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Oriented diameter and rainbow connection number of a graph

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The oriented diameter of a bridgeless graph $G$ is $\min\{\text{diam}(H) \mid H \text{ is an orientation of } G\}$. A path in an edge-colored graph $G$, where adjacent edges may have the same color, is called rainbow if no two edges of the path are colored the same. The rainbow connection number $rc(G)$ of $G$ is the smallest integer number $k$ for which there exists a $k$-edge-coloring of $G$ such that every two distinct vertices of $G$ are connected by a rainbow path. In this paper, we obtain upper bounds for the oriented diameter and the rainbow connection number of a graph in terms of $\text{rad}(G)$ and $\eta(G)$, where $\text{rad}(G)$ is the radius of $G$ and $\eta(G)$ is the smallest integer number such that every edge of $G$ is contained in a cycle of length at most $\eta(G)$. We also obtain constant bounds of the oriented diameter and the rainbow connection number for a (bipartite) graph $G$ in terms of the minimum degree of $G$.

Keywords: Diameter, Radius, Oriented diameter, Rainbow connection number, Cycle length, Bipartite graph.

1 Introduction

All graphs in this paper are undirected, finite and simple. We refer to the book [2] for notation and terminology not described here. A path $u = u_1, u_2, \ldots, u_k = v$ is called a $P_{u,v}$ path. Denote by $u_i Pu_j$ the subpath $u_i, u_{i+1}, \ldots, u_j$ for $i \leq j$. The length $\ell(P)$ of a path $P$ is the number of edges in $P$. The distance between two vertices $x$ and $y$ in $G$, denoted by $d_G(x,y)$, is the length of a shortest path between them. The eccentricity of a vertex $x$ in $G$ is $\text{ecc}_G(x) = \max_{y \in V(G)} d(x,y)$. The radius and diameter of $G$ are $\text{rad}(G) = \min_{x \in V(G)} \text{ecc}(x)$ and $\text{diam}(G) = \max_{x \in V(G)} \text{ecc}(x)$, respectively. A vertex $u$ is a center of a graph $G$ if $\text{ecc}(u) = \text{rad}(G)$. The oriented diameter of a bridgeless graph $G$ is $\min \{\text{diam}(H) \mid H \text{ is an orientation of } G\}$, and the oriented radius of a bridgeless graph $G$ is $\min \{\text{rad}(H) \mid H \text{ is an orientation of } G\}$. For any graph $G$ with edge-connectivity $\lambda(G) = 0, 1$, $G$ has oriented radius (resp. diameter) $\infty$.

In 1939, Robbins solved the One-Way Street Problem and proved that a graph $G$ admits a strongly connected orientation if and only if $G$ is bridgeless, that is, $G$ does not have any cut-edge. Naturally, one hopes that the oriented diameter of a bridgeless graph is as small as possible. Bondy and Murty suggested

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to study the quantitative variations on Robbins’ theorem. In particular, they conjectured that there exists a function \( f \) such that every bridgeless graph with diameter \( d \) admits an orientation of diameter at most \( f(d) \).


**Theorem 1 (Chvátal and Thomassen 1978 [5])** For every bridgeless graph \( G \), there exists an orientation \( H \) of \( G \) such that

\[
\text{rad}(H) \leq \text{rad}(G)^2 + \text{rad}(G),
\]

\[
\text{diam}(H) \leq 2\text{rad}(G)^2 + 2\text{rad}(G).
\]

Moreover, the above bounds are optimal.

There exists a minor error when they constructed the graph \( G_d \) which arrives at the upper bound when \( d \) is odd. Kwok, Liu and West gave a slight correction in [11]. They also showed that determining whether an arbitrary graph can be oriented so that its diameter is at most 2 is NP-complete. Bounds for the oriented diameter of graphs have also been studied in terms of other parameters, for example, radius, dominating number [5, 6, 11, 18], etc. Some classes of graphs have also been studied in [6, 7, 8, 9, 14].

Let \( \eta(G) \) be the smallest integer such that every edge of \( G \) belongs to a cycle of length at most \( \eta(G) \).

In this paper, we show the following result.

**Theorem 2** For every bridgeless graph \( G \), there exists an orientation \( H \) of \( G \) such that

\[
\text{rad}(H) \leq \sum_{i=1}^{\text{rad}(G)} \min\{2i, \eta(G) - 1\} \leq \text{rad}(G)(\eta(G) - 1),
\]

\[
\text{diam}(H) \leq 2\sum_{i=1}^{\text{rad}(G)} \min\{2i, \eta(G) - 1\} \leq 2\text{rad}(G)(\eta(G) - 1).
\]

Note that \( \sum_{i=1}^{\text{rad}(G)} \min\{2i, \eta(G) - 1\} \leq \text{rad}(G)^2 + \text{rad}(G) \) and \( \text{diam}(H) \leq 2\text{rad}(H) \). So our result implies Chvátal and Thomassen’s Theorem 1.

A path in an edge-colored graph \( G \), where adjacent edges may have the same color, is called rainbow if no two edges of the path are colored the same. An edge-coloring of a graph \( G \) is a rainbow edge-coloring if every two distinct vertices of the graph \( G \) are connected by a rainbow path. The rainbow connection number \( rc(G) \) of \( G \) is the minimum integer \( k \) for which there exists a rainbow \( k \)-edge-coloring of \( G \). It is easy to see that \( \text{diam}(G) \leq rc(G) \) for any connected graph \( G \). The rainbow connection number was introduced by Chartrand et al. in [4]. It is of great use in transferring information of high security in multicomputer networks. We refer the readers to [3] for details.

Chakraborty et al. [3] investigated the hardness and algorithms for the rainbow connection number, and showed that given a graph \( G \), deciding if \( rc(G) = 2 \) is \( NP \)-complete. Bounds for the rainbow connection number of a graph have also been studied in terms of other graph parameters, for example, radius, dominating number, minimum degree, connectivity, etc. [11, 4, 10]. Cayley graphs and line graphs were studied in [12] and [13], respectively.

A subgraph \( H \) of a graph \( G \) is called isometric if the distance between any two distinct vertices in \( H \) is the same as their distance in \( G \). The size of a largest isometric cycle in \( G \) is denoted by \( \zeta(G) \).
Clearly, every isometric cycle is an induced cycle and thus \( \zeta(G) \) is not larger than the chordality, where chordality is the length of a largest induced cycle in \( G \). In [1], Basavaraju, Chandran, Rajendraprasad and Ramaswamy got the following sharp upper bound for the rainbow connection number of a bridgeless graph \( G \) in terms of \( \text{rad}(G) \) and \( \zeta(G) \).

**Theorem 3 (Basavaraju et al. [1])** For every bridgeless graph \( G \),

\[
\text{rc}(G) \leq \sum_{i=1}^{\text{rad}(G)} \min\{2i + 1, \zeta(G)\} \leq \text{rad}(G)\zeta(G).
\]

In this paper, we show the following result.

**Theorem 4** For every bridgeless graph \( G \),

\[
\text{rc}(G) \leq \sum_{i=1}^{\text{rad}(G)} \min\{2i + 1, \eta(G)\} \leq \text{rad}(G)\eta(G).
\]

From Lemma 2 of Section 2, we will see that \( \eta(G) \leq \zeta(G) \). Thus our result implies Theorem 3.

This paper is organized as follows: in Section 2, we introduce some definitions and show several lemmas. In Section 3, we prove Theorem 2 and study upper bounds for the oriented radius (resp. diameter) of plane graphs, edge-transitive graphs and general (bipartite) graphs. In Section 4, we prove Theorem 4 and study upper for the rainbow connection number of plane graphs, edge-transitive graphs and general (bipartite) graphs.

## 2 Preliminaries

In this section, we introduce some definitions and show several lemmas.

**Definition 1** For any \( x \in V(G) \) and \( k \geq 0 \), the \( k \)-step open neighborhood is \( \{y \mid d(x, y) = k\} \) and denoted by \( N_k(x) \), the \( k \)-step closed neighborhood is \( \{y \mid d(x, y) \leq k\} \) and denoted by \( N_k[x] \). If \( k = 1 \), we simply write \( N(x) \) and \( N[x] \) for \( N_1(x) \) and \( N_1[x] \), respectively.

**Definition 2** Let \( G \) be a graph and \( H \) be a subset of \( V(G) \) (or a subgraph of \( G \)). The edges between \( H \) and \( G \setminus H \) are called legs of \( H \). An \( H \)-ear is a path \( P = (u_0, u_1, \ldots, u_k) \) in \( G \) such that \( V(H) \cap V(P) = \{u_0, u_k\} \). The vertices \( u_0, u_k \) are called the feet of \( P \) in \( H \) and \( u_0u_1, u_{k-1}u_k \) are called the legs of \( P \). The length of an \( H \)-ear is the length of the corresponding path. If \( u_0 = u_k \), then \( P \) is called a closed \( H \)-ear. For any leg \( e \) of \( H \), denote by \( \ell(e) \) the smallest number such that there exists an \( H \)-ear of length \( \ell(e) \) containing \( e \), and such an \( H \)-ear is called an optimal \((H, e)\)-ear.

Note that for any optimal \((H, e)\)-ear \( P \) and every pair \((x, y) \neq (u_0, u_k)\) of distinct vertices of \( P \), \( x \) and \( y \) are adjacent on \( P \) if and only if \( x \) and \( y \) are adjacent in \( G \).

**Definition 3** For any two paths \( P \) and \( Q \), the joint of \( P \) and \( Q \) are the common vertex and edge of \( P \) and \( Q \). Paths \( P \) and \( Q \) have \( k \) continuous common segments if the common vertex and edge are \( k \) disjoint paths.

A common segment is trivial if it has only one vertex.
Definition 4 Let $P$ and $Q$ be two paths in $G$. Call $P$ and $Q$ independent if they have no common internal vertex.

Lemma 1 Let $n \geq 1$ be an integer, and let $G$ be a graph, $H$ be a subgraph of $G$ and $e_i = u_i, v_i$ be a leg of $H$ and $P_i = P_{u_i, w_i}$ be an optimal $(G, e_i)$-ear, where $1 \leq i \leq n$ and $u_i, v_i$ are the foot of $P_i$. Then for any leg $e_j = u_j, v_j$ such that $e_j \neq e_i$ and $e_j \not\in E(P_i)$, where $i \in \{1, 2, \ldots, n\}$, there exists an optimal $(H, e_j)$-ear $P_j = P_{u_j, w_j}$ such that either $P_i$ and $P_j$ are independent for any $P_i, 1 \leq i \leq n$, or $P_i$ and $P_j$ have only one continuous common segment containing $w_j$ for some $P_i$.

Proof: Let $P_j$ be an optimal $(H, e_j)$-ear. If $P_i$ and $P_j$ are independent for any $i$, then we are done. Suppose that $P_i$ and $P_j$ have $m$ continuous common segments for some $i$, where $m \geq 1$. When $m \geq 2$, we first construct an optimal $(H, e_j)$-ear $P_j^*$ such that $P_i$ and $P_j^*$ has only one continuous common segment. Let $P_{i_1}, P_{i_2}, \ldots, P_{i_m}$ be the $m$ continuous common segments of $P_i$ and $P_j$ and they appear in $P_i$ in that order. See Figure 1 for details. Furthermore, suppose that $x_{i_k}$ and $y_{i_k}$ are the two ends of the path $P_{i_k}$ and they appear in $P_i$ successively. We say that the following claim holds.

Claim 1: $\ell(y_k P_i x_{k+1}) = \ell(y_k P_j x_{k+1})$ for any $1 \leq k \leq m - 1$.

If not, that is, there exists an integer $k$ such that $\ell(y_k P_i x_{k+1}) \neq \ell(y_k P_j x_{k+1})$. Without loss of generality, we assume $\ell(y_k P_i x_{k+1}) < \ell(y_k P_j x_{k+1})$. Then we shall get a more shorter path $H$-ear containing $e_j$ by replacing $y_k P_i x_{k+1}$ with $y_k P_j x_{k+1}$, a contradiction. Thus $\ell(y_k P_i x_{k+1}) = \ell(y_k P_j x_{k+1})$ for any $k$.

Let $P_j^*$ be the path obtained from $P_j$ by replacing $y_k P_j x_{k+1}$ with $y_k P_i x_{k+1}$, and let $P_j^* = P_j^*$ if the continuous common segment of $P_i$ and $P_j$ does not contain $w_j$. Suppose $x$ and $y$ are the two ends of the common segment such that $x$ and $y$ appeared on $P$ starting from $u_i$ to $w_i$ successively. Similar to Claim 1, $\ell(y P_j w_i) = \ell(y P_i w_i)$. Let $P_j^*$ be the path obtained from $P_j$ by replacing $y P_j w_j$ with $y P_i w_i$. Clearly, $P_j^*$ is our desired optimal $(H, u_j v_j)$-ear.

Lemma 2 For every bridgeless graph $G$, $\eta(G) \leq \zeta(G)$.

Proof: Suppose that there exists an edge $e$ such that the length $\ell(C)$ of the smallest cycle $C$ containing $e$ is larger than $\zeta(G)$. Then, $C$ is not an isometric cycle since the length of a largest isometric cycle is
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Thus there exist two vertices $u$ and $v$ on $C$ such that $d_G(u, v) < d_C(u, v)$. Let $P$ be a shortest path between $u$ and $v$ in $G$. Then a closed trial $C'$ containing $e$ is obtained from the segment of $C$ containing $e$ between $u$ and $v$ by adding $P$. Clearly, the length $\ell(C')$ is less than $\ell(C)$. We can get a cycle $C''$ containing $e$ from $C'$. Thus there exists a cycle $C''$ containing $e$ with length less than $\ell(C)$, a contradiction. Therefore $\eta(G) \leq \zeta(G)$.

Lemma 3 Let $G$ be a bridgeless graph and $u$ be a center of $G$. For any $i \leq \text{rad}(G) - 1$ and every leg $e$ of $N_i(u)$, there exists an optimal $(N_i[u], e)$-ear with length at most $\min\{2\text{rad}(G) - i + 1, \eta(G)\}$.

Proof: Let $P$ be an optimal $(N_i[u], e)$-ear. Since $e$ belongs to a cycle with length at most $\eta(G)$, $\ell(P) \leq \eta(G)$. On the other hand, if $\ell(P) \geq 2(\text{rad}(G) - i) + 1$, then the middle vertex of $P$ has distance at least $\text{rad}(G) - i + 1$ from $N_i[u]$, a contradiction.

3 Oriented diameter

At first, we have the following observation.

Observation 1 Let $G$ be a bridgeless graph and $H$ be a bridgeless spanning subgraph of $G$. Then the oriented radius (resp. diameter) of $G$ is not larger than the oriented radius (resp. diameter) of $H$.

Proof of Theorem 2 We only need to show that $G$ has an orientation $H$ such that

$$\text{rad}(H) \leq \sum_{i=1}^{\text{rad}(G)} \min\{2i, \eta(G) - 1\} \leq \text{rad}(G)(\eta(G) - 1).$$

Let $u$ be a center of $G$ and let $H_0$ be the trivial graph with vertex set $\{u\}$. We assert that there exists a subgraph $G_i$ of $G$ such that $N_i[u] \subseteq V(G_i)$ and $G_i$ has an orientation $H_i$ satisfying that $\text{rad}(H_i) \leq \text{ecc}_{H_i}(u) \leq \sum_{j=1}^{2(\text{rad}(G) - j)} \min\{2(\text{rad}(G) - j), \eta(G) - 1\}$.

Basic step: When $i = 1$, we omit the proof since the proof of this step is similar to that of the following induction step.

Induction step: Assume that the above assertion holds for $i - 1$. Next we show that the above assertion also holds for $i$. For any $v \in N_i(u)$, either $v \in V(H_{i-1})$ or $v \in N_G(V(H_{i-1}))$ since $N_{i-1}[u] \subseteq V(H_{i-1})$. If $N_i(u) \subseteq V(H_{i-1})$, then let $H_i = H_{i-1}$ and we are done. Thus, we suppose $N_i(u) \not\subseteq V(H_{i-1})$ in the following.

Let $X = N_i(u) \setminus V(H_{i-1})$. Pick $x_1 \in X$, let $y_1$ be a neighbor of $x_1$ in $H_{i-1}$ and let $P_1 = P_{y_1z_1}$ be an optimal $(H_{i-1}, x_1y_1)$-ear. We orient $P$ such that $P_1$ is a directed path. Pick $x_2 \in X$ satisfying that all incident edges of $x_2$ are not oriented. Let $y_2$ be a neighbor of $x_2$ in $H_{i-1}$. If there exists an optimal $(H_{i-1}, x_2y_2)$-ear $P_2$ such that $P_1$ and $P_2$ are independent, then we can orient $P_2$ such that $P_2$ is a directed path. Otherwise, by Lemma 3 there exists an optimal $(H_{i-1}, x_2y_2)$-ear $P_2 = P_{y_2z_2}$ such that $P_1$ and $P_2$ has only one continuous common segment containing $z_2$. Clearly, we can orient the edges in $E(P_2) \setminus E(P_1)$ such that $P_2$ is a directed path. We can pick the vertices of $X$ and oriented optimal $H$-ears similar to the above method until that for any $x \in X$, at least two incident edges of $x$ are oriented. Let $H_i$ be the graph obtained from $H_{i-1}$ by adding vertices in $V(G) \setminus V(H_{i-1})$, which has at least two new oriented incident edges, and adding the new oriented edges. Clearly, $N_i[u] \subseteq V(H_i) = V(G_i)$. 
Now we show that $\text{rad}(H_i) \leq \sum_{j=1}^{i} \min\{2(\text{rad}(G) - i), \eta(G) - 1\}$. It suffices to show that for every vertex $x$ of $H_i$, $d_{H_i}(H_{i-1}, x) \leq \min\{2(\text{rad}(G) - i), \eta(G) - 1\}$ and $d_{H_i}(x, H_{i-1}) \leq \min\{2(\text{rad}(G) - i), \eta(G) - 1\}$. If $x \in V(H_{i-1})$, then the assertion holds by inductive hypothesis. If $x \notin V(H_{i-1})$, let $P$ be a directed optimal $(H_i, e)$-ear containing $x$, where $e$ is some leg of $H_{i-1}$ (such a leg and such an ear exists by the definition of $H_i$). By Lemma 3, $\ell(P) \leq \min\{2(\text{rad}(G) - i) + 1, \eta(G)\}$. Thus, $d_{H_i}(x, H_{i-1}) \leq \min\{2(\text{rad}(G) - i), \eta(G) - 1\}$ and $d_{H_i}(H_{i-1}, x) \leq \min\{2(\text{rad}(G) - i), \eta(G) - 1\}$. Therefore, $\text{rad}(H_i) \leq \sum_{j=1}^{i} \min\{2(\text{rad}(G) - j), \eta(G) - 1\}$. \hfill \qed

**Remark 1** The above theorem is optimal since it implies Chvátal and Thomassen's optimal Theorem\cite{7}.

The following example shows that our result is better than that of Theorem\cite{1}.

**Example 1** Let $F_3$ be a triangle with one of its vertices designated as a root. In order to construct $F_r$, take two copies of $F_{r-1}$. Let $H_r$ be the graph obtained from the triangle $u_0, u_1, u_2$ by identifying the root of first (resp. second) copy of $F_{r-1}$ with $u_1$ (resp. $u_2$), and $u_0$ be the root of $F_r$. Let $G_r$ be the graph obtained by taking two copies of $F_r$ and identifying their roots. See Figure 2 for details. It is easy to check that $G_r$ has radius $r$ and every edge belongs to a cycle of length $\eta(G) = 3$. By Theorem\cite{1} $G_r$ has an orientation $H_r$ such that $\text{rad}(H_r) \leq r^2 + r$ and $\text{diam}(H_r) \leq 2r^2 + 2r$. But, by Theorem\cite{2} $G_r$ has an orientation $H_r$ such that $\text{rad}(G) \leq 2r$ and $\text{diam}(G) \leq 4r$. On the other hand, it is easy to check that all the strong orientations of $G_r$ has radius $2r$ and diameter $4r$.

![Fig. 2: The graph $G_3$ which has oriented radius 6 and oriented diameter 12](image-url)

We have the following result for plane graphs.

**Theorem 5** Let $G$ be a plane graph. If the length of the boundary of every face is at most $k$, then $G$ has an oriented $H$ such that $\text{rad}(H) \leq \text{rad}(G)(k-1)$ and $\text{diam}(H) \leq 2\text{rad}(G)(k-1)$.

Since every edge of a maximal plane (resp. outerplane) graph belongs to a cycle with length 3, the following corollary holds.

**Corollary 1** Let $G$ be a maximal plane (resp. outerplane) graph. Then there exists an orientation $H$ of $G$ such that $\text{rad}(H) \leq 2\text{rad}(G)$ and $\text{rad}(H) \leq 4\text{rad}(G)$.

A graph $G$ is *edge-transitive* if for any $e_1, e_2 \in E(G)$, there exists an automorphism $g$ such that $g(e_1) = e_2$. We have the following result for edge-transitive graphs.
Theorem 6 Let $G$ be a bridgeless edge-transitive graph. Then $G$ has an orientation $H$ such that $\text{rad}(H) \leq \text{rad}(G)(g(G) - 1)$ and $\text{diam}(H) \leq 2\text{rad}(G)(g(G) - 1)$, where $g(G)$ is the girth of $G$, that is, the length of a smallest cycle.

For general bipartite graphs, the following theorem holds.

Theorem 7 Let $G = (V_1 \cup V_2, E)$ be a bipartite graph with $|V_1| = n$ and $|V_2| = m$. If $d(x) \geq k > \lfloor m/2 \rfloor$ for any $x \in V_1$, $d(y) \geq r > \lfloor n/2 \rfloor$ for any $y \in V_2$, then there exists an orientation $H$ of $G$ such that $\text{rad}(H) \leq 9$.

Proof: It suffices to show that $\text{rad}(G) \leq 3$ and $\eta(G) \leq 4$ by Theorem 2.

First, we show that $\text{rad}(G) \leq 3$. Fix a vertex $x$ in $G$, and let $y$ be any vertex different from $x$ in $G$. If $x$ and $y$ belong to the same part, without loss of generality, say $x, y \in V_1$. Let $X$ and $Y$ be neighborhoods of $x$ and $y$ in $V_2$, respectively. If $X \cap Y = \emptyset$, then $|V_2| \geq |X| + |Y| \geq 2k > m$, a contradiction. Thus $X \cap Y \neq \emptyset$, that is, there exists a path between $x$ and $y$ of length two. If $x$ and $y$ belong to different parts, without loss of generality, say $x \in V_1, y \in V_2$. Suppose $x$ and $y$ are nonadjacent, otherwise there is nothing to prove. Let $X$ and $Y$ be the neighborhoods of $x$ and $y$ in $G$, and let $X'$ be the set of neighbors of $X$ except for $x$ in $G$. If $X' \cap Y = \emptyset$, then $|V_1| \geq 1 + |Y| + |X'| \geq 1 + r + (r - 1) = 2r > n$, a contradiction (Note that $|X'| \geq r - 1$). Thus $X' \cap Y \neq \emptyset$, that is, there exists a path between $x$ and $y$ of length three in $G$.

Next we show that $\eta(G) \leq 4$. Let $xy$ be any edge in $G$. Let $X$ be the set of neighbors of $x$ except for $y$ in $G$, let $Y$ be the set of neighbors of $y$ except for $x$ in $G$, let $X'$ be the set of neighbors of $X$ except for $x$ in $G$. If $X' \cap Y = \emptyset$, then $|V_1| \geq 1 + |Y| + |X'| \geq 1 + r + (r - 1) = 2r - 1 > n$, a contradiction (Note that $|X'| \geq r - 1$). Thus $X' \cap Y \neq \emptyset$, that is, there exists a cycle containing $xy$ of length four in $G$. \qed

Remark 2 The degree condition is optimal. Let $m, n$ be two even numbers with $n, m \geq 2$. Since $K_{n/2, m/2} \cup K_{n/2, m/2}$ is disconnected, the oriented radius (resp. diameter) of $K_{n/2, m/2} \cup K_{n/2, m/2}$ is $\infty$.

For equal bipartition $k$-regular graph, the following corollary holds.

Corollary 2 Let $G = (V_1 \cup V_2, E)$ be a $k$-regular bipartite graph with $|V_1| = |V_2| = n$. If $k > n/2$, then there exists an orientation $H$ of $G$ such that $\text{rad}(H) \leq 9$.

The following theorem holds for general graphs.

Theorem 8 Let $G$ be a graph of order $n$.

(i) If there exists an integer $k \geq 2$ such that $|N_k(u)| > n/2 - 1$ for every vertex $u$ in $G$, then $G$ has an orientation $H$ such that $\text{rad}(H) \leq 4k^2$ and $\text{diam}(H) \leq 8k^2$.

(ii) If $\delta(G) > n/2$, then $G$ has an orientation $H$ such that $\text{rad}(H) \leq 4$ and $\text{diam}(H) \leq 8$.

Proof: Since methods of the proofs of (i) and (ii) are similar, we only prove (i). For (i), it suffices to show that $\text{rad}(G) \leq 2k$ and $\eta(G) \leq 2k + 1$ by Theorem 2.

We first show $\text{rad}(G) \leq 2k$. Fix $u$ in $G$, for every $v \in V(G)$, if $v \in N_k[u]$, then $d(u, v) \leq k$. Suppose $v \notin N_k[u]$, we have $N_k(u) \cap N_k(v) \neq \emptyset$. If not, that is, $N_k(u) \cap N_k(v) = \emptyset$, then $|N_k(u) + |N_k(v)| + 2 > n$ (a contradiction). Thus $d(u, v) \leq 2k$. 


Next we show $\eta(G) \leq 2k + 1$. Let $e = uv$ be any edge in $G$. If $N_k(u) \cap N_k(v) = \emptyset$, then $|V(G)| \geq |N_k(u)| + |N_k(v)| + 2 > n$, a contradiction. Thus $N_k(u) \cap N_k(v) \neq \emptyset$. Pick $w \in N_k(u) \cap N_k(v)$, and let $P$ (resp. $Q$) be a path between $u$ and $w$ (resp. between $v$ and $w$). Then $e$ belongs to a close trial $uPwQvu$ of length $2k + 1$. Therefore, $e$ belongs to a cycle of length at most $2k + 1$. □

**Remark 3** The above condition is almost optimal since $K_{n/2} \cup K_{n/2}$ is disconnected even $n$.

**Corollary 3** Let $G$ be a graph with minimum degree $\delta(G)$ and girth $g(G)$. If there exists an integer $k$ such that $k \leq g(G)/2$ and $\delta(G)(\delta(G) - 1)^{k-1} > n/2 - 1$, then $G$ has an orientation $H$ such that $rad(H) \leq 4k^2$.

**Proof:** Let $k$ be an integer such that $k \leq g(G)/2$ and $\delta(G)(\delta(G) - 1)^{k-1} > n/2 - 1$. For any vertex $u$ of $G$, let $1 \leq i < k$ be any integer and $x, y, z \in N_i(u)$. If $x$ and $y$ have a common neighbor $z$ in $N_{i+1}(u)$, then $G$ has a cycle of length at most $2i < 2k \leq g(G)/2$, a contradiction. Thus $x$ and $y$ has no common neighbor in $N_{i+1}(u)$. Therefore, $|N_i(u)| \geq \delta(G)(\delta(G) - 1)^{k-1} > n/2 - 1$. By Theorem 2 $G$ has an orientation $H$ such that $rad(H) \leq 4k^2$. □

## 4 Upper bound for rainbow connection number

At first, we have the following observation.

**Observation 2** Let $G$ be a graph and $H$ be a spanning subgraph of $G$. Then $rc(G) \leq rc(H)$.

**Proof of Theorem 4** Let $u$ be a center of $G$ and let $H_0$ be the trivial graph with vertex set $\{u\}$. We assert that there exists a subgraph $H_i$ of $G$ such that $N_i[u] \subseteq V(H_i)$ and $rc(H_i) \leq \sum_{j=1}^{i} \min\{2(rad(G) - j) + 1, \eta(G)\}$.

**Basic step:** When $i = 1$, we omit the proof since the proof of this step is similar to that of the following induction step.

**Induction step:** Assume that the above assertion holds for $i - 1$ and $c$ is a $rc(H_{i-1})$-rainbow coloring of $H_{i-1}$. Next we show that the above assertion holds for $i$. For any $v \in N_i(u)$, either $v \in V(H_{i-1})$ or $v \in N_G(V(H_{i-1})$ since $N_{i-1}[u] \subseteq V(H_{i-1})$. If $N_i(u) \subseteq V(H_{i-1})$, then let $H_i = H_{i-1}$ and we are done. Thus we suppose $N_i(u) \not\subseteq V(H_{i-1})$ in the following.

Let $C_1 = \{\alpha_1, \alpha_2, \cdots\}$ and $C_2 = \{\beta_1, \beta_2, \cdots\}$ be two pools of colors, none of which are used to color $H_{i-1}$, and there exists no common color in $C_1$ and $C_2$. An edge-coloring of an $H$-ear $P = (u_0, u_1, \cdots, u_k)$ is a symmetrical coloring if its edges are colored by $\alpha_1, \alpha_2, \cdots, \alpha[k/2], \beta[k/2], \cdots, \beta_2, \beta_1$ in that order or $\beta_1, \beta_2, \cdots, \beta[k/2], \alpha[k/2], \cdots, \alpha_2, \alpha_1$ in that order.

Let $X = N_i(u) \setminus V(H_{i-1})$ and $m = \min\{2(rad(G) - i) + 1, \eta(G)\}$. Pick $x_1 \in X$, $y_1$ be a neighbor of $x_1$ in $H_{i-1}$ and $P_1$ be an optimal $(H_{i-1}, x_1y_1)$-ear. We can color $P$ symmetrically with colors $\alpha_1, \alpha_2, \cdots, \alpha[k/2\prime], \beta[k/2\prime], \cdots, \beta_2, \beta_1$. Pick $x_2 \in X$ satisfying that all the incident edges of $x_2$ are not colored. Let $y_2$ be a neighbor of $x_2$ in $H_{i-1}$. If there exists an optimal $(H_{i-1}, x_2y_2)$-ear $P_2$ such that $P_1$ and $P_2$ are independent, then we can color $P_2$ symmetrically with colors $\alpha_1, \alpha_2, \cdots, \alpha[k/2\prime], \beta[k/2\prime], \cdots, \beta_2, \beta_1$. Otherwise, by Lemma there exists an optimal $(H_{i-1}, x_2y_2)$-ear $P_2 = P_2^{\alpha_2\beta_2}$ such that $P_1$ and $P_2$ have only one continuous common segment containing $z_2$, where $z_2$ is the other foot of $P_2$. Thus we can color $P_2$ symmetrically with colors
Oriented diameter and rainbow connection number of a graph

Hence, assume that $P$ is colored from $x$ to $y$ by inductive hypothesis. If exactly one of $x$ and $y$ belongs to $V(H_{i-1})$, say $x$. Let $P$ be a symmetrical colored $H_{i-1}$-ear containing $y$, and $y'$ be a foot of $P$. There exists a rainbow path $Q$ between $x$ and $y'$ in $H_{i-1}$ by inductive hypothesis. Thus, $xQy'/Py$ is a rainbow path between $x$ and $y$ in $H_i$.

Suppose none of $x$ and $y$ belongs to $H_{i-1}$. Let $P$ and $Q$ be symmetrical colored $H_{i-1}$-ear containing $x$ and $y$, respectively. Furthermore, let $x'$, $x''$ be the feet of $P$ and $y'$, $y''$ be the feet of $Q$. Without loss of generality, assume that $P$ is colored from $x'$ to $x''$ by $\alpha_1, \alpha_2, \ldots, \alpha_{\ell(P)/2}, \beta_{\ell(P)/2}, \ldots, \beta_2, \beta_1$ in that order, and $Q$ is colored from $y'$ to $y''$ by $\alpha_3, \alpha_4, \ldots, \alpha_{\ell(Q)/2}, \beta_{\ell(Q)/2}, \ldots, \beta_2, \beta_1$ in that order. If $\ell(x'Px) \leq \ell(y'Qy)$, let $R$ be a rainbow path between $x'$ and $y''$ in $H_{i-1}$. Then $xPx'Ry''Qy$ is a rainbow path between $x$ and $y$ in $H_i$. Otherwise, $\ell(x'Px) > \ell(y'Qy)$. Let $R$ be a rainbow path between $y'$ and $x''$ in $H_{i-1}$. Then $yQy'Rx''Px$ is a rainbow path between $x$ and $y$ in $H_i$. Thus, there exists a rainbow path between any two distinct vertices in $H_i$, that is, $H_i$ is $(\sum_{j=1}^{m} \min\{2(\text{rad}(G) - j) + 1, \eta(G)\})$-rainbow connected. \(\Box\) Readers can see 11 for an optimal example. The following example shows that our result is better than that of Theorem 3.

Example 2 Let $r \geq 3$, $k \geq 2r$ be two integers, and $W_k = C_k \vee K_1$ be an wheel, where $V(C_k) = \{u_1, u_2, \ldots, u_k\}$ and $V(K_1) = \{u\}$. Let $H$ be the graph obtained from $W_k$ by inserting $r - 1$ vertices between every edge $u_{i-1}u_i$, $1 \leq i \leq k$. For every edge $e = xy$ of $H$, add a new vertex $v_e$ and new edges $v_ex, v_ey$. Denote by $G$ the resulting graph. It is easy to check that $\text{rad}(G) = r$, $\text{diam}(G) = 2r$, $\eta(G) = 3$ and $\zeta(G) = 2r - 1$. By Theorem 3, we have $rc(G) \leq \sum_{i=1}^{r} \min\{2i + 1, \zeta(G)\} \leq r^2 + 2r - 2$. But, by Theorem 4, we have $rc(G) \leq 3r$. On the other hand, $rc(G) \geq 2r$ since $\text{diam}(G) = 2r$.

The remaining results are similar to those in Section 3.

Theorem 9 Let $G$ be a plane graph. If the length of the boundary of every face is at most $k$, then $rc(G) \leq k \text{rad}(G)$.

Corollary 4 Let $G$ be a maximal plane (resp. outerplane) graph. Then $rc(G) \leq 3\text{rad}(G)$.

Theorem 10 Let $G$ be a bridgeless edge-transitive graph. Then $rc(G) \leq \text{rad}(G)g(G)$, where $g(G)$ is the girth of $G$.

Theorem 11 Let $G = (V_1 \cup V_2, E)$ be a bipartite graph with $|V_1| = n$ and $|V_2| = m$. If $d(x) \geq k > \lceil m/2 \rceil$ for any $x \in V_1$, $d(y) \geq r > \lceil n/2 \rceil$ for any $y \in V_2$, then $rc(G) \leq 12$.

Remark 4 The degree condition is optimal. Let $m, n$ be two even numbers with $n, m \geq 2$. Since $K_{n/2,m/2} \cup K_{n/2,m/2}$ is disconnected, $rc(K_{n/2,m/2} \cup K_{n/2,m/2}) = \infty$.

Corollary 5 Let $G = (V_1 \cup V_2, E)$ be a k-regular bipartite graph with $|V_1| = |V_2| = n$. If $k > \lceil n/2 \rceil$, then $rc(G) \leq 12$.

The following theorem holds for general graphs.
Theorem 12 Let $G$ be a graph.

(i) If there exists an integer $k \geq 2$ such that $|N_k(u)| > n/2-1$ for every vertex $u$ in $G$, then $rc(G) \leq 4k^2 + 2k$.

(ii) If $\delta(G) > n/2$, then $rc(G) \leq 6$.

Remark 5 The above condition is almost optimal since $K_{n/2} \cup K_{n/2}$ is disconnected for even $n$.

Corollary 6 Let $G$ be a graph with minimum degree $\delta(G)$ and girth $g(G)$. If there exists an integer $k$ such that $k < g(G)/2$ and $\delta(G)(\delta(G) - 1)^{k-1} > n/2 - 1$, then $rc(G) \leq 4k^2 + 2k$.

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