, $2P_2 \oplus P_2$, $2P_2 \circ P_2$, and $3P_2$, and the possible minimum padding is only $P_2$. 

...
Lemma 4.1  \( K_{5,5} \) has no \( (C_4, S_4) \)-covering with padding \( 3P_2 \).

Proof: It suffices to show that \( K_{5,5} + 3\{a_0b_0\} \) is not \( (C_4, S_4) \)-decomposable. Suppose, to the contrary of the conclusion, that there exists a \( (C_4, S_4) \)-decomposition \( \mathcal{D} \) of \( K_{5,5} + 3\{a_0b_0\} \). Since there are at most two star with center \( a_0 \) (or \( b_0 \)) and each edge joining \( a_0 \) and \( b_0 \) lies in exactly one subgraph in \( \mathcal{D} \), there are exactly three possibilities for the edges joining \( a_0 \) and \( b_0 \) to lie in the decomposition: in four 4-cycles, in three 4-cycles and a 4-star, or in two 4-cycles and two 4-stars. Let \( G_1 \) be the graph obtained from \( K_{5,5} + 3\{a_0b_0\} \) by deleting the edges of four 4-cycles, and let \( G_2 \) be the graph obtained from \( K_{5,5} + 3\{a_0b_0\} \) by deleting the edges of three 4-cycles or deleting the edges of two 4-cycles. Note that \( \deg_{G_1} x = 3 \) for \( x \notin \{a_0, b_0\} \), which implies that there is no 4-star in \( G_1 \). Since \( \deg_{G_2} x \leq 3 \) for \( x \in \{a_0, b_0\} \), there is no 4-star with center at \( a_0 \) or \( b_0 \) in \( G_2 \). This leads to a contradiction and completes the proof.

We summarize the results discussed above in Table 1.

<table>
<thead>
<tr>
<th>( \lambda ) \mod 4</th>
<th>( n ) \mod 2</th>
<th>( \lambda \equiv 0 ) or ( n \equiv 0 )</th>
<th>( \lambda \equiv 1 ) and ( n \equiv 1 )</th>
<th>( \lambda \equiv 2 ) and ( n \equiv 1 )</th>
<th>( \lambda \equiv 3 ) and ( n \equiv 1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Leave</td>
<td>( \emptyset )</td>
<td>( S_3, P_4, P_3 \square P_2, M_3, 2P_2 \square P_2, 2P_2 \square P_2, 3P_2 ) (3P_2 for ( \lambda \neq 1 ))</td>
<td>( P_2 )</td>
<td>( P_2 )</td>
<td>( P_3, M_2, 2P_2 )</td>
</tr>
<tr>
<td>Padding</td>
<td>( \emptyset )</td>
<td>( S_3, P_4, P_3 \square P_2, M_3, 2P_2 \square P_2, 2P_2 \square P_2, 3P_2 ) (3P_2 for ( \lambda \neq 1 ))</td>
<td>( P_2 )</td>
<td>( P_2 )</td>
<td>( P_2 )</td>
</tr>
</tbody>
</table>

Lemma 4.2: Let \( r \in \{1, 2, 3, 5\} \).

(a) There exists a \( (C_4, S_4) \)-packing of \( rK_{5,5} \) with leave \( L \) where

\[
L = \begin{cases} \emptyset & \text{if } r = 1 \text{ or } r = 5, \\ \{P_3, M_2, 2P_2\} & \text{if } r = 2, \\ \{S_3, P_4, P_3 \square P_2, M_3, 2P_2 \square P_2, 2P_2 \square P_2, 3P_2\} & \text{if } r = 3. \end{cases}
\]

(b) There exists a \( (C_4, S_4) \)-covering of \( rK_{5,5} \) with padding \( R \) where

\[
R = \begin{cases} \emptyset & \text{if } r = 1, \\ \{S_3, P_4, P_3 \square P_2, M_3, 2P_2 \square P_2, 2P_2 \square P_2\} & \text{if } r = 2, \\ \{P_3, M_2, 2P_2\} & \text{if } r = 3, \\ \{R_2\} & \text{if } r = 5. \end{cases}
\]

Proof: The proof is divided into four parts according to the value of \( r \).

Case 1. \( r = 1 \).

Let \( A_1 = \{a_1, a_2, a_3, a_4\} \) and \( B_1 = \{b_1, b_2, b_3, b_4\} \), and let \( H = K_{5,5}[A_1 \cup B_1] \). Trivially, \( H \) is isomorphic to \( K_{4,4} \). By Corollary 3.12, there exists a \( (C_4, S_4) \)-decomposition \( \mathcal{D} \) of \( K_{4,4} \).
of 200

Now we give the required coverings of $K_{5,5}$. Note that $\mathcal{P} \cup \{(a_0; b_0, b_1, b_2, b_3, b_4)\}$ is a $(C_4, S_4)$-covering of $K_{5,5}$ with padding $S_3 : \{(a_0; b_1, b_2, b_3)\}$, and $\mathcal{P} \cup \{(a_0, b_1, a_1, b_0)\}$ is a $(C_4, S_4)$-covering of $K_{5,5}$ with padding $P_4 : \{a_0 b_1 a_1 b_0\}$. Without loss of generality, we assume that $\mathcal{P}$ contains a 4-star $(a_4; b_1, b_2, b_3, b_4)$. Thus $\mathcal{P} - \{(a_0; b_1, b_2, b_3, b_4, (a_4; b_1, b_2, b_3, b_4))\} \cup \{(a_0, b_1, a_4, b_4), (a_0; b_1, b_2, b_3, b_4)\}$ is a $(C_4, S_4)$-covering of $K_{5,5}$ with padding $P_3 \cup P_2 : \{a_0 b_4 a_4, a_0 b_3\}$. In addition, $\{(a_0, b_4, a_4, b_4), (a_4; b_0, b_2, b_3, b_4), (a_1; b_0, b_1, b_3, b_4), (a_2; b_1, b_2, b_3, b_4), (a_0; a_2, a_3, a_4), (b_1; a_0, a_1, a_4, a_4), (b_2; a_1, a_2, a_3, a_4)\}$ is a $(C_4, S_4)$-covering of $K_{5,5}$ with padding $M_3 : \{a_0 b_0, a_1 b_1, a_2 b_2\}$, $\mathcal{P} - \{(a_4; b_1, b_2, b_3, b_4)\} \cup \{(a_0, b_1, a_4, b_4), (a_4; b_0, b_1, b_2, b_3)\}$ is a $(C_4, S_4)$-covering of $K_{5,5}$ with padding $2P_2 \cup P_2 : 2\{b_0 a_4\}$, and $\mathcal{P} - \{(a_0; b_1, b_2, b_3, b_4)\} \cup \{(a_0, b_0, a_4, b_4), (a_0; b_1, b_2, b_3, b_4)\}$ is a $(C_4, S_4)$-covering of $K_{5,5}$ with padding $2P_2 \cup P_2 : 2\{b_0 a_4\}$. 

Case 2. $r = 2$.

First, we use $\mathcal{P}$ to construct the required packings of $2K_{5,5}$. Exchanging $b_0$ with $b_1$ in $\mathcal{P}$, we obtain a packing $\mathcal{P}'$ of $K_{5,5}$ with leave $a_0 b_1$. Let $\mathcal{P}_1 = \mathcal{P} \cup \mathcal{P}'$. One can see that $\mathcal{P}_1$ is a packing of $2K_{5,5}$ with leave $P_3 : \{b_0 a_0 b_1\}$. Next, rename the vertices $a_0, a_1, b_0, b_1$ in $\mathcal{P}_1$ to $a_0, a_0, b_1, b_0$ respectively, we obtain a packing $\mathcal{P}''$ of $K_{5,5}$ with leave $a_1 b_1$. Let $\mathcal{P}_2 = \mathcal{P} \cup \mathcal{P}''$. It is easy to see that $\mathcal{P}_2$ is a packing of $2K_{5,5}$ with leave $M_2 : \{a_0 b_0, a_1 b_1\}$. Finally, $2\mathcal{P}$ is clearly a packing of $2K_{5,5}$ with leave $2P_2 : 2\{a_0 b_0\}$.

Now we use packings to construct the required packings of $2K_{5,5}$. Note that $\mathcal{P}_1 \cup \{(a_0; b_1, b_2, b_3, b_4)\}$ is a $(C_4, S_4)$-covering of $2K_{5,5}$ with padding $P_3 : \{b_0 a_0 b_3\}$, and $\mathcal{P}_2 \cup \{(a_0, b_0, a_1, b_1)\}$ is a $(C_4, S_4)$-covering of $2K_{5,5}$ with padding $M_3 : \{a_0 b_0, a_1 b_0\}$, and $\mathcal{P} - \{(a_4; b_1, b_2, b_3, b_4)\} \cup \{(a_0, b_1, a_4, b_4)\}$ is a $(C_4, S_4)$-covering of $2K_{5,5}$ with padding $M_3 : \{a_0 b_0, a_1 b_0\}$, and $\mathcal{P} - \{(a_4; b_1, b_2, b_3, b_4)\} \cup \{(a_0, b_1, a_4, b_4)\}$ is a $(C_4, S_4)$-covering of $2K_{5,5}$ with padding $2P_2 \cup P_2 : 2\{a_0 b_4\}$.

Case 3. $r = 3$.

First, we use packings of $K_{5,5}$ and $2K_{5,5}$ to construct the required packings of $3K_{5,5}$. Exchanging $b_0$ with $b_2$ in $\mathcal{P}$, we obtain a packing $\mathcal{P}'$ of $K_{5,5}$ with leave $a_0 b_2$. Hence $\mathcal{P} \cup \mathcal{P}'$ is a packing of $3K_{5,5}$ with leave $S_3 : \{(a_0; b_0, b_1, b_2)\}$. Next, rename the vertices $a_0, a_2, b_0, b_2$ in $\mathcal{P}'$ to $a_0, a_0, b_2, b_2$, respectively, we obtain a packing $\mathcal{P}''$ of $K_{5,5}$ with leave $a_2 b_2$. Thus $\mathcal{P}_2 \cup \mathcal{P}''$ is a packing of $3K_{5,5}$ with leave $M_3 : \{a_0 b_0, a_1 b_1, a_2 b_2\}$. Note that $\mathcal{P}_1 \cup \mathcal{P}''$ is a packing of $3K_{5,5}$ with leave $P_3 \cup P_2 : \{b_0 a_0 b_1\}$.

In addition, $\mathcal{P}_1 \cup \mathcal{P}''$ is a packing of $3K_{5,5}$ with leave $P_3 \cup P_2 : \{b_0 a_0 b_1\}$, $2P_2 \cup \mathcal{P}''$ is a packing of $3K_{5,5}$ with leave $2P_2 \cup P_2 : 2\{a_0 b_0\}$, and $\mathcal{P} \cup \mathcal{P}''$ is a packing of $3K_{5,5}$ with leave $2P_2 \cup P_2 : 2\{a_0 b_0\}$, and $\mathcal{P} \cup \mathcal{P}''$ is clearly a packing of $3K_{5,5}$ with leave $3P_3 : 3\{a_0 b_0\}$.

Finally, since $3(5-4)^2 = 3 < 4$, there exists a $(C_4, S_4)$-covering of $3K_{5,5}$ with leave $2P_2$ by Lemma 3.6

Case 4. $r = 5$.

By Corollary 3.12 $(C_4, S_4) \mid 4K_{5,5}$. Since $5K_{5,5} = K_{5,5} \cup 4K_{5,5}$, it suffices to show that there exists a $(C_4, S_4)$-covering of $5K_{5,5}$ with padding $3P_2$. Note that $5K_{5,5} = 2K_{5,5} \cup 3K_{5,5}$. Since $2K_{5,5}$ has a $(C_4, S_4)$-covering with padding $2P_2 : 2\{a_0 a_4\}$ and $3K_{5,5}$ has a $(C_4, S_4)$-covering with padding $P_2$ (say $\{b_0 a_4\}$), we have the required covering.

Lemma 4.3 Let $r$ be a positive integer and let $m$ be a positive odd integer with $m \geq 5$. If $rK_{m,m}$ has a $(C_4, S_4)$-packing (resp. $(C_4, S_4)$-covering) with leave $L$ (resp. padding $R$), then $rK_{m+2, m+2}$ also has a $(C_4, S_4)$-packing (resp. $(C_4, S_4)$-covering) with leave $L$ (resp. padding $R$).

Proof: Let $m = 2t + 1$ where $t$ is a positive integer with $t \geq 2$. Let $A_1 = \{a_0, a_1, \ldots, a_{2t}\}$ and
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Let \( B_1 = \{ b_0, b_1, \ldots, b_{2t} \} \). Letting \( G_1 = K_{m+2,m+2}[A_1 \cup B_1] \) and \( G_2 = K_{m+2,m+2} - E(G_1) \). Clearly, \( G_1 \) is isomorphic to \( K_{m,m} \). Note that \( \{(a_{2i+1}, b_{2i+2}, b_{2i+1}), (a_{2i}, b_{2i-2}, b_{2i+2}), (a_{2i+1}, b_{2i+2}, b_{2i+1}), (a_{2i}, b_{2i-1}, b_{2i+2}), (b_{2i+2}, a_{2i-2}, a_{2i+2}), b_{2i+2}, a_{2i-1}, a_{2i}, a_{2i+2}) \} \) is a \((C_4, S_1)\)-decomposition of \( G_2 \). Since \( rK_{m+2,m+2} = rG_1 \cup rG_2 \), \( rK_{m+2,m+2} \) has the required packings and coverings.

Now, we are ready for the main result of this section.

**Theorem 4.4** Let \( \lambda \) and \( n \) be positive integers with \( n \geq 4 \).

(A) \( \lambda K_{n,n} \) has a maximum \((C_4, S_1)\)-packing with leave \( L \) if and only if

\[
\begin{align*}
L &= \emptyset & \text{if } & \lambda n^2 \equiv 0 \pmod{4}, \\
L &= \{ P_3, M_2, 2P_2 \} & \text{if } & \lambda n^2 \equiv 1 \pmod{4}, \\
L &= \{ S_3, P_1, P_3 \cup P_2, M_3, 2P_2 \cup P_2, 2P_2 \circ P_2, 3P_2 \} & \text{if } & \lambda n^2 \equiv 2 \pmod{4}, \\
L &= \{ S_3, P_1, P_3 \cup P_2, M_3, 2P_2 \cup P_2, 2P_2 \circ P_2, 3P_2 \} & \text{if } & \lambda n^2 \equiv 3 \pmod{4}.
\end{align*}
\]

(B) \( \lambda K_{n,n} \) has a minimum \((C_4, S_1)\)-covering with padding \( R \) if and only if

\[
\begin{align*}
R &= \emptyset & \text{if } & \lambda n^2 \equiv 0 \pmod{4}, \\
R &= \{ S_3, P_1, P_3 \cup P_2, M_3, 2P_2 \cup P_2, 2P_2 \circ P_2 \} & \text{if } & \lambda n^2 \equiv 1 \pmod{4} \text{ and } \lambda = 1, \\
R &= \{ S_3, P_1, P_3 \cup P_2, M_3, 2P_2 \cup P_2, 2P_2 \circ P_2, 3P_2 \} & \text{if } & \lambda n^2 \equiv 1 \pmod{4} \text{ and } \lambda \geq 5, \\
R &= \{ P_3, M_2, 2P_2 \} & \text{if } & \lambda n^2 \equiv 2 \pmod{4}, \\
R &= \{ P_3, M_2, 2P_2 \} & \text{if } & \lambda n^2 \equiv 3 \pmod{4}.
\end{align*}
\]

**Proof:** The necessity follows from the arguments above Table 1. It suffices to show that \( \lambda K_{n,n} \) has required packings and coverings. The result for \( \lambda n^2 \equiv 0 \pmod{4} \) follows from Corollary 3.12 immediately. So it remains to consider the case \( \lambda n^2 \equiv r \pmod{4} \) for \( r \in \{ 1, 2, 3 \} \). Note that \( \lambda n^2 \equiv r \pmod{4} \) if and only if \( \lambda \equiv r \pmod{4} \) and \( n \equiv 1 \pmod{2} \). When \( \lambda \in \{ 1, 2, 3, 5 \} \), the result for \( n = 5 \) follows from Lemma 4.2, and the result for \( n > 5 \) can be obtained by using Lemma 4.3 recursively. Now consider \( \lambda \equiv r \pmod{4} \) and \( \lambda \geq 5 \). Note that \( \lambda K_{n,n} = rK_{n,n} \cup (\lambda - r)K_{n,n} \). Since \((\lambda - r)K_{n,n}\) is \((C_4, S_1)\)-decomposable by Corollary 3.12, we have the result. \( \square \)

**References**

1. A. Abueida, S. Clark, and D. Leach, Multidecomposition of the complete graph into graph pairs of order 4 with various leaves. *Ars Combin.*, 93:403–407, 2009.


