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Bounding the monomial index and $(1, l)$-weight choosability of a graph

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Let $G = (V, E)$ be a graph. For each $e \in E(G)$ and $v \in V(G)$, let L_e and L_v, respectively, be a list of real numbers. Let w be a function on $V(G) \cup E(G)$ such that $w(e) \in L_e$ for each $e \in E(G)$ and $w(v) \in L_v$ for each $v \in V(G)$, and let c_w be the vertex colouring obtained by $c_w(v) = w(v) + \sum_{e \ni v} w(e)$. A graph is (k, l)-weight choosable if there exists a weighting function w for which c_w is proper whenever $|L_v| \geq k$ and $|L_e| \geq l$ for every $v \in V(G)$ and $e \in E(G)$.

A sufficient condition for a graph to be $(1, l)$-weight choosable was developed by Bartnicki, Grytczuk and Niwczyk (2009), based on the Combinatorial Nullstellensatz, a parameter which they call the monomial index of a graph, and matrix permanents. This paper extends their method to establish the first general upper bound on the monomial index of a graph, and thus to obtain an upper bound on l for which every admissible graph is $(1, l)$-weight choosable. Let $\partial_2(G)$ denote the smallest value s such that every induced subgraph of G has vertices at distance 2 whose degrees sum to at most s. We show that every admissible graph has monomial index at most $\partial_2(G)$ and hence that such graphs are $(1, \partial_2(G)+1)$-weight choosable. While this does not improve the best known result on $(1, l)$-weight choosability, we show that the results can be extended to obtain improved bounds for some graph products; for instance, it is shown that $G \square K_n$ is $(1, nd + 3)$-weight choosable if G is d-degenerate.

Keywords: 1-2-3 Conjecture, Combinatorial Nullstellensatz, permanents, graph colouring, graph labelling

1 Introduction

A graph $G = (V, E)$ will be simple and loopless unless otherwise stated. An **edge k-weighting**, w, of G is an assignment of a number from $[k] := \{1, 2, \ldots, k\}$ to each $e \in E(G)$, that is $w : E(G) \to [k]$. Karoński, Łuczak, and Thomason [5] conjecture that, for every graph without a component isomorphic to K_2, there is an edge 3-weighting such that the function $S : V(G) \to \mathbb{Z}$ given by $S(v) = \sum_{e \ni v} w(e)$ is a proper colouring of $V(G)$ (in other words, any two adjacent vertices have different sums of incident edge weights). If such a proper colouring S exists, then w is a **vertex colouring by sums**. Let $\chi^e_\Sigma(G)$ be the smallest value of k such that a graph G has an edge k-weighting which is a vertex colouring by sums. A graph G is **nice** if it contains no component isomorphic to K_2. Karoński, Łuczak, and Thomason's conjecture (frequently called the "1-2-3 Conjecture") may be expressed as follows:

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1-2-3 Conjecture. *If G is a nice graph, then $\chi^e_\Sigma(G) \leq 3$.*

The best known upper bound on $\chi^e_\Sigma(G)$ is due to Kalkowski, Karoński and Pfender [4], who show that $\chi^e_\Sigma(G) \leq 5$ if G is nice.

In [2], Bartnicki, Grytczuk and Niwczyk consider a list variation of the 1-2-3 Conjecture. Assign to each edge $e \in E(G)$ a list of k real numbers, say L_e, and choose a weight $w(e) \in L_e$ for each $e \in E(G)$. The resulting function $w : E(G) \to \cup_{e \in E(G)} L_e$ is called an **edge k-list-weighting**. Given a graph G, the smallest k such that any assignment of lists of size k to $E(G)$ permits an edge k-list-weighting which is a vertex colouring by sums is denoted $\mathrm{ch}^e_\Sigma(G)$ and called the **edge weight choosability number** of G. The following, stronger, conjecture is proposed in [2]:

List 1-2-3 Conjecture. *If G is a nice graph, then $\mathrm{ch}^e_\Sigma(G) \leq 3$.*

A similar problem to the List 1-2-3 Conjecture for graphs is solved for digraphs in [2], where a constructive method is used to show that $\mathrm{ch}^e_\Sigma(D) \leq 2$ for any digraph D. An alternate proof which uses Alon's Combinatorial Nullstellensatz [1] may be found in [6].

Another variant of the 1-2-3 Conjecture allows each vertex $v \in V(G)$ to receive a weight $w(v)$; the colour of v is then $w(v) + \sum_{e \ni v} w(e)$ rather than $\sum_{e \ni v} w(e)$. Such a function $w : V \cup E \to [k]$ is called a **total k-weighting**. The smallest k for which G has a total k-weighting that is a proper colouring by sums is denoted $\chi^t_\Sigma(G)$. A similar list generalization as above may be considered; the smallest k such that the list version holds is denoted $\mathrm{ch}^t_\Sigma(G)$. The following two conjectures are posed in [9] and [10, 14] respectively:

1-2 Conjecture. *If G is any graph, then $\chi^t_\Sigma(G) \leq 2$.*

List 1-2 Conjecture. *If G is any graph, then $\mathrm{ch}^t_\Sigma(G) \leq 2$.*

Though the 1-2 Conjecture remains open, Kalkowski [3] has shown that a total weighting w of G which properly colours $V(G)$ by sums always exists with $w(v) \in \{1, 2\}$ and $w(e) \in \{1, 2, 3\}$ for all $v \in V(G), e \in E(G)$.

In [14], Wong and Zhu study (k, l)**-total list-assignments**, which are assignments of lists of size k to the vertices of a graph and lists of size l to the edges. If any (k, l)-total list-assignment of G permits a total weighting which is a vertex colouring by sums, then G is (k, l)**-weight choosable**. Obviously, if a graph G is (k, l)-weight choosable, then $\mathrm{ch}^t_\Sigma(G) \leq \max\{k, l\}$. The List 1-2 Conjecture is equivalent to the statement that every graph is $(2, 2)$-choosable. Wong and Zhu [14] further conjecture that every nice graph is $(1, 3)$-weight choosable, a strengthening of the List 1-2-3 Conjecture. A recent breakthrough by Wong and Zhu [13] shows that every graph is $(2, 3)$-weight choosable and hence $\mathrm{ch}^t_\Sigma(G) \leq 3$ for every graph G. There is a good deal of literature on graph classes which are (k, l)-weight choosable for small values of k and l (see [2, 8, 10, 11, 12, 14]). Of particular note, it is shown in [8] that every nice d-degenerate graph is $(1, 2d)$-weight choosable. However, whether or not there exists a constant l such that every nice graph is $(1, l)$-weight choosable remains an open question.

The purpose of this paper is show how the methods used in [2] can be extended to obtain the following general result. Let $G = (V, E)$ be a nice graph, with $E(G) = \{e_1, \ldots, e_m\}$ and $V(G) = \{v_1, \ldots, v_n\}$. Let $\partial_2(G)$ denote the smallest value s such that every induced subgraph of G has vertices at distance 2 whose degrees sum to at most s. Associate with each e_i the variable x_i, with each v_j the variable x_{m+j}, and let $X_{v_j} = \sum_{e_i \ni v_j} x_i$ for each v_j. We show that the polynomial $\prod_{(u,v) \in E(D)} (X_v - X_u)$ has a monomial term in its expansion with non-zero coefficient for which no exponent exceeds $\partial_2(G)$. Based on the work of [2], this implies that every nice graph is $(1, \partial_2(G) + 1)$-weight choosable.

The structure of the paper is as follows. In Section 2, we present a Combinatorial Nullstellensatz approach to the List 1-2-3 and List 1-2 Conjectures, which relies on calculating permanents of matrices which arise from natural colouring polynomials (one of which is given in the previous paragraph). Section 3 contains some intermediary lemmas on matrix permanents and colouring polynomials. Sections 2 and 3 are largely reliant on the results found in [2, 10]; results are presented in near full detail, with examples, in the interest of keeping the article self-contained. Some results are generalized where necessary. Section 4 contains the main result of this paper, given above. The result that every nice graph is $(1, \partial_2(G)+1)$-weight choosable is, unfortunately, weaker than that of Pan and Yang [8], however we are able to obtain improved bounds for some graph products in Section 5.

2 The permanent method and Alon's Nullstellensatz

Let $G = (V, E)$ be a graph, with $E(G) = \{e_1, \dots, e_m\}$ and $V(G) = \{v_1, \dots, v_n\}$. Associate with each e_i the variable x_i and with each v_j the variable x_{m+j}. Define two more variables for each $v_j \in V(G)$: $X_{v_j} = \sum_{e_i \ni v_j} x_i$ and $Y_{v_j} = x_{m+j} + X_{v_j}$. For an orientation D of G, define the following two polynomials, where $l = m + n$:

$$P_D(x_1, \dots, x_m) = \prod_{(u,v) \in E(D)} (X_v - X_u) \tag{1}$$

$$T_D(x_1, \dots, x_l) = \prod_{(u,v) \in E(D)} (Y_v - Y_u). \tag{2}$$

Let w be an edge weighting of G. By letting $x_i = w(e_i)$ for $1 \leq i \leq m$, w is a proper vertex colouring by sums if and only if $P_D(w(e_1), \dots, w(e_m)) \neq 0$. A similar conclusion can be made about T_D if w is a total weighting of G. This leads us to the problem of determining when the polynomials P_D and T_D do not vanish everywhere, i.e., when there exist values of the variables for which the polynomial is non-zero. Alon's famed Combinatorial Nullstellensatz gives sufficient conditions to guarantee that a polynomial does not vanish everywhere.

Combinatorial Nullstellensatz (Alon [1])**.** *Let $\mathbb{F}$ be an arbitrary field, and let $f = f(x_1, \dots, x_n)$ be a polynomial in $\mathbb{F}[x_1, \dots, x_n]$. Suppose the total degree of f is $\sum_{i=1}^n t_i$, where each t_i is a nonnegative integer, and suppose the coefficient of $\prod_{i=1}^n x_i^{t_i}$ in f is nonzero. If $S_1, \dots, S_n$ are subsets of $\mathbb{F}$ with $|S_i| > t_i$, then there are $s_1 \in S_1, s_2 \in S_2, \dots, s_n \in S_n$ so that*

$$f(s_1, \dots, s_n) \neq 0.$$

For a polynomial $P \in \mathbb{F}[x_1, \dots, x_l]$ and a monomial term M of P, let $h(M)$ be the largest exponent of any variable in M. The **monomial index** of P, denoted $\mathrm{mind}(P)$, is the minimum $h(M)$ taken over all monomials of P. Define the graph parameters $\mathrm{mind}(G) := \mathrm{mind}(P_D)$ and $\mathrm{tmind}(G) := \mathrm{mind}(T_D)$, where D is an orientation of G. Note that, given a graph G and two orientations D and D', $P_D(x_1, \dots, x_l) = \pm P_{D'}(x_1, \dots, x_l)$; a similar argument holds for T_D. The parameters $\mathrm{mind}(G)$ and $\mathrm{tmind}(G)$ are hence well-defined. Note that, for any graph G, $\mathrm{tmind}(G) \leq \mathrm{mind}(G)$.

The following lemma is obtained by applying the Combinatorial Nullstellensatz to P_D and T_D:

Lemma 2.1. *Let G be a graph and k a positive integer.*

1. (Bartnicki, Grytczuk, Niwczyk [2]) *If G is nice and* $\mathrm{mind}(G) \leq k$*, then* $\mathrm{ch}^e_\Sigma(G) \leq k + 1$.

2. (Przybyło, Woźniak [10]) *If* $\operatorname{tmind}(G) \leq k$, *then* $\operatorname{ch}^t_\Sigma(G) \leq k+1$.

More generally, Wong and Zhu show the following:

Lemma 2.2 (Wong, Zhu [14])**.** *Let G be a nice graph, D an orientation of G, and*

$$M = c x_1^{a_1} \cdots x_m^{a_m} x_{m+1}^{a_{m+1}} \cdots x_l^{a_l}$$

a monomial term of $T_D(x_1, \ldots, x_l)$ *(equation (2)) with* $c \neq 0$. *If* $\max\{a_i : 1 \leq i \leq m\} = l$ *and* $\max\{a_i : m+1 \leq i \leq l\} = k$, *then G is* $(k+1, l+1)$*-weight choosable.*

This leads to the following simple corollary:

Corollary 2.3. *If G is a nice graph, then G is* $(1, \operatorname{mind}(G)+1)$*-weight choosable.*

The following proposition allows us to consider only connected graphs.

Proposition 2.4. *If G is a graph with connected components* $G_1, G_2, \ldots, G_k$, *then*

$$\operatorname{mind}(G) = \max\{\operatorname{mind}(G_i) : 1 \leq i \leq k\}.$$

In [2], Bartnicki et al. show how one may study the permanent of particular $\{-1, 0, 1\}$-matrices in order to gain insight on $\operatorname{mind}(G)$ and $\operatorname{tmind}(G)$. Let $\mathbb{M}(m, n)$ denote the set of all real valued matrices with m rows and n columns, and $\mathbb{M}(m)$ denote the set of square $m \times m$ matrices. The permanent of a matrix $A \in \mathbb{M}(m)$, denoted $\operatorname{per} A$, is calculated as follows:

$$\operatorname{per} A = \sum_{\sigma \in S_m} \prod_{i=1}^{m} a_{i,\sigma(i)}.$$

The permanent may also be defined for a general matrix $A \in \mathbb{M}(m, n)$ if $n \geq m$. Let $Q_{m,n}$ denote the set of sequences of length m with entries from $[n]$ which contain no repetition of elements; such sequences are also known as m**-permutations** from $[n]$. For example, $Q_{2,3} = \{(1,2), (1,3), (2,1), (2,3), (3,1), (3,2)\}$. The permanent of A is defined as follows:

$$\operatorname{per} A = \sum_{\alpha \in Q_{m,n}} \prod_{i=1}^{m} a_{i,\alpha(i)} = \sum_{i=1}^{\binom{n}{m}} \operatorname{per} B_i,$$

where $\{B_i \mid 1 \leq i \leq \binom{n}{m}\}$ is the set of all $m \times m$ submatrices of A.

The **permanent rank** of a matrix A (not necessarily square) is the size of the largest square submatrix of A having nonzero permanent. Let $A^{(k)} = [AA \cdots A]$ denote the matrix formed of k consecutive copies of A. If A has size $m \times l$, then the **permanent index** of A is the smallest k, if it exists, such that $A^{(k)}$ has permanent rank m. This parameter is denoted $\operatorname{pind}(A)$. If such a k does not exist, then $\operatorname{pind}(A) := \infty$. Alternately, $\operatorname{pind}(A)$ is the smallest k such that a square matrix of size m having nonzero permanent can be constructed by taking columns from A, each column taken no more than k times.

There are three matrices related to directed graphs which will be of interest:

Definition 2.5. *Let* $G = (V, E)$ *be a graph,* $V(G) = \{v_1, \ldots, v_n\}$, $E(G) = \{e_1, \ldots, e_m\}$. *For an orientation D of G, define the matrices* $A_D \in \mathbb{M}(m)$, $B_D \in \mathbb{M}(m, n)$, *and* $M_D \in \mathbb{M}(m, m+n)$ *as follows:*

- $A_D = (a_{i,j})$ *where* $a_{i,j} = \begin{cases} 1 & \text{if } e_j \text{ is incident with the head of } e_i \\ -1 & \text{if } e_j \text{ is incident with the tail of } e_i \\ 0 & \text{otherwise} \end{cases}$

- $B_D = (b_{i,j})$ *where* $b_{i,j} = \begin{cases} 1 & \text{if } v_j \text{ is the head of } e_i \\ -1 & \text{if } v_j \text{ is the tail of } e_i \\ 0 & \text{otherwise} \end{cases}$

- $M_D = (A_D \mid B_D)$.

The following lemmas, which relate the matrices A_D, B_D, and M_D to the polynomials P_D and T_D, provide the fundamental link between the graphic polynomials of interest and matrix permanents:

Lemma 2.6 (Bartnicki, Grytczuk, Niwczyk [2])**.** *Let* $A = (a_{ij}) \in \mathbb{M}(m)$ *have finite permanent index. If* $P(x_1, \ldots, x_m) = \prod_{i=1}^{m}(a_{i1}x_1 + \ldots + a_{im}x_m)$*, then* $\text{mind}(P) = \text{pind}(A_D)$.

The proof is omitted, but the result follows from the fact that the coefficient of $x_1^{k_1} x_2^{k_2} \cdots x_m^{k_m}$ in the expansion of P is equal to $\frac{\text{per}\,(M)}{k_1! \cdots k_m!}$ where M is the $m \times m$ matrix where column a_j from A appears k_j times. Lemma 2.6 immediately implies the following vital link between the (total) monomial index of a graph G and the permanent index of A_D (respectively, T_D) for any orientation D of G:

Lemma 2.7. *Let D be an orientation of a graph G.*

1. (Bartnicki, Grytczuk, Niwczyk [2]) *If G is nice, then* $\text{mind}(G) = \text{pind}(A_D)$.

2. (Przybyło, Woźniak [10]) *For any graph G,* $\text{tmind}(G) = \text{pind}(M_D)$.

Lemmas 2.1 and 2.7 imply that $\text{ch}^e_\Sigma(G) \le \text{pind}(A_D) + 1$ (if G is nice) and $\text{ch}^t_\Sigma(G) \le \text{pind}(M_D) + 1$.

We note the following general result of Wong and Zhu [14]. Let G be a graph, D an orientation of G, and M_D the matrix defined above. Suppose a matrix $M \in \mathbb{M}(m)$ has only columns taken from M_D and has $\text{per}\,(M) \neq 0$. If no column associated with an edge $e \in E(G)$ appears more than l times and no column associated with a vertex $v \in V(G)$ appears more than k times, then G is $(k+1, l+1)$-weight choosable. However, in light of the theorem in [13] stating that every graph is $(2, 3)$-weight choosable, it is now sufficient to consider only $(1, l)$-weight choosability (and hence to bounding $\text{mind}(G)$) and $(2, 2)$-weight choosability (and hence trying to show that $\text{tmind}(G) = \text{pind}(M_D) = 1$).

In summary, to determine upper bounds on $\text{ch}^e_\Sigma(G)$, $\text{ch}^t_\Sigma(G)$ or values of k and l for which G is (k, l)-weight choosable, it is sufficient to consider the permanents of matrices obtained by replicating columns from M_D for some orientation D of G. Consider the following illustrative example. Let D be the digraph in Figure 1 and let G be its underlying simple graph.

The associated polynomial, P_D, is

$$\begin{aligned} P_D(x_1, \ldots, x_6) = &(x_1 + x_4 - x_1 - x_2 - x_3) \times (x_2 + x_5 - x_1 - x_2 - x_3) \\ &\times (x_1 + x_2 + x_3 - x_3 - x_5 - x_6) \times (x_1 + x_4 - x_4 - x_6) \\ &\times (x_3 + x_5 + x_6 - x_2 - x_5) \times (x_4 + x_6 - x_3 - x_5 - x_6) \\ = &(x_4 - x_2 - x_3) \times (x_5 - x_1 - x_3) \times (x_1 + x_2 - x_5 - x_6) \\ &\times (x_1 - x_6) \times (x_3 + x_6 - x_2) \times (x_4 - x_3 - x_5). \end{aligned}$$

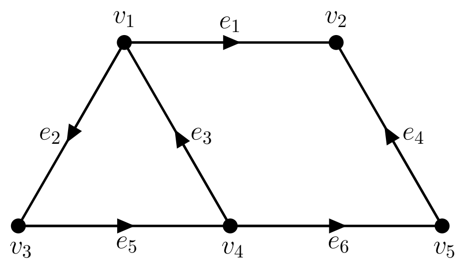

Fig. 1: A digraph used to illustrate A_D, B_D, and M_D

Note that for each factor f of P_D, the coefficients of $x_1, \ldots, x_6$ are equal to the entries appearing on the row of A_D corresponding to f. We have:

$$M_D = [A_D \,|\, B_D] = \left(\begin{array}{cccccc|ccccc} 0 & -1 & -1 & 1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 & 1 & 0 & -1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & -1 & -1 & 1 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & -1 \\ 0 & -1 & 1 & 0 & 0 & 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 & -1 & 0 & 0 & 0 & 0 & -1 & 1 \end{array} \right).$$

Since $\operatorname{per} A_D = -4 \neq 0$, we have $\operatorname{pind}(A_D) = 1$ (each column from A_D is chosen once). By Lemma 2.7(1), $\operatorname{mind}(G) = 1$, and so G is $(1,2)$-weight choosable by Corollary 2.3. Note that no graph is $(1,1)$-weight choosable, so $l = 2$ is the minimum value for which G is $(1,l)$-weight choosable. We also clearly have that $\operatorname{ch}^e_\Sigma(G) \leq 2$. Since there are adjacent vertices of equal degree in G, we have $\chi^e_\Sigma(G) \neq 1$, implying $\operatorname{ch}^e_\Sigma(G) \neq 1$ and so $\operatorname{ch}^e_\Sigma(G) = 2$.

3 Some intermediary results on permanent indices and monomial indices

The major results of this paper are proven by establishing bounds on $\operatorname{mind}(G)$ using the permanent method outlined in the previous section. One important tool is the following lemma, a generalization of a similar result in [2]:

Lemma 3.1 (Przybyło, Woźniak [10]). *Let A be an $m \times l$ matrix, and let L be an $m \times m$ matrix where each column of L is a linear combination of columns of A. Let n_j denote the number of columns of L in which the j^{th} column of A appears with nonzero coefficient. If* $\operatorname{per} L \neq 0$, *then* $\operatorname{pind}(A) \leq \max\{n_j \mid j = 1, \ldots l\}$.

We will also find the following theorem useful, which gives a method for constructing graphs in a way that preserves the property of having low monomial index:

Theorem 3.2 (Bartnicki, Grytczuk, Niwczyk [2]). *Let G be a simple graph with* $\operatorname{mind}(G) \leq 2$. *Let U be a nonempty subset of $V(G)$. If F is a graph obtained by adding two new vertices u, v to $V(G)$ and joining them to each vertex of U, and H is a graph obtained from F by joining u and v, then* $\operatorname{mind}(F), \operatorname{mind}(H) \leq 2$.

As a consequence, the following graph classes have low monomial index and hence small values of $\mathrm{ch}^e_\Sigma(G)$ by Corollary 2.1(1):

Corollary 3.3 (Bartnicki, Grytczuk, Niwczyk [2])**.** *If G is a complete graph, a complete bipartite graph, or tree, then* $\mathrm{mind}(G) \le 2$*.*

Proposition 3.4. *If $G = C_n$, then* $\mathrm{mind}(G) \le 2$*.*

Proof: Let $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $E(G) = \{v_1v_2, v_2v_3, \ldots, v_{n-1}v_n, v_nv_1\}$. Let D be the orientation of G with $A(D) = \{(v_1, v_2), (v_2, v_3), \ldots, (v_{n-1}, v_n), (v_n, v_1)\}$. Consider the colouring polynomial

$$P_D = (x_2 - x_n)(x_3 - x_1)(x_4 - x_2)\cdots(x_n - x_{n-2})(x_1 - x_{n-1}).$$

Since each variable appears in exactly two factors of P_D, no exponent in the expansion of P_D exceeds 2, and hence $\mathrm{mind}(G) \le 2$. $\square$

In order to prove our major results in Section 4, the following generalization of Theorem 3.2 is required:

Lemma 3.5. *Let G be a graph with finite monomial index* $\mathrm{mind}(G) \ge 1$*. Let U be a nonempty subset of $V(G)$. If F is a graph obtained by adding two new vertices u, v to $V(G)$ and joining them to each vertex of U, and F^* is a graph obtained from F by joining u and v, then* $\mathrm{mind}(F), \mathrm{mind}(F^*) \le \max\{2, \mathrm{mind}(G)\}$*.*

The proof which follows is an adaptation of the proof of Theorem 3.2 found in [2]. Given a matrix A with columns $a_1, a_2, \ldots, a_n$ and a sequence of (not necessarily distinct) column indices $K = (i_1, i_2, \ldots, i_k)$, $A(K)$ is defined to be the matrix $A(K) = \begin{pmatrix} a_{i_1} & a_{i_2} & \cdots & a_{i_k} \end{pmatrix}$.

Proof: Let $U = \{u_1, \ldots, u_k\}$ be the subset of $V(G)$ stated in the theorem. Let $E_u = \{e_1, e_3, \ldots, e_{2k-1}\}$ and $E_v = \{e_2, e_4, \ldots, e_{2k}\}$ be the sets of edges incident to the vertices u and v, respectively. Assume that these edges are oriented toward U, and that for each $i = 1, 2, \ldots, k$ the edges e_{2i-1} and e_{2i} have the same head.

Let D be an orientation of F, D' the induced orientation of G, and consider the matrices A_D and $A_{D'}$. Let $A_1, \ldots, A_{2k}$ be the first $2k$ columns of A_D, corresponding to $\{e_1, e_2, \ldots, e_{2k}\}$. If we write $A = (A_1 \ \cdots \ A_{2k})$, then $A_D = (A \ B)$ where $B = \begin{pmatrix} X \\ A_{D'} \end{pmatrix}$.

Let Y be the $(2k) \times (2k)$ matrix and Z the $(|E(F)| - 2k) \times (2k)$ matrix such that $A = \begin{bmatrix} Y \\ Z \end{bmatrix}$. Since the edges e_{2i-1} and e_{2i} have the same head for each $i = 1, 2, \ldots, k$, the columns A_{2i-1} and A_{2i} agree on Z. Furthermore, Y may be written as a block matrix, where $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ occupies the diagonals and $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ is everywhere else, as seen in Figure 2.

There exists a matrix of columns from $A_{D'}$, with no column used more than $\mathrm{mind}(G)$ times, with nonzero permanent. Let K denote the sequence of edges of G which index this matrix. Consider a new matrix

$$M = \begin{pmatrix} M_1 & M_1 & M_2 & M_2 & \cdots & M_k & M_k & B(K) \end{pmatrix},$$

where $M_j = A_{2j-1} - A_{2j}$ for $j = 1, 2, \ldots, k$.

The properties of the columns of A outlined above imply that the matrix M can be written as follows:

$$Y = \begin{pmatrix} 0 & 1 & -1 & 0 & & -1 & 0 \\ 1 & 0 & 0 & -1 & \cdots & 0 & -1 \\ -1 & 0 & 0 & 1 & & -1 & 0 \\ 0 & -1 & 1 & 0 & & 0 & -1 \\ & \vdots & & & \ddots & & \\ -1 & 0 & -1 & 0 & & 0 & 1 \\ 0 & -1 & 0 & -1 & & 1 & 0 \end{pmatrix}.$$

Fig. 2: The block matrix Y

$M = \left(\begin{smallmatrix} R & X(K) \\ 0 & A_{D'}(K) \end{smallmatrix}\right)$, where R has all constant rows:

$$R = \begin{pmatrix} -1 & -1 & -1 & & -1 \\ 1 & 1 & 1 & \cdots & 1 \\ -1 & -1 & -1 & & -1 \\ & \vdots & & \ddots & \\ 1 & 1 & 1 & & 1 \end{pmatrix}.$$

Since $\operatorname{per} M = \operatorname{per} R \times \operatorname{per} A_{D'}(K)$, each of $\operatorname{per} R$ and $\operatorname{per} A_G(K)$ are nonzero, and any column of A appears in the linear combination of at most 2 columns of M, Lemma 3.1 implies that $\operatorname{mind}(F) \leq \max\{2, \operatorname{mind}(G)\}$.

We now consider F^*. Let H be an orientation of F^* with $e_0 = uv$ oriented from v to u. The matrix A_H is precisely A_D with a row and column added for e_0 (say, as the first row and column). It can be depicted in block form $A_H = \left(\begin{smallmatrix} Y' & X' \\ Z' & A_G \end{smallmatrix}\right)$, where Y' and Z' are the matrices depicted in Figure 3.

$$Y' = \begin{pmatrix} 0 & 1 & -1 & \cdots & 1 & -1 \\ -1 & & & & & \\ -1 & & & & & \\ \vdots & & & Y & & \\ -1 & & & & & \\ -1 & & & & & \end{pmatrix}, \quad Z' = \begin{pmatrix} 0 & \\ 0 & \\ \vdots & Z \\ 0 & \\ 0 & \end{pmatrix}$$

Fig. 3: The matrices Y' and Z'

Let $A_0, A_1, \ldots, A_{2k}$ denote the first $2k+1$ columns of A_H, corresponding to the edges $e_0, e_1, \ldots, e_{2k}$. Form a new matrix

$$N = (N_0 \ N_0 \ N_1 \ N_2 \ N_2 \ \cdots \ N_k \ N_k \ B(K)),$$

so that $N_0 = A_0$ and $N_j = A_{2j-1} - A_{2j}$ for $j = 1, 2, \ldots, k$. Arguing as before, $N = \left(\begin{smallmatrix} R' & X'(K) \\ 0 & A_G(K) \end{smallmatrix}\right)$, where R' is the following square matrix:

$$R' = \begin{pmatrix} 0 & 0 & 2 & 2 & & 2 \\ -1 & -1 & -1 & -1 & \cdots & -1 \\ -1 & -1 & 1 & 1 & & 1 \\ & \vdots & \vdots & & \ddots & \\ -1 & -1 & 1 & 1 & & 1 \end{pmatrix}.$$

It is shown in [2] that per $R' \neq 0$. Hence per $N =$ per $R' \times$ per $A_G(K) \neq 0$, and since any column of A appears in the linear combination of at most 2 columns of N, Lemma 3.1 implies thats $\text{mind}(H) \leq \max\{2, \text{mind}(G)\}$. □

4 A general bound for a graph's monomial index

Armed with the Combinatorial Nullstellensatz and the permanent method, we may now proceed with our main results.

Recall that a graph G is *d-degenerate* if every induced subgraph of G has a vertex of degree at most d. If G and H are graphs, we write $H \leq_i G$ to denote that H is an induced subgraph of G. The degeneracy of a graph G, which we denote by $\partial(G)$, is the smallest d for which G is d-degenerate; that is $\partial(G) = \max\{\delta(H) | H \leq_i G\}$. We extend the notion of degeneracy to pairs of vertices at a given distance. Given an integer $r \geq 1$, let $\delta_r(G)$ denote the minimum value of $d(u) + d(v)$ for two vertices $u, v \in V(G)$ at distance exactly r in G. The *r-degeneracy* of G, denoted $\partial_r(G)$, is

$$\partial_r(G) = \max\{\delta_r(H) | H \leq_i G\}.$$

If no induced subgraph of G has vertices at distance exactly r (for example, $G = K_n$ and $r \geq 2$), then we adopt the convention that $\partial_r(G) = 2\Delta(G)$.

We now show that $\text{mind}(G)$ is at most $\partial_2(G)$. The result is achieved by carefully orienting the edges of a graph and applying the lemmas from the previous sections to show that our desired matrix has non-zero permanent.

Theorem 4.1. *If G is a nice graph, then* $\text{mind}(G) \leq \partial_2(G)$.

Proof: We may assume that G is connected, since Proposition 2.4 states that $\text{mind}(G)$ is at most the largest monomial index of its components. If G is a tree, cycle, or complete graph, then $\text{mind}(G) \leq 2$ by Corollary 3.3 and Proposition 3.4, and hence the theorem holds for the following graphs: P_3, K_3, P_4, $K_{1,3}$, C_4, and K_4. If G is isomorphic to K_3 with a leaf or C_4 with a chord, then one may check that the theorem holds for G by straightforward computation of the associated colouring polynomial P_D for any orientation D. Hence, the theorem holds for any connected graph on 3 or 4 vertices.

We proceed now by induction on $|V(G)|$. Let G be a connected graph on at least 5 vertices, and for any graph H with $|V(H)| < |V(G)|$, assume that $\text{mind}(H) \leq \partial_2(H)$.

If G is a complete graph, then the theorem holds by Corollary 3.3. Assume that G is not complete. There exist $u, v, w \in V(G)$ such that the induced subgraph $G[\{u, v, w\}]$ is a path of length 2 (or, uvw is an induced 2-path). Choose this 2-path such that $d(u) + d(w)$ is minimum (and, hence, $d(u) + d(w) \leq \partial_2(G)$). The ultimate goal will be to apply an inductive argument to $G - \{u, w\}$, however we must concern ourselves with whether or not this subgraph of G is nice. To this end, we define the following sets of edges:

$$\begin{aligned} \mathcal{F} &= \text{ the edges of those components in } G-\{u,w\} \text{ isomorphic to } K_2 \\ E_u &= \{e \in E(G) \,|\, e \ni u, e \neq uv\} \\ E_w &= \{e \in E(G) \,|\, e \ni w, e \neq vw\} \\ E_v &= \{e \in E(G) \,|\, e \ni v, e \neq uv, vw\} \\ E^* &= E(G) \setminus (E_u \cup E_v \cup E_w \cup \{uv, vw\}) \end{aligned}$$

The path uvw and the sets of edges E_u, E_v, E_w are shown in Figure 4.

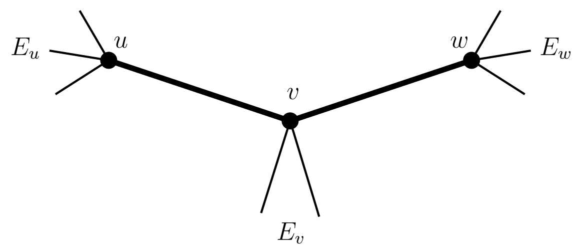

Fig. 4: The induced 2-path uvw in G

Case 1: $E_v \cap \mathcal{F} \neq \emptyset$

If $E_v \cap \mathcal{F} \neq \emptyset$, then there can be only one edge in this intersection, otherwise the connected component containing v in $G-\{u,w\}$ would have two or more edges. Since $uv, vw \in E(G)$, it follows that $N_G(v) = \{u,w,x\}$ for some vertex $x \in V(G)$. Since $\{v,x\}$ induces a graph isomorphic to K_2 in $G-\{u,w\}$, we have that $N_G(x) \subseteq \{u,v,w\}$.

If x is adjacent to both u and w, then v and x are adjacent twins. Suppose that $G \setminus \{v,x\}$ is not nice; we will show that this contradicts our choice of uvw which minimizes $d_G(u) = d_G(w)$. If G is not nice, then u, w, or both u and w are adjacent to exactly one vertex in G other than v and x; without loss of generality, suppose that $uy \in E(G)$, $y \neq v, x$. Since $y \notin N_G(x)$, the vertices y, u, x induce a 2-path; furthermore, $d_G(y) + d_G(x) = 1 + 3 = 4$. This contracts our choice of uvw, since $d(u) + d(w) \geq 3 + 2 = 5$. Thus, $G - \{v,x\}$ is a nice graph, and so, by Lemma 3.5, mind$(G) \leq \max\{2, \text{mind}(G-\{v,x\})\}$. By the induction hypothesis, mind$(G-\{v,x\}) \leq \partial_2(G-\{v,x\})$, and so

$$\text{mind}(G) \leq \max\{2, \text{mind}(G-\{v,x\})\} \leq \max\{2, \partial_2(G-\{v,x\})\} \leq \partial_2(G).$$

We may now assume that x is not adjacent to at least one of u and w. If $w \notin N_G(x)$, then both uvw and xvw are induced 2-paths in G. By the minimality of $d(u)+d(w)$, we must have that $d(u) \leq d(x)$. If u is adjacent to x, then $d(x) = 2$ and, since u is adjacent to v as well,

$d(u) = 2$ and $N_G(u) = \{v, x\}$. Otherwise, if u is not adjacent to x, then $d_G(u) = d_G(x) = 1$ and $N_G(u) = \{v\}$. In either case, u and x are twins. If $G - \{u, x\}$ is not nice, then the only edge not incident to u or x is the edge vw, contradicting our choice of G with $|V(G)| \geq 5$. Assume that $G - \{u, x\}$ is nice. By Lemma 3.5, mind$(G) \leq \max\{2, \text{mind}(G - \{u, x\})\}$, and by the induction hypothesis, mind$(G - \{u, x\}) \leq \partial_2(G - \{u, x\}))$. Thus,

$$\text{mind}(G) \leq \max\{2, \text{mind}(G - \{u, x\})\} \leq \max\{2, \partial_2(G - \{u, x\})\} \leq \partial_2(G).$$

If $u \notin N_G(x)$ and $w \in N_G(x)$, then the exact same argument holds as for $u \in N_G(x)$ and $w \notin N_G(x)$. Having considered all possible neighbourhoods of x, we conclude that if $E_v \cap \mathcal{F}$ is nonempty, then mind$(G) \leq \partial_2(G)$.

Case 2: $E_v \cap \mathcal{F} = \emptyset$

Suppose that $E_v \cap \mathcal{F} = \emptyset$. The argument proceeds as follows: after choosing a "good" orientation D of G, we will construct a matrix whose columns are linear combinations of A_D with no column of A_D being used more than $\partial_2(G)$ times and with nonzero permanent. The result then follows by Lemma 3.1.

Let D be an orientation of G where the edges of $E_u \cup \{uv\}$ and E_v are oriented toward u and v, respectively, and the edges of $E_w \cup \{vw\}$ are oriented away from w; see Figure 5. Let

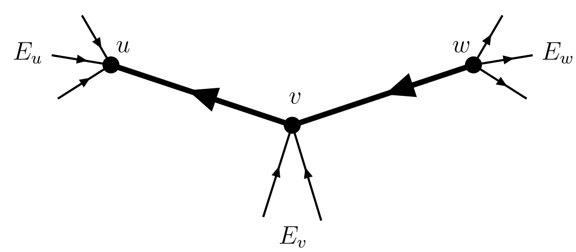

Fig. 5: An orientation D of a graph G with an induced 2-path uvw

c_{uv} and c_{vw} be the columns of A_D associated with the edges uv and vw, respectively, and let $c = c_{uv} - c_{vw}$; see Figure 6.

We must still concern ourselves with the possibility that deleting u and w from G gives a graph which is not nice. If a component of $G - \{u, w\}$ is isomorphic to K_2, then one vertex of this component must be adjacent to either u or w in G. Let $\mathcal{F} = \{f_1, \ldots, f_k\}$ be the set of edges belonging to the k connected components of $G - \{u, w\}$ that are isomorphic to K_2. For each $f_i \in \mathcal{F}$, let e_i be an edge from E_u or E_w to which f_i is adjacent. Let F denote this collection of edges from $E_u \cup E_w$, and let $F_u = \{e : e \in E_u \cap F\}$ and $F_w = \{e : e \in E_w \cap F\}$. Each edge $f_i \in \mathcal{F}$ will be oriented away from its shared endpoint with e_i.

Let $H = G - \{u, w\} - \mathcal{F}$ and $D(H)$ be the corresponding sub-digraph of D. Since we have removed all components isomorphic to K_2, H is nice. Since H has fewer vertices than G, by

$$
\begin{array}{rc}
 & \begin{array}{ccc} c_{uv} & c_{vw} & \cdots \end{array} \\
\begin{array}{r} uv \\ vw \\ \\ E_u \\ \\ \\ E_w \\ \\ \\ E_v \\ \\ \\ E^* \\ \\ \end{array} &
\left(\begin{array}{ccc}
0 & -1 & \\
1 & 0 & \\
1 & 0 & \\
\vdots & \vdots & \\
1 & 0 & \\
0 & -1 & \\
\vdots & \vdots & \cdots \\
0 & -1 & \\
1 & 1 & \\
\vdots & \vdots & \\
1 & 1 & \\
0 & 0 & \\
\vdots & \vdots & \\
0 & 0 &
\end{array}\right)
\end{array}
\Longrightarrow \quad c = \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \\ 1 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}
$$

Fig. 6: An operation on two columns of A_D

the induction hypothesis, $\mathrm{mind}(H) \leq \partial_2(H)$. Hence, there exists a matrix L_H consisting of columns of $A_{D(H)}$, none repeated more than $\partial_2(H)$ times, with $\mathrm{per}\,(L_H) \neq 0$. Let K denote the sequence of edges which indexes the columns of L_H. For an $m \times n$ matrix A, recall that $A^{(k)}$ is the $m \times kn$ matrix consisting of k consecutive copies of A (see page 176). Let L_G be the following block matrix:

$$
L_G = \left(c^{(d(u)+d(w))} \;\middle|\; A_D(F) \;\middle|\; A_D(K) \right) = \begin{array}{r} {\scriptstyle E_u\cup E_w\cup\{uv,vw\}} \\ {\scriptstyle \mathcal{F}} \\ {\scriptstyle E(H)} \end{array} \begin{pmatrix} J_{d(u)+d(w)} & K_1 & X_1 \\ 0 & K_2 & X_2 \\ 0 & 0 & L_H \end{pmatrix},
$$

where the blocks are as follows:

- $J_{d(u)+d(w)}$ is the $(d(u)+d(w)) \times (d(u)+d(w))$ all 1's matrix.
- $K = \binom{K_1}{K_2}$ having entries depending on whether the column is indexed by $e_i \in F_u$ or $e_i \in F_w$. If the column is indexed by $e_i \in F_u$, then the column will have (i) 1 in each row indexed by the other edges from E_u, (ii) 1 in the row indexed by uv, (iii) -1 in the row indexed by f_i, and (iv) 0 in all other entries. Otherwise, the entries follow the same pattern with the signs swapped. Since the column associated with e_i has only one non-zero entry in the rows indexed by $\mathcal{F}$, K_2 is diagonal with $|F_u|$ entries being -1 and $|F_w|$ entries being 1.
- $X = \binom{X_1}{X_2}$, the $(|E(G)| - |E(H)|) \times |E(H)|$ submatrix of $A_D(K)$ whose rows are indexed by $E(G) \setminus E(H)$; and
- L_H, is the matrix with $\mathrm{per}\,(L_H) \neq 0$ defined above.

Since $J_{d(u)+d(w)}$, K_2, and L_H are all square matrices,

$$\begin{aligned}\text{per}\,(L_G) &= \text{per}\,\left(J_{d(u)+d(w)}\right)\cdot \text{per}\,(K_2)\cdot \text{per}\,(L_H)\\ &= (d(u)+d(w))!\cdot(-1)^{|F_u|}\,(1)^{|F_w|}\cdot \text{per}\,(L_H)\neq 0.\end{aligned}$$

Since the sets $\{uv, vw\}$, F, and $E(H)$ are pairwise disjoint, no column is used more than $\max\{d(u)+d(w),1,\text{mind}(H)\}$ times. Lemma 3.1 implies that $\text{pind}(A_D)\leq\max\{d(u)+d(w),1,\text{mind}(H)\}$. Since $\text{pind}(A_D)=\text{mind}(G)$ (Lemma 2.7.1) and $\text{mind}(H)\leq\partial_2(H)$ by induction,

$$\text{mind}(G)\leq\max\{d(u)+d(w),1,\text{mind}(H)\}\leq\max\{\partial_2(G),1,\partial_2(H)\}\leq\partial_2(G).$$

□

We immediately obtain the following result by Corollary 2.3:

Corollary 4.2. *If G is a nice graph, then it is $(1,\partial_2(G)+1)$-weight choosable.*

It is not hard to see that, if G is d-degenerate, then $2d\leq\partial_2(G)\leq\Delta(G)+d$, and so Corollary 4.2 does not improve upon the constructive result of Pan and Yang [8] that every nice d-degenerate graph is $(1,2d)$-weight choosable. However, we hope that the extension of the algebraic methods established by Bartnicki et al given in this section will serve as motivation for subsequent improvements. In particular, the proof relied on finding a "good" induced subgraph whose columns had cancellation properties that could be exploited in calculating the permanent index of the matrix A_D. It is conceivable that a more clever choice of induced subgraph might yield a better result than that given in Theorem 4.1.

5 Monomial indices of graph products

We now consider some classes of graphs where we can improve upon the result on Theorem 4.1, in particular the cartesian product of two graphs. The following decomposition lemma on $\text{mind}(G)$ provides an approach for such graphs:

Lemma 5.1. *Let G be a graph, and let H be an induced subgraph of G containing a 2-factor. Let X be a minimal edge cut separating $V(H)$ from $V(G)\setminus V(H)$. If the components of $G-H-X$ are $C_1,\ldots,C_k$, then* $\text{mind}(G)\leq\max\{\text{mind}(H)+|X|,\text{mind}(C_1),\ldots,\text{mind}(C_k)\}$.

Proof: Let $|V(H)|=v$ and $F=\{e_1,\ldots,e_v\}$ be a 2-factor of H. Let D be an orientation of G such that the cycles of F are directed. Define the column vector $c=\sum_{i=1}^{v}c_i$ where c_i is the column of A_D corresponding to e_i. For each $e\in E(H)\setminus F$ there are two edges of F incident to each of the head and tail of e, and for each $e\in F$ there is one edge of F incident to each of the head and tail of e. Hence, the entries of c are nonzero in the rows indexed by the edges of X and 0 in all other entries.

There exists a matrix L_{G-X} consisting of columns of A_{G-X} with no column of A_D repeated more than $\text{mind}(G-X)$ times and $\text{per}\,(L_{G-X})\neq 0$. Let K denote the sequence of edges of $G-X$ which index A_{G-X}. Consider the following matrix:

$$L=\begin{pmatrix}c^{(|X|)} & A_D(K)\end{pmatrix}=\begin{pmatrix}M & N\\ 0 & L_{G-X}\end{pmatrix},$$

where $(M\ N)$ is indexed by X, each row of M is constant, and every entry of M is nonzero. Any column indexed by $e \in E(G) \setminus F$ is used at most $\text{mind}(G - X)$ times in the construction of L, and any edge from F is used at most $|X| + \text{mind}(H)$ times. Clearly, $\text{per}\,(L) = \text{per}\,(M)\text{per}\,(L_G - X) \neq 0$, and hence $\text{pind}(A_D) \leq \max\{|X| + \text{mind}(H), \text{mind}(G - X)\}$. Since $\text{mind}(G - X) = \max\{\text{mind}(C_1), \ldots, \text{mind}(C_k), \text{mind}(H)\}$ by Proposition 2.4 and $\text{mind}(G) = \text{pind}(A_D)$ by Lemma 2.7(1), the result follows. □

Recall that the Cartesian product of two graphs G and H, denoted by $G \,\square\, H$, is defined as the graph having vertex set $V(G) \times V(H)$ where two vertices (u, u') and (v, v') are adjacent if and only if either $u = v$ and u' is adjacent to v' in H or $u' = v'$ and u is adjacent to v in G. Some results on $\chi^e_\Sigma(G)$ for Cartesian products of graphs are given in [7]; for instance, if G and H are regular and bipartite, then $\chi^e_\Sigma(K_n \,\square\, H)$, $\chi^e_\Sigma(C_t \,\square\, H)$, and $\chi^e_\Sigma(G \,\square\, H)$ are at most 2 for $n \geq 4$, and $t \geq 4, t \neq 5$. Lemma 5.1 may be used to bound $\text{ch}^e_\Sigma(G \,\square\, H)$ for many more graphs G and H. Note that, for the graph $G \,\square\, H$ and vertex $v \in V(G)$, the subgraph induced by the set of vertices $\{(v, x) : x \in V(H)\}$ is denoted (v, H).

Theorem 5.2. *Let H be a regular graph on $n \geq 3$ vertices which contains a 2-factor. If G is a d-degenerate graph, then* $\text{mind}(G \,\square\, H) \leq nd + \text{mind}(H)$.

Proof: We may assume that G is connected. The proof of (1) is by induction on $|V(G)|$; the statement is true when G is a single vertex, since $d = 0$ and $\text{ch}^e_\Sigma(H) \leq \text{mind}(H) + 1$ is guaranteed by Lemma 2.1.

Suppose $|V(G)| \geq 2$. Let $v \in V(G)$ have degree at most d, and let X be the minimal edge cut for (v, H). Since $|X| = n \cdot d_G(v)$ and $G - v$ is d-degenerate, Lemma 5.1 implies that

$$\begin{aligned}
\text{mind}(G \,\square\, H) &\leq \max\{\text{mind}(H) + nd_G(v), \text{mind}((G \,\square\, H) - X)\}\\
&\leq \max\{\text{mind}(H) + nd, \text{mind}((G \,\square\, H) - (v, H)\}\\
&\leq \max\{\text{mind}(H) + nd, \text{mind}((G - v) \,\square\, H)\}\\
&\leq \max\{\text{mind}(H) + nd, nd + \text{mind}(H)\}\\
&\leq \text{mind}(H) + nd.
\end{aligned}$$

□

Corollary 5.3. *Let H be a regular graph on $n \geq 3$ vertices which contains a 2-factor. If G is a d-degenerate graph, then $(G \,\square\, H)$ is $(1, nd + \text{mind}(H) + 1)$-weight choosable.*

Since $\text{mind}(K_n)$ is at most 2 by Corollary 3.3, the following corollary is obtained:

Corollary 5.4. *For any integer $n \geq 3$ and any d-degenerate graph G, the graph $G \,\square\, K_n$ is $(1, nd + 3)$-weight choosable.*

Recall that Pan and Yang [8] showed that every nice d-degenerate graph is $(1, 2d)$-weight choosable. Since $G \,\square\, K_n$ is $d(n - 1)$ degenerate, where d is the degeneracy of G, Corollary 5.4 represents an improvement for these graphs.

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References

[1] Noga Alon. Combinatorial Nullstellensatz. *Combin. Probab. Comput.*, 8(1-2):7–29, 1999. Recent trends in combinatorics (Mátraháza, 1995).

[2] Tomasz Bartnicki, Jarosław Grytczuk, and Stanisław Niwczyk. Weight choosability of graphs. *J. Graph Theory*, 60(3):242–256, 2009.

[3] Maciej Kalkowski. A note on the 1,2-conjecture. Preprint, private communication, 2009.

[4] Maciej Kalkowski, Michał Karoński, and Florian Pfender. Vertex-coloring edge-weightings: towards the 1-2-3-conjecture. *J. Combin. Theory Ser. B*, 100(3):347–349, 2010.

[5] Michał Karoński, Tomasz Łuczak, and Andrew Thomason. Edge weights and vertex colours. *J. Combin. Theory Ser. B*, 91(1):151–157, 2004.

[6] Mahdad Khatirinejad, Reza Naserasr, Mike Newman, Ben Seamone, and Brett Stevens. Digraphs are 2-weight choosable. *Electron. J. Combin.*, 18(1):Paper 21, 4, 2011.

[7] Mahdad Khatirinejad, Reza Naserasr, Mike Newman, Ben Seamone, and Brett Stevens. Vertex-colouring edge-weightings with two edge weights. *Discrete Mathematics & Theoretical Computer Science*, 14(1), 2012.

[8] Haili Pan and Daqing Yang. On total weight choosability of graphs. *J. Comb. Optim.*, 25(4):766–783, 2013.

[9] Jakub Przybyło and Mariusz Woźniak. On a 1,2 conjecture. *Discrete Math. Theor. Comput. Sci.*, 12(1):101–108, 2010.

[10] Jakub Przybyło and Mariusz Woźniak. Total weight choosability of graphs. *Electron. J. Combin.*, 18(1):Paper 112, 11, 2011.

[11] Tsai-Lien Wong, Jiaojiao Wu, and Xuding Zhu. Total weight choosability of cartesian product of graphs. *European Journal of Combinatorics*, 33(8):1725 – 1738, 2012.

[12] Tsai-Lien Wong, Daqing Yang, and Xuding Zhu. List total weighting of graphs. In *Fete of combinatorics and computer science*, volume 20 of *Bolyai Soc. Math. Stud.*, pages 337–353. János Bolyai Math. Soc., Budapest, 2010.

[13] Tsai-Lien Wong and Xuding Zhu. Every graph is (2,3)-choosable. Preprint (*submitted*).

[14] Tsai-Lien Wong and Xuding Zhu. Total weight choosability of graphs. *J. Graph Theory*, 66(3):198–212, 2011.