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*Oriented diameter and rainbow connection number of a graph**

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The oriented diameter of a bridgeless graph G is $\min\{diam(H) \mid H$ is a strang orientation of $G\}$. A path in an edge-colored graph G, where adjacent edges may have the same color, is called rainbow if no two edges of the path are colored the same. The rainbow connection number $rc(G)$ of G is the smallest integer number k for which there exists a k-edge-coloring of G such that every two distinct vertices of G are connected by a rainbow path. In this paper, we obtain upper bounds for the oriented diameter and the rainbow connection number of a graph in terms of $rad(G)$ and $\eta(G)$, where $rad(G)$ is the radius of G and $\eta(G)$ is the smallest integer number such that every edge of G is contained in a cycle of length at most $\eta(G)$. We also obtain constant bounds of the oriented diameter and the rainbow connection number for a (bipartite) graph G in terms of the minimum degree of G.

Keywords: Diameter, Radius, Oriented diameter, Rainbow connection number, Cycle length, Bipartite graph.

1 Introduction

All graphs in this paper are undirected, finite and simple. We refer to the book [2] for notation and terminology not described here. A path $u = u_1, u_2, \dots, u_k = v$ is called a $P_{u,v}$ path. Denote by u_iPu_j the subpath $u_i, u_{i+1}, \dots, u_j$ for $i \leq j$. The *length* $\ell(P)$ of a path P is the number of edges in P. The *distance* between two vertices x and y in G, denoted by $d_G(x, y)$, is the length of a shortest path between them. The *eccentricity* of a vertex x in G is $ecc_G(x) = max_{y\in V(G)}d(x, y)$. The *radius* and *diameter* of G are $rad(G) = \min_{x\in V(G)} ecc(x)$ and $diam(G) = \max_{x\in V(G)} ecc(x)$, respectively. A vertex u is a *center* of a graph G if $ecc(u) = rad(G)$. The oriented diameter of a bridgeless graph G is $\min\{diam(H) \mid H$ is an orientation of $G\}$, and the oriented radius of a bridgeless graph G is $\min\{rad(H) \mid H$ is an orientation of $G\}$. For any graph G with edge-connectivity $\lambda(G) = 0, 1$, G has oriented radius (resp. diameter) ∞.

In 1939, Robbins solved the One-Way Street Problem and proved that a graph G admits a strongly connected orientation if and only if G is bridgeless, that is, G does not have any cut-edge. Naturally, one hopes that the oriented diameter of a bridgeless graph is as small as possible. Bondy and Murty suggested

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to study the quantitative variations on Robbins' theorem. In particular, they conjectured that there exists a function f such that every bridgeless graph with diameter d admits an orientation of diameter at most $f(d)$.

In 1978, Chvátal and Thomassen [5] obtained some general bounds.

Theorem 1 (Chvátal and Thomassen 1978 [5]) *For every bridgeless graph G, there exists an orientation H of G such that*

$$rad(H) \leq rad(G)^2 + rad(G),$$

$$diam(H) \leq 2rad(G)^2 + 2rad(G).$$

Moreover, the above bounds are optimal.

There exists a minor error when they constructed the graph G_d which arrives at the upper bound when d is odd. Kwok, Liu and West gave a slight correction in [11].

They also showed that determining whether an arbitrary graph can be oriented so that its diameter is at most 2 is NP-complete. Bounds for the oriented diameter of graphs have also been studied in terms of other parameters, for example, radius, dominating number [5, 6, 11, 18], etc. Some classes of graphs have also been studied in [6, 7, 8, 9, 14].

Let $\eta(G)$ be the smallest integer such that every edge of G belongs to a cycle of length at most $\eta(G)$. In this paper, we show the following result.

Theorem 2 *For every bridgeless graph G, there exists an orientation H of G such that*

$$rad(H) \leq \sum_{i=1}^{rad(G)} \min\{2i, \eta(G) - 1\} \leq rad(G)(\eta(G) - 1),$$

$$diam(H) \leq 2 \sum_{i=1}^{rad(G)} \min\{2i, \eta(G) - 1\} \leq 2rad(G)(\eta(G) - 1).$$

Note that $\sum_{i=1}^{rad(G)} \min\{2i, \eta(G) - 1\} \leq rad(G)^2 + rad(G)$ and $diam(H) \leq 2rad(H)$. So our result implies Chvátal and Thomassen's Theorem 1.

A path in an edge-colored graph G, where adjacent edges may have the same color, is called $rainbow$ if no two edges of the path are colored the same. An edge-coloring of a graph G is a $rainbow\ edge\text{-}coloring$ if every two distinct vertices of the graph G are connected by a rainbow path. The $rainbow\ connection\ number\ rc(G)$ of G is the minimum integer k for which there exists a rainbow k-edge-coloring of G. It is easy to see that $diam(G) \leq rc(G)$ for any connected graph G. The rainbow connection number was introduced by Chartrand et al. in [4]. It is of great use in transferring information of high security in multicomputer networks. We refer the readers to [3] for details.

Chakraborty et al. [3] investigated the hardness and algorithms for the rainbow connection number, and showed that given a graph G, deciding if $rc(G) = 2$ is NP-complete. Bounds for the rainbow connection number of a graph have also been studies in terms of other graph parameters, for example, radius, dominating number, minimum degree, connectivity, etc. [1, 4, 10]. Cayley graphs and line graphs were studied in [12] and [13], respectively.

A subgraph H of a graph G is called $isometric$ if the distance between any two distinct vertices in H is the same as their distance in G. The size of a largest isometric cycle in G is denoted by $\zeta(G)$.

Clearly, every isometric cycle is an induced cycle and thus $\zeta(G)$ is not larger than the chordality, where $chordality$ is the length of a largest induced cycle in G. In [1], Basavaraju, Chandran, Rajendraprasad and Ramaswamy got the the following sharp upper bound for the rainbow connection number of a bridgeless graph G in terms of $rad(G)$ and $\zeta(G)$.

Theorem 3 (Basavaraju et al. [1]) *For every bridgeless graph G,*

$$rc(G) \leq \sum_{i=1}^{rad(G)} \min\{2i+1, \zeta(G)\} \leq rad(G)\zeta(G).$$

In this paper, we show the following result.

Theorem 4 *For every bridgeless graph G,*

$$rc(G) \leq \sum_{i=1}^{rad(G)} \min\{2i+1, \eta(G)\} \leq rad(G)\eta(G).$$

From Lemma 2 of Section 2, we will see that $\eta(G) \leq \zeta(G)$. Thus our result implies Theorem 3.

This paper is organized as follows: in Section 2, we introduce some new definitions and show several lemmas. In Section 3, we prove Theorem 2 and study upper bounds for the oriented radius (resp. diameter) of plane graphs, edge-transitive graphs and general (bipartite) graphs. In Section 4, we prove Theorem 4 and study upper for the rainbow connection number of plane graphs, edge-transitive graphs and general (bipartite) graphs.

2 Preliminaries

In this section, we introduce some definitions and show several lemmas.

Definition 1 For any $x \in V(G)$ and $k \geq 0$, the k-$step\ open\ neighborhood$ is $\{y \mid d(x,y) = k\}$ and denoted by $N_k(x)$, the k-$step\ closed\ neighborhood$ is $\{y \mid d(x,y) \leq k\}$ and denoted by $N_k[x]$. If $k = 1$, we simply write $N(x)$ and $N[x]$ for $N_1(x)$ and $N_1[x]$, respectively.

Definition 2 Let G be a graph and H be a subset of $V(G)$ (or a subgraph of G). The edges between H and $G \setminus H$ are called $legs$ of H. An H-ear is a path $P = (u_0, u_1, \ldots, u_k)$ in G such that $V(H) \cap V(P) = \{u_0, u_k\}$. The vertices u_0, u_k are called the $feet$ of P in H and u_0u_1, $u_{k-1}u_k$ are called the $legs$ of P. The $length$ of an H-ear is the length of the corresponding path. If $u_0 = u_k$, then P is called a $closed\ H$-ear. For any leg e of H, denote by $\ell(e)$ the smallest number such that there exists an H-ear of length $\ell(e)$ containing e, and such an H-ear is called an $optimal$ (H, e)-ear.

Note that for any optimal (H, e)-ear P and every pair $(x, y) \neq (u_0, u_k)$ of distinct vertices of P, x and y are adjacent on P if and only if x and y are adjacent in G.

Definition 3 For any two paths P and Q, the joint of P and Q are the common vertex and edge of P and Q. Paths P and Q have $k\ continuous\ common\ segments$ if the common vertex and edge are k disjoint paths.

A common segment is trivial if it has only one vertex.

Definition 4 Let P and Q be two paths in G. Call P and Q $independent$ if they has no common internal vertex.

Lemma 1 *Let $n \geq 1$ be an integer, and let G be a graph, H be a subgraph of G and $e_i = u_iv_i$ be a leg of H and $P_i = P_{u_iw_i}$ be an optimal (G, e_i)-ear, where $1 \leq i \leq n$ and u_i, w_i are the foot of P_i. Then for any leg $e_j = u_jv_j$ such that $e_j \neq e_i$ and $e_j \notin E(P_i)$, where $i \in \{1, 2, \ldots, n\}$, there exists an optimal (H, e_j)-ear $P_j = P_{u_jw_j}$ such that either P_i and P_j are independent for any P_i, $1 \leq i \leq n$, or P_i and P_j have only one continuous common segment containing w_j for some P_i.*

Proof: Let P_j be an optimal (H, e_j)-ear. If P_i and P_j are independent for any i, then we are done. Suppose that P_i and P_j have m continuous common segments for some i, where $m \geq 1$. When $m \geq 2$, we first

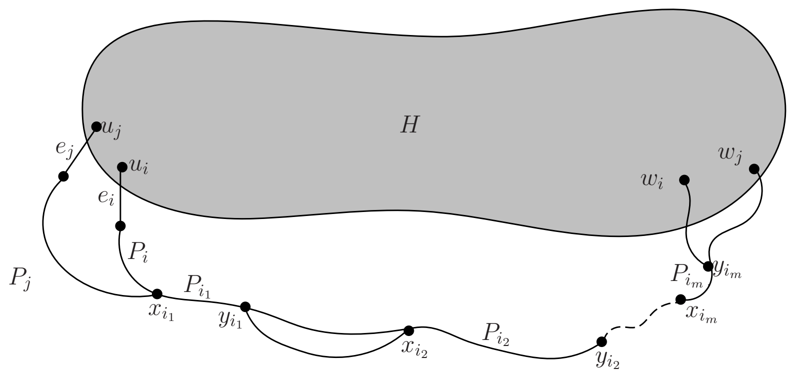

Fig. 1: Two H-ears P_i and P_j

construct an optimal (H, e_j)-ear P_j^* such that P_i and P_j^* has only one continuous common segment. Let $P_{i_1}, P_{i_2}, \ldots, P_{i_m}$ be the m continuous common segments of P_i and P_j and they appear in P_i in that order. See Figure 1 for details. Furthermore, suppose that x_{i_k} and y_{i_k} are the two ends of the path P_{i_k} and they appear in P_i successively. We say that the following claim holds.

Claim 1: $\ell(y_kP_ix_{k+1}) = \ell(y_kP_jx_{k+1})$ *for any* $1 \leq k \leq m - 1$.

If not, that is, there exists an integer k such that $\ell(y_kP_ix_{k+1}) \neq \ell(y_kP_jx_{k+1})$. Without loss of generality, we assume $\ell(y_kP_ix_{k+1}) < \ell(y_kP_jx_{k+1})$. Then we shall get a more shorter path H-ear containing e_j by replacing $y_kP_jx_{k+1}$ with $y_kP_ix_{k+1}$, a contradiction. Thus $\ell(y_kP_ix_{k+1}) = \ell(y_kP_jx_{k+1})$ for any k.

Let P_j^* be the path obtained from P_j by replacing $y_kP_jx_{k+1}$ with $y_kP_ix_{k+1}$, and let $P_j = P_j^*$. If the continuous common segment of P_i and P_j does not contain w_j. Suppose x and y are the two ends of the common segment such that x and y appeared on P starting from u_i to w_i successively. Similar to Claim 1, $\ell(yP_iw_i) = \ell(yP_jw_j)$. Let P_j^* be the path obtained from P_j by replacing yP_jw_j with yP_iw_i. Clearly, P_j^* is our desired optimal (H, u_jv_j)-ear. □

Lemma 2 *For every bridgeless graph G, $\eta(G) \leq \zeta(G)$.*

Proof: Suppose that there exists an edge e such that the length $\ell(C)$ of the smallest cycle C containing e is larger than $\zeta(G)$. Then, C is not an isometric cycle since the length of a largest isometric cycle is

$\zeta(G)$. Thus there exist two vertices u and v on C such that $d_G(u,v) < d_C(u,v)$. Let P be a shortest path between u and v in G. Then a closed trial C' containing e is obtained from the segment of C containing e between u and v by adding P. Clearly, the length $\ell(C')$ is less than $\ell(C)$. We can get a cycle C'' containing e from C'. Thus there exists a cycle C'' containing e with length less than $\ell(C)$, a contradiction. Therefore $\eta(G) \leq \zeta(G)$. □

Lemma 3 *Let G be a bridgeless graph and u be a center of G. For any $i \leq rad(G) - 1$ and every leg e of $N_i(u)$, there exists an optimal $(N_i[u], e)$-ear with length at most* $\min\{2(rad(G) - i) + 1, \eta(G)\}$.

Proof: Let P be an optimal $(N_i[u], e)$-ear. Since e belongs to a cycle with length at most $\eta(G)$, $\ell(P) \leq \eta(G)$. On the other hand, if $\ell(P) \geq 2(rad(G) - i) + 1$, then the middle vertex of P has distance at least $rad(G) - i + 1$ from $N_i[u]$, a contradiction. □

3 Oriented diameter

At first, we have the following observation.

Observation 1 *Let G be a bridgeless graph and H be a bridgeless spanning subgraph of G. Then the oriented radius (resp. diameter) of G is not larger than the oriented radius (resp. diameter) of H.*

Proof of Theorem 2: We only need to show that G has an orientation H such that

$$rad(H) \leq \sum_{i=1}^{rad(G)} \min\{2i, \eta(G) - 1\} \leq rad(G)(\eta(G) - 1).$$

Let u be a center of G and let H_0 be the trivial graph with vertex set $\{u\}$. We assert that *there exists a subgraph G_i of G such that $N_i[u] \subseteq V(G_i)$ and G_i has an orientation H_i satisfying that $rad(H_i) \leq ecc_{H_i}(u) \leq \Sigma_{j=1}^{i} \min\{2(rad(G) - j), \eta(G) - 1\}$.*

Basic step: When $i = 1$, we omit the proof since the proof of this step is similar to that of the following induction step.

Induction step: Assume that the above assertion holds for $i - 1$. Next we show that the above assertion also holds for i. For any $v \in N_i(u)$, either $v \in V(H_{i-1})$ or $v \in N_G(V(H_{i-1}))$ since $N_{i-1}[u] \subseteq V(H_{i-1})$. If $N_i(u) \subseteq V(H_{i-1})$, then let $H_i = H_{i-1}$ and we are done. Thus, we suppose $N_i(u) \not\subseteq V(H_{i-1})$ in the following.

Let $X = N_i(u) \setminus V(H_{i-1})$. Pick $x_1 \in X$, let y_1 be a neighbor of x_1 in H_{i-1} and let $P_1 = P_{y_1 z_1}$ be an optimal $(H_{i-1}, x_1 y_1)$-ear. We orient P such that P_1 is a directed path. Pick $x_2 \in X$ satisfying that all incident edges of x_2 are not oriented. Let y_2 be a neighbor of x_2 in H_{i-1}. If there exists an optimal $(H_{i-1}, x_2 y_2)$-ear P_2 such that P_1 and P_2 are independent, then we can orient P_2 such that P_2 is a directed path. Otherwise, by Lemma 1 there exists an optimal $(H_{i-1}, x_2 y_2)$-ear $P_2 = P_{y_2 z_2}$ such that P_1 and P_2 has only one continuous common segment containing z_2. Clearly, we can orient the edges in $E(P_2) \setminus E(P_1)$ such that P_2 is a directed path. We can pick the vertices of X and oriented optimal H-ears similar to the above method until that for any $x \in X$, at least two incident edges of x are oriented. Let H_i be the graph obtained from H_{i-1} by adding vertices in $V(G) \setminus V(H_{i-1})$, which has at least two new oriented incident edges, and adding the new oriented edges. Clearly, $N_i[u] \subseteq V(H_i) = V(G_i)$.

Now we show that $rad(H_i) \leq \Sigma_{j=1}^{i} \min\{2(rad(G)-i), \eta(G)-1\}$. It suffices to show that for every vertex x of H_i, $d_{H_i}(H_{i-1}, x) \leq \min\{2(rad(G)-i), \eta(G)-1\}$ and $d_{H_i}(x, H_{i-1}) \leq \min\{2(rad(G)-i), \eta(G)-1\}$. If $x \in V(H_{i-1})$, then the assertion holds by inductive hypothesis. If $x \notin V(H_{i-1})$. Let P be a directed optimal (H_i, e)-ear containing x, where e is some leg of H_{i-1} (such a leg and such an ear exists by the definition of H_i. By Lemma 3, $\ell(P) \leq \min\{2(rad(G)-i)+1, \eta(G)\}$. Thus, $d_{H_i}(x, H_{i-1}) \leq \min\{2(rad(G)-i), \eta(G)-1\}$ and $d_{H_i}(H_{i-1}, x) \leq \min\{2(rad(G)-i), \eta(G)-1\}$. Therefore, $rad(H_i) \leq \Sigma_{j=1}^{i} \min\{2(rad(G)-j), \eta(G)-1\}$. □

Remark 1 *The above theorem is optimal since it implies Chvátal and Thomassen's optimal Theorem 1. Readers can see [5, 11] for optimal examples.*

The following example shows that our result is better than that of Theorem 1.

Example 1 Let F_3 be a triangle with one of its vertices designated as a root. In order to construct F_r, take two copies of F_{r-1}. Let H_r be the graph obtained from the triangle u_0, u_1, u_2 by identifying the root of first (resp. second) copy of F_{r-1} with u_1 (resp. u_2), and u_0 be the root of F_r. Let G_r be the graph obtained by taking two copies of F_r and identifying their roots. See Figure 2 for details. It is easy to check that G_r has radius r and every edge belongs to a cycle of length $\eta(G) = 3$. By Theorem 1, G_r has an orientation H_r such that $rad(H_r) \leq r^2 + r$ and $diam(H_r) \leq 2r^2 + 2r$. But, by Theorem 2, G_r has an orientation H_r such that $rad(G) \leq 2r$ and $diam(G) \leq 4r$. On the other hand, it is easy to check that all the strong orientations of G_r has radius $2r$ and diameter $4r$.

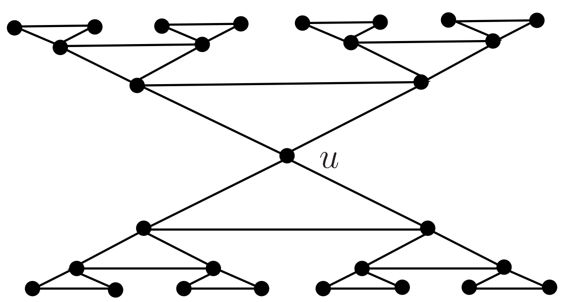

Fig. 2: The graph G_3 which has oriented radius 6 and oriented diameter 12

We have the following result for plane graphs.

Theorem 5 *Let G be a plane graph. If the length of the boundary of every face is at most k, then G has an oriented H such that $rad(H) \leq rad(G)(k-1)$ and $diam(H) \leq 2rad(G)(k-1)$.*

Since every edge of a maximal plane (resp. outerplane) graph belongs to a cycle with length 3, the following corollary holds.

Corollary 1 *Let G be a maximal plane (resp. outerplane) graph. Then there exists an orientation H of G such that $rad(H) \leq 2rad(G)$ and $rad(H) \leq 4rad(G)$.*

A graph G is $edge\text{-}transitive$ if for any $e_1, e_2 \in E(G)$, there exists an automorphism g such that $g(e_1) = e_2$. We have the following result for edge-transitive graphs.

Theorem 6 *Let G be a bridgeless edge-transitive graph. Then G has an orientation H such that $rad(H) \leq rad(G)(g(G)-1)$ and $diam(H) \leq 2rad(G)(g(G)-1)$, where $g(G)$ is the girth of G, that is, the length of a smallest induced cycle.*

For general bipartite graphs, the following theorem holds.

Theorem 7 *Let $G = (V_1 \cup V_2, E)$ be a bipartite graph with $|V_1| = n$ and $|V_2| = m$. If $d(x) \geq k > \lceil m/2 \rceil$ for any $x \in V_1$, $d(y) \geq r > \lceil n/2 \rceil$ for any $y \in V_2$, then there exists an orientation H of G such that $rad(H) \leq 9$.*

Proof: It suffices to show that $rad(G) \leq 3$ and $\eta(G) \leq 4$ by Theorem 2.

First, we show that $rad(G) \leq 3$. Fix a vertex x in G, and let y be any vertex different from x in G. If x and y belong to the same part, without loss of generality, say $x, y \in V_1$. Let X and Y be neighborhoods of x and y in V_2, respectively. If $X \cap Y = \emptyset$, then $|V_2| \geq |X| + |Y| \geq 2k > m$, a contradiction. Thus $X \cap Y \neq \emptyset$, that is, there exists a path between x and y of length two. If x and y belong to different parts, without loss of generality, say $x \in V_1, y \in V_2$. Suppose x and y are nonadjacent, otherwise there is nothing to prove. Let X and Y be the neighborhoods of x and y in G, and let X' be the set of neighbors of X except for x in G. If $X' \cap Y = \emptyset$, then $|V_1| \geq 1 + |Y| + |X'| \geq 1 + r + (r-1) = 2r > n$, a contradiction (Note that $|X'| \geq r - 1$). Thus $X' \cap Y \neq \emptyset$, that is, there exists a path between x and y of length three in G.

Next we show that $\eta(G) \leq 4$. Let xy be any edge in G. Let X be the set of neighbors of x except for y in G, let Y be the set of neighbors of y except for x in G, let X' be the set of neighbors of X except for x in G. If $X' \cap Y = \emptyset$, then $|V_1| \geq 1 + |Y| + |X'| \geq 1 + (r-1) + (r-1) = 2r - 1 > n$, a contradiction (Note that $|X'| \geq r - 1$). Thus $X' \cap Y \neq \emptyset$, that is, there exists a cycle containing xy of length four in G. □

Remark 2 *The degree condition is optimal. Let m, n be two even numbers with $n, m \geq 2$. Since $K_{n/2,m/2} \cup K_{n/2,m/2}$ is disconnected, the oriented radius (resp. diameter) of $K_{n/2,m/2} \cup K_{n/2,m/2}$ is ∞.*

For equal bipartition k-regular graph, the following corollary holds.

Corollary 2 *Let $G = (V_1 \cup V_2, E)$ be a k-regular bipartite graph with $|V_1| = |V_2| = n$. If $k > n/2$, then there exists an orientation H of G such that $rad(H) \leq 9$.*

The following theorem holds for general graphs.

Theorem 8 *Let G be a graph of order n.*

(i) If there exists an integer $k \geq 2$ such that $|N_k(u)| > n/2 - 1$ for every vertex u in G, then G has an orientation H such that $rad(H) \leq 4k^2$ and $diam(H) \leq 8k^2$.

(ii) If $\delta(G) > n/2$, then G has an orientation H such that $rad(H) \leq 4$ and $diam(H) \leq 8$.

Proof: Since methods of the proofs of (i) and (ii) are similar, we only prove (i). For (i), it suffices to show that $rad(G) \leq 2k$ and $\eta(G) \leq 2k + 1$ by Theorem 2.

We first show $rad(G) \leq 2k$. Fix u in G, for every $v \in V(G)$, if $v \in N_k[u]$, then $d(u, v) \leq k$. Suppose $v \notin N_k[u]$, we have $N_k(u) \cap N_k(v) \neq \emptyset$. If not, that is, $N_k(u) \cap N_k(v) = \emptyset$, then $|N_k(u)| + |N_k(v)| + 2 > n$ (a contradiction). Thus $d(u, v) \leq 2k$.

Next we show $\eta(G) \leq 2k+1$. Let $e = uv$ be any edge in G. If $N_k(u) \cap N_k(v) = \emptyset$, then $|V(G)| \geq |N_k(u)| + |N_k(v)| + 2 > n$, a contradiction. Thus $N_k(u) \cap N_k(v) \neq \emptyset$. Pick $w \in N_k(u) \cap N_k(v)$, and let P (resp. Q) be a path between u and w (resp. between v and w). Then e belongs a close trial $uPwQvu$ of length $2k+1$. Therefore, e belongs a cycle of length at most $2k+1$. $\square$

Remark 3 *The above condition is almost optimal since $K_{n/2} \cup K_{n/2}$ is disconnected for even n.*

Corollary 3 *Let G be a graph with minimum degree $\delta(G)$ and girth $g(G)$. If there exists an integer k such that $k \leq g(G)/2$ and $\delta(G)(\delta(G)-1)^{k-1} > n/2 - 1$, then G has an orientation H such that $rad(H) \leq 4k^2$.*

Proof: Let k be an integer such that $k \leq g(G)/2$ and $\delta(G)(\delta(G)-1)^{k-1} > n/2 - 1$. For any vertex u of G, let $1 \leq i < k$ be any integer and $x, y \in N_i(u)$. If x and y have a common neighbor z in $N_{i+1}(u)$, then G has a cycle of length at most $2i < 2k \leq g(G)/2$, a contradiction. Thus x and y has no common neighbor in $N_{i+1}(u)$. Therefore, $|N_k(u)| \geq \delta(G)(\delta(G)-1)^{k-1} > n/2 - 1$. By Theorem 2, G has an orientation H such that $rad(H) \leq 4k^2$. $\square$

4 Upper bound for rainbow connection number

At first, we have the following observation.

Observation 2 *Let G be a graph and H be a spanning subgraph of G. Then $rc(G) \leq rc(H)$.*

Proof of Theorem 4: Let u be a center of G and let H_0 be the trivial graph with vertex set $\{u\}$. We assert that *there exists a subgraph H_i of G such that $N_i[u] \subseteq V(H_i)$ and $rc(H_i) \leq \Sigma_{j=1}^{i} \min\{2(rad(G) - j) + 1, \eta(G)\}$.*

Basic step: When $i = 1$, we omit the proof since the proof of this step is similar to that of the following induction step.

Induction step: Assume that the above assertion holds for $i-1$ and c is a $rc(H_{i-1})$-rainbow coloring of H_{i-1}. Next we show that the above assertion holds for i. For any $v \in N_i(u)$, either $v \in V(H_{i-1})$ or $v \in N_G(V(H_{i-1})$ since $N_{i-1}[u] \subseteq V(H_{i-1})$. If $N_i(u) \subseteq V(H_{i-1})$, then let $H_i = H_{i-1}$ and we are done. Thus, we suppose $N_i(u) \nsubseteq V(H_{i-1})$ in the following.

Let $C_1 = \{\alpha_1, \alpha_2, \cdots\}$ and $C_2 = \{\beta_1, \beta_2, \cdots\}$ be two pools of colors, none of which are used to color H_{i-1}, and there exists no common color in C_1 and C_2. An edge-coloring of an H-ear $P = (u_0, u_1, \cdots, u_k)$ is a $symmetrical\ coloring$ if its edges are colored by $\alpha_1, \alpha_2, \cdots, \alpha_{\lceil k/2 \rceil}, \beta_{\lfloor k/2 \rfloor}, \cdots, \beta_2, \beta_1$ in that order or $\beta_1, \beta_2, \cdots, \beta_{\lfloor k/2 \rfloor}, \alpha_{\lceil k/2 \rceil} \cdots, \alpha_2, \alpha_1$ in that order.

Let $X = N_i(u) \setminus V(H_{i-1})$ and $m = \min\{2(rad(G) - i) + 1, \eta(G)\}$. Pick $x_1 \in X$, Let y_1 be a neighbor of x_1 in H_{i-1} and P_1 be an optimal (H_{i-1}, x_1y_1)-ear. We can color P symmetrically with colors α_1, α_2, $\cdots$, $\alpha_{\lceil \ell(P)/2 \rceil}$, $\beta_{\lfloor \ell(P)/2 \rfloor}$, $\cdots$, β_2, β_1. Pick $x_2 \in X$ satisfying that all the incident edges of x_2 are not colored. Let y_2 be a neighbor of x_2 in H_{i-1}. If there exists an optimal (H_{i-1}, x_2y_2)-ear P_2 such that P_1 and P_2 are independent, then we can color P_2 symmetrically with colors $\alpha_1, \alpha_2, \cdots, \alpha_{\lceil \ell(P_2)/2 \rceil}, \beta_{\lfloor \ell(P_2)/2 \rfloor}, \ldots, \beta_2, \beta_1$. Otherwise, by Lemma 1, there exists an optimal (H_{i-1}, x_2y_2)-ear $P_2 = P_{y_2z_2}$ such that P_1 and P_2 have only one continuous common segment containing z_2, where z_2 is the other foot of P_2. Thus we can color P_2 symmetrically with colors

$\alpha_1, \alpha_2, \cdots, \alpha_{\lceil \ell(P_2)/2 \rceil}$, $\beta_{\lfloor \ell(P_2)/2 \rfloor}, \ldots, \beta_2, \beta_1$ by preserving the coloring of P_1. We can pick the vertices of X and color optimal H_i-ears until that for any $x \in X$, at least two incident edges of x are colored. Since for any leg e of H_{i-1}, $\ell(e) \leq m$ by Lemma 3, we use at most m coloring in the above coloring process.

Let H_i be the graph obtained from H_{i-1} by adding all the vertices in $V(G) \setminus V(H_{i-1})$, which have at least two new colored incident edges, and adding the new colored edges. Clearly, $N_i[u] \subseteq V(H_i)$. It is suffices to show that H_i is rainbow connected. Let x and y be two distinct vertices in H_i. If $x, y \in V(H_{i-1})$, then there exists a rainbow path between x and y by inductive hypothesis. If exactly one of x and y belongs to $V(H_{i-1})$, say x. Let P be a symmetrical colored H_{i-1}-ear containing y, and y' be a foot of P. There exists a rainbow path Q between x and y' in H_{i-1} by inductive hypothesis. Thus, $xQy'Py$ is a rainbow path between x and y in H_i.

Suppose none of x and y belongs to H_{i-1}. Let P and Q be symmetrical colored H_{i-1}-ear containing x and y, respectively. Furthermore, let x', x'' be the feet of P and y', y'' be the feet of Q. Without loss of generality, assume that P is colored from x' to x'' by $\alpha_1, \alpha_2, \cdots, \alpha_{\lceil \ell(P)/2 \rceil}, \beta_{\lfloor \ell(P)/2 \rfloor}, \ldots, \beta_2, \beta_1$ in that order, and Q is colored from y' to y'' by $\alpha_1, \alpha_2, \cdots, \alpha_{\lceil \ell(Q)/2 \rceil}, \beta_{\lfloor \ell(Q)/2 \rfloor}, \ldots, \beta_2, \beta_1$ in that order. If $\ell(x'Px) \leq \ell(y'Qy)$, let R be a rainbow path between x' and y'' in H_{i-1}. Then $xPx'Ry''Qy$ is a rainbow path between x and y in H_i. Otherwise, $\ell(x'Px) > \ell(y'Qy)$. Let R be a rainbow path between y' and x'' in H_{i-1}. Then $yQy'Rx''Px$ is a rainbow path between x and y in H_i. Thus, there exists a rainbow path between any two distinct vertices in H_i, that is, H_i is $(\Sigma_{j=1}^{i} \min\{2(rad(G) - j) + 1, \eta(G)\})$-rainbow connected. □ Readers can see [1] for an optimal example. The following example shows that our result is better than that of Theorem 3.

Example 2 Let $r \geq 3, k \geq 2r$ be two integers, and $W_k = C_k \vee K_1$ be an wheel, where $V(C_k) = \{u_1, u_2, \ldots, u_k\}$ and $V(K_1) = \{u\}$. Let H be the graph obtained from W_k by inserting $r - 1$ vertices between every edge uu_i, $1 \leq i \leq k$. For every edge $e = xy$ of H, add a new vertex v_e and new edges $v_e x, v_e y$. Denote by G the resulting graph. It is easy to check that $rad(G) = r$, $diam(G) = 2r$, $\eta(G) = 3$ and $\zeta(G) = 2r - 1$. By Theorem 3, we have $rc(G) \leq \sum_{i=1}^{r} \min\{2i + 1, \zeta(G)\} \leq r^2 + 2r - 2$. But, by Theorem 4 we have $rc(G) \leq 3r$. On the other hand, $rc(G) \geq 2r$ since $diam(G) = 2r$.

The remaining results are similar to those in Section 3.

Theorem 9 *Let G be a plane graph. If the length of the boundary of every face is at most k, then $rc(G) \leq k\, rad(G)$.*

Corollary 4 *Let G be a maximal plane (resp. outerplane) graph. Then $rc(G) \leq 3rad(G)$.*

Theorem 10 *Let G be a bridgeless edge-transitive graph. Then $rc(G) \leq rad(G)g(G)$, where $g(G)$ is the girth of G.*

Theorem 11 *Let $G = (V_1 \cup V_2, E)$ be a bipartite graph with $|V_1| = n$ and $|V_2| = m$. If $d(x) \geq k > \lceil m/2 \rceil$ for any $x \in V_1$, $d(y) \geq r > \lceil n/2 \rceil$ for any $y \in V_2$, then $rc(G) \leq 12$.*

Remark 4 *The degree condition is optimal. Let m, n be two even numbers with $n, m \geq 2$. Since $K_{n/2,m/2} \cup K_{n/2,m/2}$ is disconnected, $rc(K_{n/2,m/2} \cup K_{n/2,m/2}) = \infty$.*

Corollary 5 *Let $G = (V_1 \cup V_2, E)$ be a k-regular bipartite graph with $|V_1| = |V_2| = n$. If $k > \lceil n/2 \rceil$, then $rc(G) \leq 12$.*

The following theorem holds for general graphs.

Theorem 12 *Let G be a graph.*

(i) *If there exists an integer $k \geq 2$ such that $|N_k(u)| > n/2 - 1$ for every vertex u in G, then $rc(G) \leq 4k^2 + 2k$.*

(ii) *If $\delta(G) > n/2$, then $rc(G) \leq 6$.*

Remark 5 *The above condition is almost optimal since $K_{n/2} \cup K_{n/2}$ is disconnected for even n.*

Corollary 6 *Let G be a graph with minimum degree $\delta(G)$ and girth $g(G)$. If there exists an integer k such that $k < g(G)/2$ and $\delta(G)(\delta(G) - 1)^{k-1} > n/2 - 1$, then then $rc(G) \leq 4k^2 + 2k$.*

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