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To cite this version:
Xiaolong Huang, Hengzhe Li, Xueliang Li, Yuefang Sun. Oriented diameter and rainbow connection number of a graph. Discrete Mathematics and Theoretical Computer Science, DMTCS, 2014, Vol. 16 no. 3 (in progress) (3), pp.51–60. <hal-01188907>

HAL Id: hal-01188907
https://hal.inria.fr/hal-01188907
Submitted on 31 Aug 2015

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Oriented diameter and rainbow connection number of a graph

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received 23rd Dec. 2012, accepted 28th May 2014.

The oriented diameter of a bridgeless graph $G$ is $\min\{\text{diam}(H) \mid H$ is a strang orientation of $G\}$. A path in an edge-colored graph $G$, where adjacent edges may have the same color, is called rainbow if no two edges of the path are colored the same. The rainbow connection number $rc(G)$ of $G$ is the smallest integer number $k$ for which there exists a $k$-edge-coloring of $G$ such that every two distinct vertices of $G$ are connected by a rainbow path. In this paper, we obtain upper bounds for the oriented diameter and the rainbow connection number of a graph in terms of $\text{rad}(G)$ and $\eta(G)$, where $\text{rad}(G)$ is the radius of $G$ and $\eta(G)$ is the smallest integer number such that every edge of $G$ is contained in a cycle of length at most $\eta(G)$. We also obtain constant bounds of the oriented diameter and the rainbow connection number for a (bipartite) graph $G$ in terms of the minimum degree of $G$.

Keywords: Diameter, Radius, Oriented diameter, Rainbow connection number, Cycle length, Bipartite graph.

1 Introduction

All graphs in this paper are undirected, finite and simple. We refer to the book [2] for notation and terminology not described here. A path $u = u_1, u_2, \ldots, u_k = v$ is called a $P_{u,v}$ path. Denote by $u_iP_{u,j}$ the subpath $u_i, u_{i+1}, \ldots, u_j$ for $i \leq j$. The length $\ell(P)$ of a path $P$ is the number of edges in $P$. The distance between two vertices $x$ and $y$ in $G$, denoted by $d_G(x,y)$, is the length of a shortest path between them. The eccentricity of a vertex $x$ in $G$ is $ecc_G(x) = \max_{y \in V(G)}d(x,y)$. The radius and diameter of $G$ are $\text{rad}(G) = \min_{x \in V(G)}ecc(x)$ and $\text{diam}(G) = \max_{x \in V(G)}ecc(x)$, respectively. A vertex $u$ is a center of a graph $G$ if $ecc(u) = \text{rad}(G)$. The oriented diameter of a bridgeless graph $G$ is $\min\{\text{diam}(H) \mid H$ is an orientation of $G\}$, and the oriented radius of a bridgeless graph $G$ is $\min\{\text{rad}(H) \mid H$ is an orientation of $G\}$. For any graph $G$ with edge-connectivity $\lambda(G) = 0, 1$, $G$ has oriented radius (resp. diameter) $\infty$.

In 1939, Robbins solved the One-Way Street Problem and proved that a graph $G$ admits a strongly connected orientation if and only if $G$ is bridgeless, that is, $G$ does not have any cut-edge. Naturally, one hopes that the oriented diameter of a bridgeless graph is as small as possible. Bondy and Murty suggested...
to study the quantitative variations on Robbins’ theorem. In particular, they conjectured that there exists a function $f$ such that every bridgeless graph with diameter $d$ admits an orientation of diameter at most $f(d)$.


**Theorem 1 (Chvátal and Thomassen 1978 [5])** For every bridgeless graph $G$, there exists an orientation $H$ of $G$ such that

$$\text{rad}(H) \leq \text{rad}(G)^2 + \text{rad}(G),$$
$$\text{diam}(H) \leq 2\text{rad}(G)^2 + 2\text{rad}(G).$$

Moreover, the above bounds are optimal.

There exists a minor error when they constructed the graph $G_d$ which arrives at the upper bound when $d$ is odd. Kwok, Liu and West gave a slight correction in [11].

They also showed that determining whether an arbitrary graph can be oriented so that its diameter is at most 2 is NP-complete. Bounds for the oriented diameter of graphs have also been studied in terms of other parameters, for example, radius, dominating number [5, 6, 11, 18], etc. Some classes of graphs have also been studied in [6, 7, 8, 9, 14].

Let $\eta(G)$ be the smallest integer such that every edge of $G$ belongs to a cycle of length at most $\eta(G)$.

In this paper, we show the following result.

**Theorem 2** For every bridgeless graph $G$, there exists an orientation $H$ of $G$ such that

$$\text{rad}(H) \leq \sum_{i=1}^{\text{rad}(G)} \min\{2i, \eta(G) - 1\} \leq \text{rad}(G)(\eta(G) - 1),$$
$$\text{diam}(H) \leq 2\sum_{i=1}^{\text{rad}(G)} \min\{2i, \eta(G) - 1\} \leq 2\text{rad}(G)(\eta(G) - 1).$$

Note that $\sum_{i=1}^{\text{rad}(G)} \min\{2i, \eta(G) - 1\} \leq \text{rad}(G)^2 + \text{rad}(G)$ and $\text{diam}(H) \leq 2\text{rad}(H)$. So our result implies Chvátal and Thomassen’s Theorem [1].

A path in an edge-colored graph $G$, where adjacent edges may have the same color, is called rainbow if no two edges of the path are colored the same. An edge-coloring of a graph $G$ is a rainbow edge-coloring if every two distinct vertices of the graph $G$ are connected by a rainbow path. The rainbow connection number $rc(G)$ of $G$ is the minimum integer $k$ for which there exists a rainbow $k$-edge-coloring of $G$. It is easy to see that $\text{diam}(G) \leq rc(G)$ for any connected graph $G$. The rainbow connection number was introduced by Chartrand et al. in [4]. It is of great use in transferring information of high security in multicomputer networks. We refer the readers to [3] for details.

Chakraborty et al. [3] investigated the hardness and algorithms for the rainbow connection number, and showed that given a graph $G$, deciding if $rc(G) = 2$ is NP-complete. Bounds for the rainbow connection number of a graph have also been studied in terms of other graph parameters, for example, radius, dominating number, minimum degree, connectivity, etc. [11, 14, 10]. Cayley graphs and line graphs were studied in [12] and [13], respectively.

A subgraph $H$ of a graph $G$ is called isometric if the distance between any two distinct vertices in $H$ is the same as their distance in $G$. The size of a largest isometric cycle in $G$ is denoted by $\zeta(G)$. 

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Clearly, every isometric cycle is an induced cycle and thus \( \zeta(G) \) is not larger than the chordality, where chordality is the length of a largest induced cycle in \( G \). In \cite{Basavaraju2011}, Basavaraju, Chandran, Rajendraprasad and Ramaswamy got the following sharp upper bound for the rainbow connection number of a bridgeless graph \( G \) in terms of \( \text{rad}(G) \) and \( \zeta(G) \).

**Theorem 3 (Basavaraju et al. \cite{Basavaraju2011})** For every bridgeless graph \( G \),

\[
\text{rc}(G) \leq \sum_{i=1}^{\text{rad}(G)} \min\{2i+1,\zeta(G)\} \leq \text{rad}(G)\zeta(G).
\]

In this paper, we show the following result.

**Theorem 4** For every bridgeless graph \( G \),

\[
\text{rc}(G) \leq \sum_{i=1}^{\text{rad}(G)} \min\{2i+1,\eta(G)\} \leq \text{rad}(G)\eta(G).
\]

From Lemma 2 of Section 2, we will see that \( \eta(G) \leq \zeta(G) \). Thus our result implies Theorem 3.

This paper is organized as follows: in Section 2, we introduce some new definitions and show several lemmas. In Section 3, we prove Theorem 3 and study upper bounds for the oriented radius (resp. diameter) of plane graphs, edge-transitive graphs and general (bipartite) graphs. In Section 4, we prove Theorem 4 and study upper for the rainbow connection number of plane graphs, edge-transitive graphs and general (bipartite) graphs.

## 2 Preliminaries

In this section, we introduce some definitions and show several lemmas.

**Definition 1** For any \( x \in V(G) \) and \( k \geq 0 \), the \textit{k-step open neighborhood} is \( \{y \mid d(x,y) = k\} \) and denoted by \( N_k(x) \), the \textit{k-step closed neighborhood} is \( \{y \mid d(x,y) \leq k\} \) and denoted by \( N_k[x] \). If \( k = 1 \), we simply write \( N(x) \) and \( N[x] \) for \( N_1(x) \) and \( N_1[x] \), respectively.

**Definition 2** Let \( G \) be a graph and \( H \) be a subset of \( V(G) \) (or a subgraph of \( G \)). The edges between \( H \) and \( G \setminus H \) are called \textit{legs} of \( H \). An \textit{H-ear} is a path \( P = (u_0, u_1, \ldots, u_k) \) in \( G \) such that \( V(H) \cap V(P) = \{u_0, u_k\} \). The vertices \( u_0, u_k \) are called the \textit{feet} of \( P \) in \( H \) and \( u_0u_1, u_{k-1}u_k \) are called the \textit{legs} of \( P \). The \textit{length} of an \( H \)-ear is the length of the corresponding path. If \( u_0 = u_k \), then \( P \) is called a \textit{closed H-ear}. For any leg \( e \) of \( H \), denote by \( \ell(e) \) the smallest number such that there exists an \( H \)-ear of length \( \ell(e) \) containing \( e \), and such an \( H \)-ear is called an \textit{optimal} \( (H,e) \)-ear.

Note that for any optimal \( (H,e) \)-ear \( P \) and every pair \( (x,y) \neq (u_0, u_k) \) of distinct vertices of \( P \), \( x \) and \( y \) are adjacent on \( P \) if and only if \( x \) and \( y \) are adjacent in \( G \).

**Definition 3** For any two paths \( P \) and \( Q \), the joint of \( P \) and \( Q \) are the common vertex and edge of \( P \) and \( Q \). Paths \( P \) and \( Q \) have \textit{k continuous common segments} if the common vertex and edge are \( k \) disjoint paths.

A common segment is trivial if it has only one vertex.
Lemma 1 Let \( n \geq 1 \) be an integer, and let \( G \) be a graph, \( H \) be a subgraph of \( G \) and \( e_i = u_i v_i \) be a leg of \( H \) and \( P_i = P_{u_i w_i} \) be an optimal \((G, e_i)\)-ear, where \( 1 \leq i \leq n \) and \( u_i, w_i \) are the foot of \( P_i \). Then for any leg \( e_j = u_j v_j \) such that \( e_j \neq e_i \) and \( e_j \notin E(P_i) \), where \( i \in \{1, 2, \ldots, n\} \), there exists an optimal \((H, e_j)\)-ear \( P_j = P_{u_j w_j} \) such that either \( P_i \) and \( P_j \) are independent for any \( P_i, 1 \leq i \leq n \), or \( P_i \) and \( P_j \) have only one continuous common segment containing \( w_j \) for some \( P_i \).

Proof: Let \( P_j \) be an optimal \((H, e_j)\)-ear. If \( P_i \) and \( P_j \) are independent for any \( i \), then we are done. Suppose that \( P_i \) and \( P_j \) have \( m \) continuous common segments for some \( i \), where \( m \geq 1 \). When \( m \geq 2 \), we first construct an optimal \((H, e_j)\)-ear \( P_j^* \) such that \( P_i \) and \( P_j^* \) has only one continuous common segment. Let \( P_{i_1}, P_{i_2}, \ldots, P_{i_m} \) be the \( m \) continuous common segments of \( P_i \) and \( P_j \) and they appear in \( P_i \) in that order. See Figure 1 for details. Furthermore, suppose that \( x_{i_k} \) and \( y_{i_k} \) are the two ends of the path \( P_{i_k} \) and they appear in \( P_i \) successively. We say that the following claim holds.

Claim 1: \( \ell(y_k P_j x_{k+1}) = \ell(y_k P_j x_{k+1}) \) for any \( 1 \leq k \leq m - 1 \).

If not, that is, there exists an integer \( k \) such that \( \ell(y_k P_j x_{k+1}) \neq \ell(y_k P_j x_{k+1}) \). Without loss of generality, we assume \( \ell(y_k P_j x_{k+1}) < \ell(y_k P_j x_{k+1}) \). Then we shall get a more shorter path \( H \)-ear containing \( e_j \) by replacing \( y_k P_j x_{k+1} \) with \( y_k P_j x_{k+1} \), a contradiction. Thus \( \ell(y_k P_j x_{k+1}) = \ell(y_k P_j x_{k+1}) \) for any \( k \).

Let \( P_j^* \) be the path obtained from \( P_j \) by replacing \( y_k P_j x_{k+1} \) with \( y_k P_j x_{k+1} \), and let \( P_j = P_j^* \). If the continuous common segment of \( P_i \) and \( P_j \) does not contain \( w_j \), suppose \( x \) and \( y \) are the two ends of the common segment such that \( x \) and \( y \) appeared on \( P \) starting from \( u_i \) to \( w_i \) successively. Similar to Claim 1, \( \ell(y P_j w_i) = \ell(y P_j w_i) \). Let \( P_j^* \) be the path obtained from \( P_j \) by replacing \( y P_j w_j \) with \( y P_j w_i \). Clearly, \( P_j^* \) is our desired optimal \((H, u_j v_j)\)-ear.

Lemma 2 For every bridgeless graph \( G \), \( \eta(G) \leq \zeta(G) \).

Proof: Suppose that there exists an edge \( e \) such that the length \( \ell(C) \) of the smallest cycle \( C \) containing \( e \) is larger than \( \zeta(G) \). Then, \( C \) is not an isometric cycle since the length of a largest isometric cycle is
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ζ(G). Thus there exist two vertices u and v on C such that \( d_G(u, v) < d_C(u, v) \). Let \( P \) be a shortest path between \( u \) and \( v \) in \( G \). Then a closed trial \( C' \) containing \( e \) is obtained from the segment of \( C \) containing \( e \) between \( u \) and \( v \) by adding \( P \). Clearly, the length \( \ell(C') \) is less than \( \ell(C) \). We can get a cycle \( C'' \) containing \( e \) from \( C' \). Thus there exists a cycle \( C'' \) containing \( e \) with length less than \( \ell(C) \), a contradiction. Therefore \( \eta(G) \leq \zeta(G) \).

Lemma 3 Let \( G \) be a bridgeless graph and \( u \) be a center of \( G \). For any \( i \leq \text{rad}(G) - 1 \) and every leg \( e \) of \( N_i(u) \), there exists an optimal \( (N_i[u], e) \)-ear with length at most \( \min\{2(\text{rad}(G) - i) + 1, \eta(G)\} \).

Proof: Let \( P \) be an optimal \( (N_i[u], e) \)-ear. Since \( e \) belongs to a cycle with length at most \( \eta(G) \), \( \ell(P) \leq \eta(G) \). On the other hand, if \( \ell(P) \geq 2(\text{rad}(G) - i) + 1 \), then the middle vertex of \( P \) has distance at least \( \text{rad}(G) - i + 1 \) from \( N_i[u] \), a contradiction.

3 Oriented diameter

At first, we have the following observation.

Observation 1 Let \( G \) be a bridgeless graph and \( H \) be a bridgeless spanning subgraph of \( G \). Then the oriented radius (resp. diameter) of \( G \) is not larger than the oriented radius (resp. diameter) of \( H \).

Proof of Theorem 2 We only need to show that \( G \) has an orientation \( H \) such that

\[
\text{rad}(H) \leq \sum_{i=1}^{\text{rad}(G)} \min\{2i, \eta(G) - 1\} \leq \text{rad}(G)(\eta(G) - 1).
\]

Let \( u \) be a center of \( G \) and let \( H_0 \) be the trivial graph with vertex set \( \{u\} \). We assert that there exists a subgraph \( G_i \) of \( G \) such that \( N_i[u] \subseteq V(G_i) \) and \( G_i \) has an orientation \( H_i \) satisfying that \( \text{rad}(H_i) \leq \text{ecc}_{H_i}(u) \leq \sum_{j=1}^{i} \min\{2(\text{rad}(G) - j), \eta(G) - 1\} \).

Basic step: When \( i = 1 \), we omit the proof since the proof of this step is similar to that of the following induction step.

Induction step: Assume that the above assertion holds for \( i - 1 \). Next we show that the above assertion also holds for \( i \). For any \( v \in N_i(u) \), either \( v \in V(H_{i-1}) \) or \( v \in N_G(V(H_{i-1})) \) since \( N_{i-1}[u] \subseteq V(H_{i-1}) \). If \( N_i(u) \subseteq V(H_{i-1}) \), then let \( H_i = H_{i-1} \) and we are done. Thus, we suppose \( N_i(u) \not\subseteq V(H_{i-1}) \) in the following.

Let \( X = N_i(u) \setminus V(H_{i-1}) \). Pick \( x_1 \in X \), let \( y_1 \) be a neighbor of \( x_1 \) in \( H_{i-1} \) and let \( P_1 = P_{y_1z_1} \) be an optimal \((H_{i-1}, x_1y_1)\)-ear. We orient \( P \) such that \( P_1 \) is a directed path. Pick \( x_2 \in X \) satisfying that all incident edges of \( x_2 \) are not oriented. Let \( y_2 \) be a neighbor of \( x_2 \) in \( H_{i-1} \). If there exists an optimal \((H_{i-1}, x_2y_2)\)-ear \( P_2 \) such that \( P_1 \) and \( P_2 \) are independent, then we can orient \( P_2 \) such that \( P_2 \) is a directed path. Otherwise, by Lemma 1 there exists an optimal \((H_{i-1}, x_2y_2)\)-ear \( P_2 = P_{y_2z_2} \) such that \( P_1 \) and \( P_2 \) has only one continuous common segment containing \( z_2 \). Clearly, we can orient the edges in \( E(P_2) \setminus E(P_1) \) such that \( P_2 \) is a directed path. We can pick the vertices of \( X \) and oriented optimal \( H \)-ears similar to the above method until that for any \( x \in X \), at least two incident edges of \( x \) are oriented. Let \( H_i \) be the graph obtained from \( H_{i-1} \) by adding vertices in \( V(G) \setminus V(H_{i-1}) \), which has at least two new oriented incident edges, and adding the new oriented edges. Clearly, \( N_i[u] \subseteq V(H_i) = V(G_i) \).
Now we show that \( \text{rad}(H_i) \leq \sum_{j=1}^{k} \min\{2(\text{rad}(G) - i), \eta(G) - 1\} \). It suffices to show that for every vertex \( x \) of \( H_i \), \( d_{H_i}(H_{i-1}, x) \leq \min\{2(\text{rad}(G) - i), \eta(G) - 1\} \) and \( d_{H_i}(x, H_{i-1}) \leq \min\{2(\text{rad}(G) - i), \eta(G) - 1\} \). If \( x \in V(H_{i-1}) \), then the assertion holds by inductive hypothesis. If \( x \notin V(H_{i-1}) \), let \( P \) be a directed optimal \((H_i, e)\)-ear containing \( x \), where \( e \) is some leg of \( H_{i-1} \) (such a leg and such an ear exists by the definition of \( H_i \)). By Lemma 3, \( \ell(P) \leq \min\{2(\text{rad}(G) - i) + 1, \eta(G)\} \). Thus, \( d_{H_i}(x, H_{i-1}) \leq \min\{2(\text{rad}(G) - i), \eta(G) - 1\} \) and \( d_{H_i}(H_{i-1}, x) \leq \min\{2(\text{rad}(G) - i), \eta(G) - 1\} \). Therefore, \( \text{rad}(H_i) \leq \sum_{j=1}^{k} \min\{2(\text{rad}(G) - j), \eta(G) - 1\} \). \( \square \)

**Remark 1** The above theorem is optimal since it implies Chvátal and Thomassen’s optimal Theorem. Readers can see [5, 11] for optimal examples.

The following example shows that our result is better than that of Theorem 1.

**Example 1** Let \( F_3 \) be a triangle with one of its vertices designated as a root. In order to construct \( F_r \), take two copies of \( F_{r-1} \). Let \( H_r \) be the graph obtained from the triangle \( u_0, u_1, u_2 \) by identifying the root of first (resp. second) copy of \( F_{r-1} \) with \( u_1 \) (resp. \( u_2 \)), and \( u_0 \) be the root of \( F_r \). Let \( G_r \) be the graph obtained by taking two copies of \( F_r \) and identifying their roots. See Figure 2 for details. It is easy to check that \( G_r \) has radius \( r \) and every edge belongs to a cycle of length \( \eta(G) = 3 \). By Theorem 1, \( G_r \) has an orientation \( H_r \) such that \( \text{rad}(H_r) \leq r^2 + r \) and \( \text{diam}(H_r) \leq 2r^2 + 2r \). But, by Theorem 5, \( G_r \) has an orientation \( H_r \) such that \( \text{rad}(G) \leq 2r \) and \( \text{diam}(G) \leq 4r \). On the other hand, it is easy to check that all the strong orientations of \( G_r \) has radius \( 2r \) and diameter \( 4r \).

![Fig. 2: The graph \( G_3 \) which has oriented radius 6 and oriented diameter 12](image)

We have the following result for plane graphs.

**Theorem 5** Let \( G \) be a plane graph. If the length of the boundary of every face is at most \( k \), then \( G \) has an oriented \( H \) such that \( \text{rad}(H) \leq \text{rad}(G)(k - 1) \) and \( \text{diam}(H) \leq 2\text{rad}(G)(k - 1) \).

Since every edge of a maximal plane (resp. outerplane) graph belongs to a cycle with length 3, the following corollary holds.

**Corollary 1** Let \( G \) be a maximal plane (resp. outerplane) graph. Then there exists an orientation \( H \) of \( G \) such that \( \text{rad}(H) \leq 2\text{rad}(G) \) and \( \text{rad}(H) \leq 4\text{rad}(G) \).

A graph \( G \) is edge-transitive if for any \( e_1, e_2 \in E(G) \), there exists an automorphism \( g \) such that \( g(e_1) = e_2 \). We have the following result for edge-transitive graphs.
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Theorem 6 Let $G$ be a bridgeless edge-transitive graph. Then $G$ has an orientation $H$ such that $\text{rad}(H) \leq \text{rad}(G)(g(G) - 1)$ and $\text{diam}(H) \leq 2\text{rad}(G)(g(G) - 1)$, where $g(G)$ is the girth of $G$, that is, the length of a smallest induced cycle.

For general bipartite graphs, the following theorem holds.

Theorem 7 Let $G = (V_1 \cup V_2, E)$ be a bipartite graph with $|V_1| = n$ and $|V_2| = m$. If $d(x) \geq k > \lceil m/2 \rceil$ for any $x \in V_1$, $d(y) \geq r > \lceil n/2 \rceil$ for any $y \in V_2$, then there exists an orientation $H$ of $G$ such that $\text{rad}(H) \leq 9$.

Proof: It suffices to show that $\text{rad}(G) \leq 3$ and $\eta(G) \leq 4$ by Theorem 2.

First, we show that $\text{rad}(G) \leq 3$. Fix a vertex $x$ in $G$, and let $y$ be any vertex different from $x$ in $G$. If $x$ and $y$ belong to the same part, without loss of generality, say $x \in V_1$. Let $X$ and $Y$ be neighborhoods of $x$ and $y$ in $V_2$, respectively. If $X \cap Y = \emptyset$, then $|V_1| \geq |X| + |Y| \geq 2k > m$, a contradiction. Thus $X \cap Y \neq \emptyset$, that is, there exists a path between $x$ and $y$ of length two. If $x$ and $y$ belong to different parts, without loss of generality, say $x \in V_1, y \in V_2$. Suppose $x$ and $y$ are nonadjacent, otherwise there is nothing to prove. Let $X$ and $Y$ be the neighborhoods of $x$ and $y$ in $G$, and let $X'$ be the set of neighbors of $X$ except for $x$ in $G$. If $X' \cap Y = \emptyset$, then $|V_1| \geq 1 + |Y| + |X'| \geq 1 + r + (r - 1) = 2r > n$, a contradiction (Note that $|X'| \geq r - 1$). Thus $X' \cap Y \neq \emptyset$, that is, there exists a path between $x$ and $y$ of length three in $G$.

Next we show that $\eta(G) \leq 4$. Let $xy$ be any edge in $G$. Let $X$ be the set of neighbors of $x$ except for $y$ in $G$, let $Y$ be the set of neighbors of $y$ except for $x$ in $G$, let $X'$ be the set of neighbors of $X$ except for $x$ in $G$. If $X' \cap Y = \emptyset$, then $|V_1| \geq 1 + |Y| + |X'| \geq 1 + (r - 1) + (r - 1) = 2r - 1 > n$, a contradiction (Note that $|X'| \geq r - 1$). Thus $X' \cap Y \neq \emptyset$, that is, there exists a cycle containing $xy$ of length four in $G$.

Remark 2 The degree condition is optimal. Let $m, n$ be two even numbers with $m, n \geq 2$. Since $K_{n/2,m/2} \cup K_{n/2,m/2}$ is disconnected, the oriented radius (resp. diameter) of $K_{n/2,m/2} \cup K_{n/2,m/2}$ is $\infty$.

For equal bipartition $k$-regular graph, the following corollary holds.

Corollary 2 Let $G = (V_1 \cup V_2, E)$ be a $k$-regular bipartite graph with $|V_1| = |V_2| = n$. If $k > n/2$, then there exists an orientation $H$ of $G$ such that $\text{rad}(H) \leq 9$.

The following theorem holds for general graphs.

Theorem 8 Let $G$ be a graph of order $n$.

(i) If there exists an integer $k \geq 2$ such that $|N_k(u)| > n/2 - 1$ for every vertex $u$ in $G$, then $G$ has an orientation $H$ such that $\text{rad}(H) \leq 4k^2$ and $\text{diam}(H) \leq 8k^2$.

(ii) If $\delta(G) > n/2$, then $G$ has an orientation $H$ such that $\text{rad}(H) \leq 4$ and $\text{diam}(H) \leq 8$.

Proof: Since methods of the proofs of (i) and (ii) are similar, we only prove (i). For (i), it suffices to show that $\text{rad}(G) \leq 2k$ and $\eta(G) \leq 2k + 1$ by Theorem 2.

We first show $\text{rad}(G) \leq 2k$. Fix $u$ in $G$, for every $v \in V(G)$, if $v \in N_k[u]$, then $d(u, v) \leq k$. Suppose $v \notin N_k[u]$, we have $N_k(u) \cap N_k(v) \neq \emptyset$. If not, that is, $N_k(u) \cap N_k(v) = \emptyset$, then $|N_k(u)| + |N_k(v)| + 2 > n$ (a contradiction). Thus $d(u, v) \leq 2k$. 


Next we show $\eta(G) \leq 2k + 1$. Let $e = uv$ be any edge in $G$. If $N_k(u) \cap N_k(v) = \emptyset$, then $|V(G)| \geq |N_k(u)| + |N_k(v)| + 2 > n$, a contradiction. Thus $N_k(u) \cap N_k(v) \neq \emptyset$. Pick $w \in N_k(u) \cap N_k(v)$, and let $P$ (resp. $Q$) be a path between $u$ and $w$ (resp. between $v$ and $w$). Then $e$ belongs to a close trial $uPwQvu$ of length $2k + 1$. Therefore, $e$ belongs to a cycle of length at most $2k + 1$.

**Remark 3** The above condition is almost optimal since $K_{n/2} \cup K_{n/2}$ is disconnected for even $n$.

**Corollary 3** Let $G$ be a graph with minimum degree $\delta(G)$ and girth $g(G)$. If there exists an integer $k$ such that $k \leq g(G)/2$ and $\delta(G)(\delta(G) - 1)^{k-1} > n/2 - 1$, then $G$ has an orientation $H$ such that $\text{rad}(H) \leq 4k^2$.

**Proof:** Let $k$ be an integer such that $k \leq g(G)/2$ and $\delta(G)(\delta(G) - 1)^{k-1} > n/2 - 1$. For any vertex $u$ of $G$, let $1 \leq i < k$ be any integer and $x, y \in N_i(u)$. If $x$ and $y$ have a common neighbor $z$ in $N_{i+1}(u)$, then $G$ has a cycle of length at most $2i < 2k \leq g(G)/2$, a contradiction. Thus $x$ and $y$ has no common neighbor in $N_{i+1}(u)$. Therefore, $|N_i(u)| \geq \delta(G)(\delta(G) - 1)^{k-1} > n/2 - 1$. By Theorem 2, $G$ has an orientation $H$ such that $\text{rad}(H) \leq 4k^2$.

## 4 Upper bound for rainbow connection number

At first, we have the following observation.

**Observation 2** Let $G$ be a graph and $H$ be a spanning subgraph of $G$. Then $rc(G) \leq rc(H)$.

**Proof of Theorem 4:** Let $u$ be a center of $G$ and let $H_0$ be the trivial graph with vertex set $\{u\}$. We assert that there exists a subgraph $H_i$ of $G$ such that $N_i[u] \subseteq V(H_i)$ and $rc(H_i) \leq \sum_{j=1}^{i} \min\{2(\text{rad}(G) - j) + 1, \eta(G)\}$.

**Basic step:** When $i = 1$, we omit the proof since the proof of this step is similar to that of the following induction step.

**Induction step:** Assume that the above assertion holds for $i - 1$ and $c$ is a $rc(H_{i-1})$-rainbow coloring of $H_{i-1}$. Next we show that the above assertion holds for $i$. For any $v \in N_i(u)$, either $v \in V(H_{i-1})$ or $v \in N_G(V(H_{i-1}))$ since $N_{i-1}[u] \subseteq V(H_{i-1})$. If $N_i(u) \subseteq V(H_{i-1})$, then let $H_i = H_{i-1}$ and we are done. Thus, we suppose $N_i(u) \not\subseteq V(H_{i-1})$ in the following.

Let $C_1 = \{\alpha_1, \alpha_2, \cdots\}$ and $C_2 = \{\beta_1, \beta_2, \cdots\}$ be two pools of colors, none of which are used to color $H_{i-1}$, and there exists no common color in $C_1$ and $C_2$. An edge-coloring of an $H$-ear $P = (u_0, u_1, \cdots, u_k)$ is a symmetrical coloring if its edges are colored by $\alpha_1, \alpha_2, \cdots, \alpha_{[k/2]}, \beta_{[k/2]}, \cdots, \beta_2, \beta_1$ in that order or $\beta_1, \beta_2, \cdots, \beta_{[k/2]}, \alpha_{[k/2]}, \cdots, \alpha_2, \alpha_1$ in that order.

Let $X = N_i(u) \setminus V(H_{i-1})$ and $m = \min\{2(\text{rad}(G) - i) + 1, \eta(G)\}$. Pick $x_1 \in X$, Let $y_1$ be a neighbor of $x_1$ in $H_{i-1}$ and $P_1$ be an optimal $(H_{i-1}, x_1y_1)$-ear. We can color $P$ symmetrically with colors $\alpha_1, \alpha_2, \cdots, \alpha_{[\ell(p)/2]}, \beta_{[\ell(p)/2]}, \cdots, \beta_2, \beta_1$. Pick $x_2 \in X$ satisfying that all the incident edges of $x_2$ are not colored. Let $y_2$ be a neighbor of $x_2$ in $H_{i-1}$. If there exists an optimal $(H_{i-1}, x_2y_2)$-ear $P_2$ such that $P_1$ and $P_2$ are independent, then we can color $P_2$ symmetrically with colors $\alpha_1, \alpha_2, \cdots, \alpha_{[\ell(p)/2]}, \beta_{[\ell(p)/2]}, \cdots, \beta_2, \beta_1$. Otherwise, by Lemma 1 there exists an optimal $(H_{i-1}, x_2y_2)$-ear $P_2 = P_{y_2z_2}$ such that $P_1$ and $P_2$ have only one continuous common segment containing $z_2$, where $z_2$ is the other foot of $P_2$. Thus we can color $P_2$ symmetrically with colors
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vertices of

\(X\)

of generality, assume that

\(P\)

between

\(\ell\)

of \(H_{i-1}\), then there exists a rainbow path between \(x\) and \(y\) by inductive hypothesis. If exactly one of \(x\) and \(y\) belongs to \(V(H_{i-1})\), say \(x\). Let \(P\) be a symmetrical colored \(H_{i-1}\)-ear containing \(y\), and \(y'\) be a foot of \(P\). There exists a rainbow path \(Q\) between \(x\) and \(y'\) in \(H_{i-1}\) by inductive hypothesis. Thus, \(xPy'y\) is a rainbow path between \(x\) and \(y\) in \(H_i\).

Suppose none of \(x\) and \(y\) belongs to \(H_{i-1}\). Let \(P\) and \(Q\) be symmetrical colored \(H_{i-1}\)-ear containing \(x\) and \(y\), respectively. Furthermore, let \(x', x''\) be the feet of \(P\) and \(y', y''\) be the feet of \(Q\). Without loss of generality, assume that \(P\) is colored from \(x'\) to \(x''\) by \(\alpha_1, \alpha_2, \ldots, \alpha_\ell\), \(P/2\), \(\beta_1, \beta_2, \beta_1\) in that order, and \(Q\) is colored from \(y'\) to \(y''\) by \(\alpha_1, \alpha_2, \ldots, \alpha_\ell\), \(P/2\), \(\beta_1, \beta_2, \beta_1\) in that order. If \(\ell(x'Px) \leq \ell(yQy')\), let \(R\) be a rainbow path between \(x'\) and \(y''\) in \(H_{i-1}\). Then \(xPx'Rx''Qy\) is a rainbow path between \(x\) and \(y\) in \(H_i\). Otherwise, \(\ell(x'Px) > \ell(yQy')\). Let \(R\) be a rainbow path between \(y'\) and \(x''\) in \(H_{i-1}\). Then \(yQy'Rx''Px\) is a rainbow path between \(x\) and \(y\) in \(H_i\). Thus, there exists a rainbow path between any two distinct vertices in \(H_i\), that is, \(H_i\) is \(\Sigma_{j=1}^{\ell} (2(rad(G) - j) + 1, \eta(G))\)-rainbow connected.

\(\square\)

Readers can see \(\square\) for an optimal example. The following example shows that our result is better than that of Theorem \(\square\).

Example 2 Let \(r \geq 3, k \geq 2r\) be two integers, and \(W_k = C_k \vee K_1\) be an wheel, where \(V(C_k) = \{u_1, u_2, \ldots, u_k\}\) and \(V(K_1) = \{u\}\). Let \(H\) be the graph obtained from \(W_k\) by inserting \(r - 1\) vertices between every edge \(u_iu_j, 1 \leq i \leq k\). For every edge \(e = xy\) of \(H\), add a new vertex \(v_e\) and new edges \(v_ex, v_ey\). Denote by \(G\) the resulting graph. It is easy to check that \(rad(G) = r\), \(diam(G) = 2r\), \(\eta(G) = 3\) and \(\zeta(G) = 2r - 1\). By Theorem \(\square\), we have \(rc(G) \leq \Sigma_{j=1}^{\ell} (2 + 1, \zeta(G)) \leq r^2 + 2r - 2\). But, by Theorem \(\square\), we have \(rc(G) \leq 3r\). On the other hand, \(rc(G) \geq 2r\) since \(diam(G) = 2r\).

The remaining results are similar to those in Section 3.

Theorem 9 Let \(G\) be a plane graph. If the length of the boundary of every face is at most \(k\), then \(rc(G) \leq krad(G)\).

Corollary 4 Let \(G\) be a maximal plane (resp. outerplane) graph. Then \(rc(G) \leq 3rad(G)\).

Theorem 10 Let \(G\) be a bridgeless edge-transitive graph. Then \(rc(G) \leq rad(G)g(G)\), where \(g(G)\) is the girth of \(G\).

Theorem 11 Let \(G = (V_1 \cup V_2, E)\) be a bipartite graph with \(|V_1| = n\) and \(|V_2| = m\). If \(d(x) \geq k > \lceil m/2 \rceil\) for any \(x \in V_1\), \(d(y) \geq r > \lceil n/2 \rceil\) for any \(y \in V_2\), then \(rc(G) \leq 12\).

Remark 4 The degree condition is optimal. Let \(m, n\) be two even numbers with \(n, m \geq 2\). Since \(K_{n/2,m/2} \cup K_{n/2,m/2}\) is disconnected, \(rc(K_{n/2,m/2} \cup K_{n/2,m/2}) = \infty\).

Corollary 5 Let \(G = (V_1 \cup V_2, E)\) be a \(k\)-regular bipartite graph with \(|V_1| = |V_2| = n\). If \(k > \lceil n/2 \rceil\), then \(rc(G) \leq 12\).

The following theorem holds for general graphs.
Theorem 12  Let $G$ be a graph.

(i) If there exists an integer $k \geq 2$ such that $|N_k(u)| > n/2 - 1$ for every vertex $u$ in $G$, then $rc(G) \leq 4k^2 + 2k$.

(ii) If $\delta(G) > n/2$, then $rc(G) \leq 6$.

Remark 5  The above condition is almost optimal since $K_{n/2} \cup K_{n/2}$ is disconnected for even $n$.

Corollary 6  Let $G$ be a graph with minimum degree $\delta(G)$ and girth $g(G)$. If there exists an integer $k$ such that $k < g(G)/2$ and $\delta(G)(\delta(G) - 1)^k - 1 > n/2 - 1$, then $rc(G) \leq 4k^2 + 2k$.

Acknowledgments

The authors are very grateful to the referees for their helpful comments and suggestions.

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