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Extending a perfect matching to a Hamiltonian cycle

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In 1993 Ruskey and Savage conjectured that in the d-dimensional hypercube, every matching M can be extended to a Hamiltonian cycle. Fink verified this for every perfect matching M, remarkably even if M contains external edges. We prove that this property also holds for sparse spanning regular subgraphs of the cubes: for every $d \geq 7$ and every k, where $7 \leq k \leq d$, the d-dimensional hypercube contains a k-regular spanning subgraph such that every perfect matching (possibly with external edges) can be extended to a Hamiltonian cycle. We do not know if this result can be extended to $k = 4, 5, 6$. It cannot be extended to $k = 3$. Indeed, there are only three 3-regular graphs such that every perfect matching (possibly with external edges) can be extended to a Hamiltonian cycle, namely the complete graph on 4 vertices, the complete bipartite 3-regular graph on 6 vertices and the 3-cube on 8 vertices. Also, we do not know if there are graphs of girth at least 5 with this matching-extendability property.

Keywords: some well classifying words

1 Introduction

The d-dimensional hypercube $\mathcal{Q}_d$ is the d-fold Cartesian product of P_2s, i.e $P_2\Box P_2\Box \ldots \Box P_2$, with P_2 appearing d times in the product. The resulting graph is of interest in many disciplines and has been widely studied for many years. It is a Cayley graph of $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \ldots \times \mathbb{Z}_2$ and is both distance-transitive and bipartite. From the definition it is readily apparent that the graphs are Hamiltonian for $d \geq 2$ and

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admit perfect matchings for $d \geq 1$. Indeed, van der Waerden's conjecture (now a theorem, see [3] and [4]) implies a large number of perfect matchings. Kreweras [14] conjectured that every perfect matching of the hypercube extends to a Hamiltonian cycle. This was confirmed in [6] where Fink showed a stronger result, namely that if $K_{\mathcal{Q}_d}$ is the complete graph on the same vertex set as $\mathcal{Q}_d$ and M is a perfect matching in $K_{\mathcal{Q}_d}$, then there is a perfect matching M' in $\mathcal{Q}_d$ such that $M \cup M'$ is a Hamiltonian cycle in $K_{\mathcal{Q}_d}$. In [5] the bound on the number of perfect matchings indicated by van der Waerden's conjecture, combined with Fink's proof of the Kreweras conjecture was used to show that there are at least $((d \log 2/(e \log \log d))(1-o(1)))^{2^d}$ Hamiltonian cycles in $\mathcal{Q}_d$. In [7] Fink's proof was refined to prove that every perfect matching in $\mathcal{Q}_d$ extends to at least $2^{2^{d-4}}$ Hamiltonian cycles.

Kreweras's conjecture is a special case of the following conjecture of Ruskey and Savage [16].

Conjecture 1 *Every matching in the hypercube $\mathcal{Q}_d$ extends to a Hamiltonian cycle.*

Fink's result [6] is a major step toward a solution of Conjecture 1. However, the case of a general matching seems much different from the special case of a perfect matching. Indeed, if one deletes just one edge from the cube, we get a graph which has a matching (namely an odd cut) which does not extend to a cycle. So it is of interest to consider which edges of $\mathcal{Q}_d$ may be discarded while maintaining the ability to extend a perfect matching in $K_{\mathcal{Q}_d}$ to a Hamiltonian cycle. Gregor [9] showed that, given a perfect matching M in $K_{\mathcal{Q}_d}$, we may extend this to a Hamiltonian cycle adding only edges from certain subcubes, depending on M. Fon-Der-Flaass [8] showed that we can delete some edges from the cube (at most one edge from each 4-cycle in the cube) and retain the property that every perfect matching extends to a Hamiltonian cycle. In the present paper we show that we may delete most of the edges in the cube and still have a graph in which each perfect matching extends to a Hamiltonian cycle.

Our approach is to extend Fink's result to show that the edges in $\mathcal{Q}_d$ that cannot be used to extend a matching M in $K_{\mathcal{Q}_d}$ to a Hamiltonian cycle in $M \cup \mathcal{Q}_d$ form a matching in $\mathcal{Q}_d$. We apply this to prove that, for every $d \geq 7$ and every k, where $7 \leq k \leq d$, the d-dimensional hypercube contains a k-regular spanning subgraph such that every perfect matching (possibly with external edges) can be extended to a Hamiltonian cycle. We do not know if this result can be extended to $k = 4, 5, 6$, see Open Problem 1. It cannot be extended to $k = 3$ as we show in Section 3. All of our spanning subgraphs of the cubes have girth 4. We do not know if there are graphs of girth at least 5 with this matching-extendability property, see Open Problem 4.

Häggkvist[11] studied extendability of perfect matchings in graphs with p vertices (p even) satisfying the Ore-type condition that the sum of degrees of any two non-adjacent vertices is at least $p+1$. He proved that in such a graph each perfect matching extends to a Hamiltonian cycle. Note that for such graphs it makes no difference if we allow some edges in the perfect matching to be external. Further results and open problems on dense graphs inspired by this result are given in [1], [12], [13].

2 Preliminaries

Before proving our main result we introduce some terminology and observations.

Definition: *If A, B is a partition of the vertex set of a graph, then the set of edges joining A, B is called a cut.* A, B are called the *sides* of the cut.

Definition: *A perfect matching M in $\mathcal{Q}_d$ is called a dimension-cut if $\mathcal{Q}_d - M$ is isomorphic to two copies of $\mathcal{Q}_{d-1}$.*

We observe that there are precisely d dimension-cuts in $\mathcal{Q}_d$ and each edge belongs to a unique dimension-cut. Dimension-cuts are the only matchings in the hypercube that contain cuts. In other words, a matching M separates $\mathcal{Q}_d$ (that is, $\mathcal{Q}_d - M$ is disconnected) if and only if M is a dimension-cut. The dimension-cuts of $\mathcal{Q}_d$ induce a proper d-edge-coloring of $\mathcal{Q}_d$. With respect to this coloring, a path between a pair of vertices is a shortest path if and only if it contains at most one edge of any color. A given pair of vertices is separated by precisely as many distinct dimension-cuts as there are edges in a shortest path joining them in $\mathcal{Q}_d$.

Definition: *For any graph G we denote by K_G the complete graph on the vertex set $V(G)$.*

Definition: *A pairing of the graph G is a perfect matching in K_G.*

Definition: *For any graph G edges in $E(K_G) - E(G)$ are called external edges.*

Clearly, a pairing of G may contain external edges.

Definition: *For any graph G, we say that a pairing M of G extends to a Hamiltonian cycle if there is a Hamiltonian cycle C of K_G in which $E(C) - M \subseteq E(G)$. If every pairing M of G extends to a Hamiltonian cycle, then we say that G has the pairing-Hamiltonian property or, for short, the PH-property.*

We may think of the Cartesian product $G\square H$ as a graph obtained from G by replacing every vertex of G by a copy of H and every edge of G by a matching joining two copies of H. We call these copies of H in $G\square H$ the *canonical* copies of H in $G\square H$. Note that there may be other copies of H in $G\square H$ but they will not be called canonical. If G is a path P_q with q vertices, then the canonical copies of H in $G\square H$ corresponding to the first and last vertices in P_q are called the first and last canonical copies of H, respectively.

If C_1, C_2 are two vertex disjoint cycles in a graph G, and G contains two edges e_1, e_2 joining two consecutive vertices in C_1 with two consecutive vertices in C_2, then we may obtain a cycle C by deleting an edge in each of C_1, C_2 and adding the edges e_1, e_2. We say that C is obtained by *merging* C_1, C_2.

If M is a matching, and C is a path or cycle, then C is *M-alternating* if every second edge of C is in M.

3 The 3-regular graphs with the PH-property.

Theorem 1 *Let G be a 3-regular graph with the PH-property. Then G is the complete graph K_4 or the complete bipartite graph $K_{3,3}$ or the 3-cube $\mathcal{Q}_3$.*

Proof of Theorem 1:

It is easy to see that G must be 2-connected. Recall that in a 3-regular graph, vertex-connectivity and edge-connectivity are the same. In the current setting, it is more natural to consider cuts consisting of edges.

If G has a cut consisting of two edges $e_1 = x_1x_2$ and $e_2 = y_1y_2$ joining the sides A_1 (containing x_1, y_1) and A_2 (containing x_2, y_2), then we extend the edges x_1y_1, x_2y_2 to a pairing of G containing no edge from A_1 to A_2. Clearly, that pairing cannot be extended to a Hamiltonian cycle. So G is 3-connected.

If G has a cut K, consisting of a matching with an odd number of edges, each joining side A_1 to side A_2, then each A_i has an even number of vertices not incident with any edge in K. We construct a pairing M of G as follows. Include all edges of K in M. Now add edges (possibly external) pairing all vertices in A_1 not incident with K. This is possible since there are evenly many vertices in A_1 not incident with edges in K. Finally, we add edges (again, possibly external) pairing all vertices in A_2 not incident with K (again there are evenly many such vertices). The only edges in M joining vertices in A_1 to vertices in A_2 are those in K. Since $|K|$ is odd, M cannot be extended to a Hamiltonian cycle. So we may assume that G has no such cut. In particular, G has girth at least 4 unless $G = K_4$.

Consider now a shortest cycle $C : x_1x_2 \dots x_kx_1$. Let y_i be the neighbor of x_i not in C, for $i = 1, 2, \dots, k$. If k is odd, then the edges leaving C form a matching which is also a cut, a contradiction. So k is even. We may assume that $y_1, y_2, \dots, y_k$ are distinct since otherwise $k = 4$ (by the minimality of C), in which case G has a cut with three edges forming a matching, unless $G = K_{3,3}$. So consider the matching $x_1x_2, y_2y_3, x_3x_4, \dots x_{k-1}x_k, y_ky_1$. We extend that matching to a perfect matching and then to a Hamiltonian cycle. It is easy to see that that Hamiltonian cycle must be $x_1x_2y_2y_3x_3x_4 \dots x_{k-1}x_ky_ky_1x_1$. In particular G has $2k$ vertices and girth k. It is well known (and easy to see) that a cubic graph with $n \geq 12$ vertices has girth strictly less than $n/2$. So G has $2k = 8$ vertices. There are only two triangle-free cubic graphs with 8 vertices, namely $\mathcal{Q}_3$ and the 8-cycle with 4 diagonals. As those 4 diagonals cannot be extended to a cycle, it follows that $G = \mathcal{Q}_3$. □

4 The Cartesian product of a path and a cycle.

$C_4 \square K_4$ contains $C_4 \square C_4 = \mathcal{Q}_4$ which has the PH-property. Seongmin Ok and Thomas Perrett (private communication) have shown that $C_q \square K_4$ does not have the PH-property when $q \geq 7$, see Section 5. However, if each matching edge is contained in one of the canonical K_4s, then the matching extends to a Hamiltonian cycle as the proof of the proposition below easily shows. The argument in that proposition will be crucial in our main result.

For this proposition we shall consider the graph $P_q \square C_4$ to be labelled so that the q canonical 4-cycles are $w_ix_iy_iz_iw_i$ for $i = 1, 2, \dots, q$ and the four canonical paths (when we think of $P_q \square C_4$ as $C_4 \square P_q$) are $w_1w_2 \dots w_q$ etc. Matchings in distinct canonical 4-cycles are said to be *similar* if they join the same canonical paths. For example, w_ix_i, y_iz_i in the i-th canonical 4-cycle is similar to the matching w_jx_j, y_jz_j in the j-th canonical 4-cycle.

Proposition 1 *Let $q \geq 3$ be an integer and let M be a perfect matching in $G = P_q \square C_4$, such that each M-edge is contained in a canonical 4-cycle. Assume that not all canonical 4-cycles have similar M-matchings. Then G has two M-alternating Hamiltonian paths each starting at w_1 and ending at diametrically opposite vertices of the last canonical 4-cycle.*

Proof of Proposition 1:

There is a unique M-alternating Hamiltonian path starting at w_1, ending in a vertex in the last canonical 4-cycle and containing precisely one edge between consecutive canonical 4-cycles. We shall prove that we can find an M-alternating Hamiltonian path ending at another vertex of the last canonical 4-cycle (which must be diametrically opposite in the last canonical 4-cycle because G is bipartite). Consider the case where M has the edges w_1x_1, y_1z_1 and w_2z_2, x_2y_2 on first two canonical 4-cycles. The above-mentioned M-alternating Hamiltonian path with precisely one edge joining each consecutive pair of canonical 4-cycles is then $w_1x_1y_1z_1z_2w_2x_2y_2y_3\ldots$. There is another one, namely $w_1x_1x_2y_2y_1z_1z_2w_2w_3\ldots$. The two Hamiltonian paths enter the third canonical 4-cycle at diametrically opposite points and continue from there in the prescribed manner. Clearly these two Hamiltonian paths must have distinct (diametrically opposite) ends in the last canonical 4-cycle. A similar variation is readily achieved if the first change of matching occurs between canonical 4-cycles j and $j+1$. □

5 The Cartesian product of a complete graph and a cycle.

We shall seek examples which are the Cartesian product of a complete graph K_m and a cycle C_q. Such a graph is $(m+1)$-regular. If m is odd, we assume that q is even because there should be a pairing.

If $m = 2$ then this Cartesian product does not have the extendability property (except if $q = 4$) because of Theorem 1.

If m is odd (and hence q is even), then the Cartesian product of K_m and C_q has an even cut consisting of a matching with $2m$ edges. If $q \geq 6$, then this matching can be chosen such that the two sides, A_1 and A_2, of the cut are odd. (Note, when $q = 4$, the only matchings that produce cuts in $K_m \Box C_4$ are those corresponding to perfect matchings in C_4 and hence both resulting components are even.) But then we extend the cut to a pairing. Since A_1 and A_2 are both odd, that pairing contains an odd number of edges between A_1 and A_2 and, as such cannot extend to a Hamiltonian cycle.

Theorem 2 *The Cartesian product of a complete graph K_m (where m is even, $m \geq 6$) and a cycle has the PH-property.*

We observe that if a graph G has the PH-property, then any graph H obtained from G by adding edges also has the PH-property. Consequently, Theorem 2 follows immediately from the following stronger result.

Theorem 3 *The Cartesian product of a complete graph K_m (where m is even, $m \geq 6$) and a path P_q with $q \geq 1$ vertices has the PH-property. Moreover, if M is a pairing in which no edge joins vertices from distinct canonical complete graphs, and e is any edge in the last canonical complete graph and not in M, then the Hamiltonian cycle containing M can be chosen such that it contains e, too.*

Proof of Theorem 3:

Let the graph we consider be denoted G.

Clearly, with $q = 1$, $G = K_m$ and the result is immediate. So we may assume $q \geq 2$.

Consider first the case where M contains at least one edge which joins two distinct canonical complete graphs K_m. In this case we adapt Fink's inductive argument from [6] to the current situation as follows. The path P_q can be divided into two paths P_r and P_{q-r} such that M has edges joining the subgraph of G corresponding to P_r with the subgraph corresponding to P_{q-r}. Since m is even, there must be an even number of such edges in M. Suppose the edges in M between $G_1 = P_r \Box K_m$ and $G_2 = P_{q-r} \Box K_m$ are $e_1, e_2, \ldots, e_{2k}$ and that edge $e_i = u_iv_i$ with u_i in $V(G_1)$ and v_i in $V(G_2)$ for each $i = 1, 2, \ldots, 2k$. We form a new pairing M_2 in G_2 by adding to those edges of M already joining pairs of vertices in G_2, the new edges $v_1v_2, v_3v_4, \ldots, v_{2k-1}v_{2k}$. Now apply induction on G_2 with M_2. The edges of M_2 not in M

occur in some fixed order on the resulting Hamiltonian cycle in G_2. Deleting the edges in M_2 (and not in M) from the Hamiltonian cycle in G_2 leaves path segments joining the vertices v_i, $i = 1, 2, 3, \ldots, 2k$ in pairs. We now pair the vertices $u_1, u_2, \ldots, u_{2k}$ according to this order and apply induction to G_1 to yield a Hamiltonian cycle in G_1 which, when merged with the Hamiltonian cycle in G_2 gives the desired Hamiltonian cycle in G.

Consider next the case where M contains no edge which joins two distinct canonical complete graphs K_m. Let e be any edge not in M but in the last canonical K_m. Now we choose another edge e' in the last canonical K_m such that

(i) e' is not in M

(ii) e' does not join the same two M-edges as the two M-edges which e joins, and

(iii) the edge e'' in the second last K_m corresponding to e' is not in M.

As $m \geq 6$ it is easy to find such an edge e'. Now we let C be a Hamiltonian cycle in the last canonical K_m containing both of e, e' and also those edges of M which are in that K_m. We apply induction to G with the last K_m removed and with e'' playing the role of e. Then we delete e'' from the resulting cycle and merge it with C to complete the proof. $\square$

It is easy to see that $P_q \square K_4$ has a perfect matching which cannot be extended to a Hamiltonian cycle when $q \geq 3$. To see this we consider each canonical K_4 to have vertices w_i, x_i, y_i, z_i, $i = 1, 2, \ldots, q$ and canonical paths P_q have vertices $w_1, w_2, \ldots, w_q$ and so on. Taking matchings w_1x_1, y_1z_1 in the first canonical K_4, w_2y_2, x_2z_2 in the second and w_3z_3, x_3y_3 in the third we can extend this to a perfect matching in $P_q \square K_4$ with arbitrarily chosen perfect matchings in each of the remaining $q-3$ canonical K_4s to yield a perfect matching that cannot extend to a Hamiltonian cycle. Seongmin Ok and Thomas Perrett (private communication) have modified this idea to show that $C_q \square K_4$ does not have the PH-property for $q \geq 7$.

6 Extensions of the PH-property for the hypercubes.

In this section we present a technical extension of Fink's result where we try to extend a matching and an additional prescribed edge to a Hamiltonian cycle.

In the proofs we refer to Fink's inductive argument [6] used to prove his result that any perfect matching M in $K_{\mathcal{Q}_d}$ can be extended to a Hamiltonian cycle in $\mathcal{Q}_d \cup M$. The argument is as follows. We divide $\mathcal{Q}_d$ in to two $(d-1)$-cubes G_1, G_2 joined by a dimension cut. As every pair of vertices in the cube can be divided by some dimension cut, we may assume that M has at least one edge from G_1 to G_2. We first apply induction to G_1. We keep those edges in M which join two vertices in G_1. We add at random a matching (which we may call new edges) between the ends of those edges in M which go from G_1 to G_2. The induction hypothesis gives us a Hamiltonian cycle in G_1 (plus some external edges). We delete the new edges and get a path system in $G_1 \cup M$. We transform this path system to new edges in G_2 and complete the proof by applying the induction hypothesis to G_2.

The following result will be useful. It is not new having previously appeared in [2, 10, 15], for example. For the sake of completeness, however, we include a short proof.

Proposition 2 *(a) If $d \geq 3$ is odd, and v, v' are diametrically opposite vertices of $\mathcal{Q}_d$, then $\mathcal{Q}_d - v - v'$ has a Hamiltonian cycle.*

(b) If $d \geq 4$ is even, and v, u are neighbors, and v, v' are diametrically opposite vertices of $\mathcal{Q}_d$, and u, u' are diametrically opposite vertices of $\mathcal{Q}_d$, then $\mathcal{Q}_d - v - v' - u - u'$ has a Hamiltonian cycle.

Proof of Proposition 2: We first prove (a). If $d = 3$, then $\mathcal{Q}_d - v - v'$ is a cycle of length 6. So assume that $d \geq 5$. We think of the d-cube $\mathcal{Q}_d$ as the product $\mathcal{Q}_{d-2} \Box \mathcal{Q}_2$. In other words, $\mathcal{Q}_d$ is obtained from $\mathcal{Q}_{d-2}$ by replacing every vertex by a canonical 4-cycle. Consider a Hamiltonian cycle C in $\mathcal{Q}_{d-2}$ and suppose C has some fixed direction imposed upon it. We transform C into a cycle in $\mathcal{Q}_d - v - v'$ as follows. In the canonical 4-cycle containing v, we build a path xyz of length 2 containing the three other vertices in this canonical 4-cycle. In the $\mathcal{Q}_{d-2}$ containing z, we proceed from z to a neighboring canonical 4-cycle using the directed edge in C. In that canonical 4-cycle we pick up all four vertices (note that we can do this in two distinct ways) and proceed to a neighboring canonical 4-cycle using the directed edge in C in our current canonical $\mathcal{Q}_{d-2}$. We proceed like this except when we reach the canonical 4-cycle containing v'. We hit this 4-cycle at a neighbor of v' because d is odd and $\mathcal{Q}_d$ is bipartite. Then we pick up the three vertices distinct from v' (this can be done in exactly one way) and continue. When we return to the first canonical 4-cycle, we hit that canonical 4-cycle either at x or z because $\mathcal{Q}_d$ is bipartite and v, v' belong to distinct partite classes. If we hit at x, we are done. If we hit at z, we can easily modify our path in some canonical 4-cycle traversing the vertices in the opposite direction around the canonical 4-cycle. In this way we terminate at x and get a cycle containing all vertices, except v, v'.

We now prove (b). We think of the d-cube $\mathcal{Q}_d$ as the union of two $(d-1)$-cubes G_1, G_2 joined by the dimension cut containing the edge vu. Then v, u' are diametrically opposite in G_1, and u, v' are diametrically opposite in G_2. By (a), $G_1 - v - u'$ has a Hamiltonian cycle C_1. Let C_2 be the corresponding Hamiltonian cycle in $G_2 - u - v'$. Then we complete the proof of (b) by merging C_1, C_2.

□

Again, we think of the d-cube $\mathcal{Q}_d$ as the product $\mathcal{Q}_{d-2} \Box \mathcal{Q}_2$. In other words, $\mathcal{Q}_d$ is obtained from $\mathcal{Q}_{d-2}$ by replacing every vertex by a canonical 4-cycle.

Proposition 3 *Let $d \geq 3$ be an integer and let M be a pairing in $\mathcal{Q}_d = \mathcal{Q}_{d-2} \Box \mathcal{Q}_2$, such that every edge in M joins two vertices in the same canonical 4-cycle. Let e be any edge in $\mathcal{Q}_d$ joining two distinct canonical 4-cycles. Then $\mathcal{Q}_d$ has a perfect matching M' such that M' contains e and $M \cup M'$ is a Hamiltonian cycle in $\mathcal{Q}_d \cup M$.*

Proof of Proposition 3: The proof is by induction on d. Suppose first that $d = 3$. Then there are two canonical 4-cycles, and e is an edge joining them. If the two matchings in the two canonical 4-cycles are non-similar, then the union of M and the four edges joining the two canonical 4-cycles form a Hamiltonian cycle. That Hamiltonian cycle contains e. If the two matchings in the two canonical 4-cycles are similar, then there is a Hamiltonian cycle containing M and two edges joining the two canonical 4-cycles and also another Hamiltonian cycle containing M and two other edges joining the two canonical 4-cycles. One of these two Hamiltonian cycles contains e.

Assume now that $d \geq 4$. We refine our view of $\mathcal{Q}_d$ further noting that each $\mathcal{Q}_{d-2} = \mathcal{Q}_1 \Box \mathcal{Q}_{d-3}$ (i.e. two copies of $\mathcal{Q}_{d-3}$ joined by a dimension cut). Furthermore, we may assume that e lies inside one of the canonical $\mathcal{Q}_{d-3}$s in this product. This decomposition imposes a natural perspective on $\mathcal{Q}_d$ as the union of two $(d-1)$-cubes G_1, G_2 joined by a dimension cut not containing e. Moreover, each G_i inherits its structure as $\mathcal{Q}_{d-3} \Box \mathcal{Q}_2$ from our original cube. We may assume that e is in G_1 and apply induction to G_1. The resulting cycle which contains e contains another edge e_1 in the same dimension cut as e. Let e_2 be the edge in G_2 corresponding to e_1. Now apply induction to G_2 with e_2 playing the role of e. Then merge the two cycles by deleting e_1, e_2 and adding two edges between G_1, G_2. □

Theorem 4 *Let $d \geq 2$ be an integer and let M be a pairing in $\mathcal{Q}_d$. Let e_1, e_2 be two edges in $\mathcal{Q}_d - M$ incident with the same vertex. Then $\mathcal{Q}_d$ has a perfect matching M' such that M' contains one of e_1, e_2 and $M \cup M'$ is a Hamiltonian cycle in $\mathcal{Q}_d \cup M$.*

Proof of Theorem 4: The proof is by induction on d. For $d = 2$, the statement is trivial. So assume that $d \geq 3$.

There is a unique 4-cycle C_1 in $\mathcal{Q}_d$ containing e_1, e_2. Now we think of $\mathcal{Q}_d$ as $\mathcal{Q}_{d-2} \Box \mathcal{Q}_2$ such that e_1, e_2 are contained in a common canonical 4-cycle C_1. Moreover, we shall label the vertices in C_1, w_1, x_1, y_1, z_1 where $e_1 = w_1x_1$ and $e_2 = w_1z_1$. We consider the remaining canonical 4-cycles with respect to this decomposition to be labelled $C_2, C_3, \ldots, C_{2^{d-2}}$ and their constituent vertices to be labelled analogously to those in C_1.

We distinguish two cases:

Case 1: Each edge in M joins two vertices in the same canonical 4-cycle;
Case 2: There is at least one edge $e = uv$ in M such that $u \in V(C_i)$ and $v \in V(C_j)$ with $i \neq j$.

Proof of Case 1:

Note, since neither e_1 nor e_2 is in M, the vertices in C_1 are paired by diagonal edges external to $\mathcal{Q}_d$. If $d = 3$, it is easy to find M'. So assume that $d \geq 4$. We think of $\mathcal{Q}_{d-2}$ as two $(d-3)$-cubes G'_1, G'_2 joined by a dimension cut. So, we may think of $\mathcal{Q}_d$ as the union of G_1, G_2 joined by a dimension cut, where each of G_1, G_2 is the Cartesian product of a $(d-3)$-cube with $\mathcal{Q}_2$. We may assume that G_1 contains e_1, e_2 and apply induction to G_1. The resulting Hamiltonian cycle contains an edge e' joining two vertices in distinct canonical 4-cycles. Let e be the corresponding edge in G_2. Now apply Proposition 3 to G_2. Then we merge the two cycles after deleting the edges e, e'. This completes the proof of Case 1.

Proof of Case 2:

We now assume that some edge of M joins two vertices in distinct canonical 4-cycles.

If $d = 3$, then M contains either two or four edges joining the two canonical 4-cycles. Fink's proof [6] shows that $\mathcal{Q}_3 \cup M$ has a Hamiltonian cycle whose edges between the canonical 4-cycles are precisely those edges in M which join these canonical 4-cycles. Such a Hamiltonian cycle must contain at least one of e_1, e_2. So assume that $d \geq 4$.

Again, we think of $\mathcal{Q}_{d-2}$ as two $(d-3)$-cubes joined by a dimension cut. This induces a decomposition of $\mathcal{Q}_d$ into G_1, G_2 joined by a dimension cut. Note that in this decomposition, each G_i is a $\mathcal{Q}_{d-3} \Box \mathcal{Q}_2$ in which the canonical $\mathcal{Q}_2$s are canonical 4-cycles denoted $C_1, C_2, \ldots C_{2^{d-2}}$. We shall refer to such decompositions as *very canonical decompostions* of $\mathcal{Q}_d$. As before, we assume that G_1 contains C_1 and hence e_1, e_2 but now we may also assume that some edge in M joins G_1, G_2.

We look to apply the inductive hypothesis to G_1 and then Fink's result to G_2. To apply the inductive argument to G_1, we form a pairing of G_1 as follows. Keep those edges in M which join two vertices in G_1. For those edges in M which join vertices in G_1 with vertices in G_2 we arbitrarily choose a matching that pairs up the (even number) of endvertices of edges in M whose other ends lie in G_2. We shall refer to this pairing in G_1 as M_1.

We distinguish two further subcases.

Case 2.1: For some very canonical decomposition of $\mathcal{Q}_d$ into G_1, G_2, with M containing edges joining G_1 to G_2, the pairing M_1 can be chosen so that $\{e_1, e_2\} \cap M_1 = \emptyset$.

Proof of Case 2.1:

In this case, we choose M_1 containing neither of e_1, e_2 and we complete the proof by applying induction to G_1 and then Fink's argument to G_2.

Case 2.2: For each very canonical decomposition of $\mathcal{Q}_d$ into G_1, G_2, with edges in M joining G_1 to G_2, the pairing M_1 is forced to contain either e_1 or e_2.

Proof of Case 2.2:

Clearly, if we have a canonical decomposition of $\mathcal{Q}_d$ in which more than two edges of M join vertices in G_1 to vertices in G_2, we can choose M_1 to avoid e_1 and e_2.

We note that for each pair of canonical 4-cycles, $C_i, C_j,\ i \neq j$, there is a canonical decomposition of $\mathcal{Q}_d$ that separates C_i from C_j. Thus, if M contains an edge $f = tt'$ with $t \in V(C_i)$ and $t' \in V(C_j)$, $i \neq j$ such that $\{t, t'\} \cap \{w_1, x_1, z_1\} = \emptyset$, then we can find a canonical decomposition of $\mathcal{Q}_d$ that puts us in Case 2.1. Consequently, we may assume that M contains precisely two edges joining vertices in different canonical 4-cycles. Moreover, one of these edges is incident with w_1 while the other is incident with either x_1 or z_1. As such, we may assume, without loss of generality that the only two edges in M joining canonical 4-cycles are $f_1 = w_1t$ and $f_2 = x_1t'$. Note, by parity, both t an t' are vertices in the same canonical 4-cycle, C_2, say.

Case 2.2.1: The very canonical decomposition of $\mathcal{Q}_d$ into G_1, G_2 can be chosen such that there are no edges in M between G_1, G_2.

Proof of Case 2.2.1:

We note that in G_2 no edge of M can join a pair of vertices in different canonical 4-cycles. Thus we apply induction to G_1, and then apply Proposition 3 to G_2 where e is an appropriate edge which can be used to merge the two cycles.

Case 2.2.2: For each very canonical decomposition of $\mathcal{Q}_d$ into G_1, G_2, (where C_1 is in G_1), there are precisely two M-edges between G_1, G_2, and these two M-edges are $f_1 = w_1t$ and $f_2 = x_1t'$, join the two ends of $e_1 = w_1x_1$ to two vertices in a single canonical 4-cycle, C_2 (i.e. $\{t, t'\} \subset \{w_2, x_2, y_2, z_2\}$).

Proof of Case 2.2.2:

Since this applies to all partitions of $\mathcal{Q}_d$, the two canonical 4-cycles C_1, C_2 correspond to diametrically opposite vertices v, v' in $\mathcal{Q}_{d-2}$.

As usual, we think of $\mathcal{Q}_d$ as being obtained from $\mathcal{Q}_{d-2}$ by replacing every vertex by a canonical 4-cycle. Let C_0 be a fixed Hamiltonian cycle in $\mathcal{Q}_{d-2}$. We use C_0 to construct a Hamiltonian cycle in $\mathcal{Q}_d \cup M$ containing M. First construct a path starting with the vertex z_1 and the edge z_1y_1 (which is in M) followed by the edge corresponding to an edge in C_0. After that we follow C_0 and pick up the two M-edges in the canonical 4-cycle corresponding to each vertex of C_0. When we encounter C_2, the canonical 4-cycle corresponding to v', we pick up the M-edge joining two vertices in C_2, and we also include the path tw_1x_1t'. In similar fashion, we follow the rest of C_0, picking up each pair of edges from M in each canonical 4-cycle along the way. If this path terminates at z_1 we have finished. So assume that it terminates at x_1. (It cannot terminate at w_1 or y_1 because the cube is bipartite.) Since it must terminate at x_1, it follows that we have no choice in this procedure. This implies that all M-edges are internal except the two edges w_1t, x_1t' and that t, t' must be consecutive on C_2.

In each canonical 4-cycle there are two perfect matchings, namely the ones contained in the dimension cut containing e_1, which we call *horizontal edges* and the ones contained in the dimension cut containing e_2 which we call *vertical edges*. We focus again on the Hamiltonian cycle C_0 in $\mathcal{Q}_{d-2}$. The proof of

Proposition 1 shows that we may assume that all matchings in the canonical 4-cycles are horizontal. Also, the edge in M joining two vertices in C_2 is horizontal. Otherwise the proof of Proposition 1 would give us a choice we could use to get the desired Hamiltonian cycle. We now consider the M-alternating cycle S of length 8 containing all vertices in the 4-cycles C_1, C_2. If $d = 4$ we first use the two non-M edges in $E(C_2) \cap E(S)$ to merge the two remaining 4-cycles, C_3, C_4, with the cycle S to form the desired Hamiltonian cycle. If $d \geq 5$ and d is odd, then, using Proposition 2 (a), we consider a Hamiltonian cycle in $\mathcal{Q}_{d-2} - v - v'$ to find an M-alternating cycle S' containing all M edges not in S. Then we merge S, S' using a vertical edge in each of S, S' . If $d \geq 6$ and d is even, then we argue in the same way, except that $\mathcal{Q}_{d-2} - v - v'$ has no Hamiltonian cycle because v, v' are in the same bipartite class. Instead we consider, using Proposition 2 (b), a Hamiltonian cycle in $\mathcal{Q}_{d-2} - v - v' - u - u'$ where also u, u' are diametrically opposite, and u, u' are neighbors to v, v', respectively, to find an M-alternating cycle S'', containing all M-edges which are neither in S nor the two canonical 4-cycles corresponding to u, u'. Then we first merge S'', S using a vertical edge in each of S'', S , and then we merge the resulting cycle with each of the canonical 4-cycles corresponding to u, u', respectively. □

Theorem 4 shows that those edges which cannot be extended together with M to a Hamiltonian cycle form a matching. We do not know how big this matching can be, see Open Problem 2 below. But the following example shows that it may have two edges. Consider a dimension cut dividing the cube into two subcubes G_1, G_2. Let $e_1 = x_1x_2$ and $e_2 = y_1y_2$ be two of the edges in the dimension cut. In the dimension cut we now replace e_1, e_2 by the two external edges x_1y_2, x_2y_1. The resulting perfect matching (pairing as it contains two external edges) can be extended to a Hamiltonian cycle, but such a Hamiltonian cycle cannot contain either of e_1, e_2.

7 Sparse regular spanning subgraphs of the hypercubes with the PH-property.

In this section we use the previous results to prove the main result that it is possible to remove most of the edges of the hypercube and preserve the remarkable PH-property. It follows that the same holds for the graphs in Theorems 2 and 3 when m is a power of 2.

Theorem 5 *Let $d \geq 5$ be an integer and let M be a pairing in $P_q \square \mathcal{Q}_d$. Then M can be extended to a Hamiltonian cycle in $M \cup P_q \square \mathcal{Q}_d$. Moreover, if each edge of M joins two vertices in the same canonical $\mathcal{Q}_d$, then the last $\mathcal{Q}_d$ contains a matching M' such that, for each edge e in the last $\mathcal{Q}_d$, but not in M', M can be extended to a Hamiltonian cycle of $M \cup P_q \square \mathcal{Q}_d$ containing e.*

Proof of Theorem 5: The proof is by induction on q. For $q = 1, 2$, Theorem 5 follows from Theorem 4 because $P_1 \square \mathcal{Q}_d$ and $P_2 \square \mathcal{Q}_d$ are cubes. So assume that $q \geq 3$.

If some M-edge joins two distinct canonical $\mathcal{Q}_d$s, then we apply Fink's inductive argument. So assume that each M-edge joins two vertices in the same canonical $\mathcal{Q}_d$.

Now apply induction to $P_{q-1} \square \mathcal{Q}_d$ which we think of as $P_q \square \mathcal{Q}_d$ with the last canonical $\mathcal{Q}_d$ deleted. The induction hypothesis results in a matching M' in the last canonical $\mathcal{Q}_d$ in $P_{q-1} \square \mathcal{Q}_d$.

Suppose now (reductio ad absurdum) that Theorem 5 is false. Then the last canonical $\mathcal{Q}_d$ in $P_q \square \mathcal{Q}_d$ contains two edges e_1, e_2 incident with the same vertex w_1 such that M cannot be extended to a Hamiltonian cycle using edges in $P_q \square \mathcal{Q}_d$, one of which is e_1 or e_2. (Clearly, we may assume that e_1, e_2 are not in M.) We shall show that this leads to a contradiction.

For this we consider a dimension cut in the last $\mathcal{Q}_d$ dividing it into two $(d-1)$-cubes, G_1, G_2, such that M has, say, r edges joining G_1, G_2, where $r \geq 2$. Assume that w_1 is in G_2.

Case 1: The dimension cut can be chosen such that e_1, e_2 belong to G_2.

Now we try to apply Theorem 4 to G_2 after the r edges in M joining vertices in G_1 to vertices in G_2 have been replaced by $r/2$ new (possibly external) edges in G_2.

Subcase 1.1: The $r/2$ new (possibly external) edges in G_2 can be chosen such that none of them is e_1 or e_2.

In this subcase we apply Theorem 4 to G_2 such that the resulting Hamiltonian cycle C contains one of e_1, e_2. Deleting the new edges in G_2, we obtain a path system in G_2. As in Fink's inductive argument, that path system gives rise to a pairing in G_1, maintaining all edges of M contained in G_1. We apply Theorem 4 to that pairing. We select any vertex x in G_1 and we select two edges e'_1, e'_2 in G_1 and incident with x such that neither of e'_1, e'_2 is a matching edge (in the new matching in G_1), and neither of e'_1, e'_2 correspond to an edge in M'. Since G_1 is a $(d-1)$-cube, this cube has $d-1$ edges incident with x. Thus e'_1, e'_2 can be chosen among $d-3$ edges incident with x. As $d \geq 5$, this is possible. So, we can apply Theorem 4 to G_1. Without loss of generality, we may assume that the resulting Hamiltonian cycle C' contains e'_1, say. Now we apply the induction hypothesis to $P_{q-1}\square\mathcal{Q}_d$ in order to get a Hamiltonian cycle C'' containing the M-edges in that graph and also containing the edge corresponding to e'_1 (as that edge is not in M'). Now we can first combine C, C' into an M-alternating cycle C''' containing all vertices in the last canonical $\mathcal{Q}_d$ and the edge e'_1. Finally, we merge C''' with C'' (after deleting e'_1 and the corresponding edge in C'') to get the desired Hamiltonian cycle in $M \cup P_q\square\mathcal{Q}_d$. This contradiction completes Subcase 1.1.

Subcase 1.2: The $r/2$ new (possibly external) edges in G_2 cannot be chosen such that none of them is e_1 or e_2.

In this case must have $r = 2$, and the two M-edges from G_2 to G_1 are incident with w_1 and an end x_1 of e_1 or e_2, say e_1. Again, we think of the last canonical d-cube $\mathcal{Q}_d$ in $P_q\square\mathcal{Q}_d$ as the product $\mathcal{Q}_{d-2}\square\mathcal{Q}_2$. In other words, $\mathcal{Q}_d$ is obtained from $\mathcal{Q}_{d-2}$ by replacing each vertex by a canonical 4-cycle. We may assume that one of these canonical 4-cycles is $C_1 : w_1x_1y_1z_1w_1$ where $e_1 = w_1x_1, e_2 = w_1z_1$. The assumption of Subcase 1.2 implies that the two M-edges incident with w_1, x_1 join C_1 to another canonical 4-cycle C_i and all other M-edges join two vertices in the same canonical 4-cycle. We now consider a Hamiltonian cycle of $\mathcal{Q}_{d-2}$. We let $C_1, C_2, \ldots, C_{2^{d-2}}, C_1$ be the corresponding cyclic sequence of canonical 4-cycles in the last canonical d-cube $\mathcal{Q}_d$ in $P_q\square\mathcal{Q}_d$. We transform that to a Hamiltonian cycle in last canonical d-cube $\mathcal{Q}_d$ with its M-edges added such that the Hamiltonian cycle contains all those M-edges. We begin with the edge from y_1 to C_2. We pick up the two M-edges joining vertices in C_2 and proceed to C_3. When we reach C_i we also pick up w_1, x_1.

Suppose that we have a choice in some C_j. Then we can make sure that we hit z_1 when we terminate in C_1 so that we obtain a Hamiltonian cycle. As any of the $d-2$ edges from y_1 to another canonical 4-cycle can play the role of the first edge in this Hamiltonian cycle, we can choose the first edge e' such that it does not correspond to an edge in M' or an edge in M in the last canonical d-cube $\mathcal{Q}_d$ in $P_{q-1}\square\mathcal{Q}_d$. So, we can extend the M-edges in $P_{q-1}\square\mathcal{Q}_d$ to a Hamiltonian cycle containing the edge e'' corresponding to e'. By deleting the edges e', e'' we now merge the two Hamiltonian cycles into one which contains e_1 and gives a contradiction. So we may assume that we have no choice when we transform the sequence $C_1, C_2, \ldots$ into a Hamiltonian cycle.

Since we have no choice, we conclude that both M-edges joining vertices in C_j are also edges in this C_j, except that precisely one M-edge is an edge in C_i. Moreover, we may assume that all these matchings the cycles C_j are similar, since otherwise, we modify the cycle in two consecutive C_j, C_{j+1} to obtain the desired choice. So, we may assume that all M-edges in the last canonical d-cube, except the two incident with w_1, x_1, are contained in the dimension cut containing $e_1 = w_1x_1$. We may also assume that the two M-edges incident with the ends of e_1 join w_1, x_1 with either w_i, x_i or x_i, w_i or y_i, z_i or z_i, y_i, respectively.

We now focus on the sequence $C_1, C_2, \ldots, C_{2^{d-2}}$ of canonical 4-cycles in the last canonical d-cube $\mathcal{Q}_d$ in $P_q \Box \mathcal{Q}_d$ (rather than the cyclic sequence $C_1, C_2, \ldots, C_{2^{d-2}}, C_1$). Let P_1 be the path $w_1z_1y_1y_2z_2z_3 \ldots x_2x_1$ such that P_1 contains all vertices in $C_1 \cup C_2 \cup \ldots \cup C_{i-1}$ and all M-edges in these cycles. So P_1 zig-zags through the y, z vertices in $C_1, \ldots C_{i-1}$ and zig-zags back through the w, x vertices in $C_{i-1} \ldots C_1$. The exact transition at C_{i-1} depends on whether i is odd or even. We let P_2 denote the path $w_iw_{i+1}x_{i+1}x_{i+2} \ldots y_{i+1}z_{i+1}z_i$ such that P_2 starts and terminates at C_i and contains all vertices of $C_{i+1} \cup C_{i+2} \cup \ldots \cup C_{2^{d-2}}$ and all M-edges in these cycles. Again, P_2 zig-zags through the w, x vertices in $C_i, \ldots C_{2^{d-2}}$ and back through the y, z vertices. The exact transition again being dependent on whether i is odd or even. Now $P_1 \cup P_2$ can be extended to the desired Hamiltonian cycle containing all edges of M. As it also contains the edge e_2 we have reached a contradiction. The exact details of how the paths P_1 and P_2 are combined depends on the M edges between $\{w_1, x_1\}$ and either $\{w_i, x_i\}$ or $\{y_i, z_i\}$. The variations are quite straightforward and are left to the reader.

Case 2: The dimension cut cannot be chosen such that e_1, e_2 belong to G_2.

In this case all M-edges in the last canonical d-cube join two vertices in the same canonical 4-cycle C_j. This case is similar to Subcase 1.2, except that it is easier since we need not worry about the C_i which appears in Subcase 1.2.

□

As $\mathcal{Q}_{r+d}$ has a spanning subgraph which is isomorphic to $C_{2^r} \Box \mathcal{Q}_d$ which is $(d+2)$-regular and has a spanning subgraph isomorphic to $P_{2^r} \Box \mathcal{Q}_d$, it follows that the graphs in Theorem 4 can also be realized by spanning subgraphs of cubes for each $k \geq 7$.

We do not know if $C_q \Box \mathcal{Q}_d$ has the PH-property for $d = 3, 4$, see Open Problem 1 below.

8 Open problems

In this section we summarize the open problems mentioned earlier.

Open Problem 1 *Does $C_q \Box \mathcal{Q}_d$ have the PH-property for $d = 3, 4$?*

Theorem 4 shows that those edges which cannot be extended together with a perfect matching M to a Hamiltonian cycle form a matching. As mentioned earlier, we do not know how big this matching can be. Specifically, we suggest the following problem.

Open Problem 2 *Let $d \geq 2$ be an integer and let M be a pairing in $\mathcal{Q}_d$. Does $\mathcal{Q}_d$ contain a matching M' with at most 100 edges such that, for every edge e in $\mathcal{Q}_d - M'$, $\mathcal{Q}_d$ contains a perfect matching M'' containing e such that $M \cup M''$ is a Hamiltonian cycle?*

In this paper we have focused primarily on sparse graphs with the PH-property which are subgraphs of the cubes. It may also be of interest to study the PH-property for general sparse graphs. In particular, it would be interesting to find a counterpart of Theorem 3 for 4-regular graphs. We suggest the following:

Open Problem 3 *For which values of p, q does $C_p \Box C_q$ have the PH-property?*

We showed in Section 5 that $P_q \Box K_4$ has a perfect matching that cannot be extended to a perfect matching when $q \geq 3$. Hence $P_q \Box C_4$ does not have the PH-property for $q \geq 3$. The same argument shows that $C_q \Box C_4$ does not have the PH-property except for finitely many q. Seongmin Ok and Thomas Perrett (private communication) have shown, by a similar argument, that even $C_q \Box K_4$ does not have the PH-property when $q \geq 7$.

The square of a cycle C_p^2 is obtained from a cycle C_p by adding all edges joining vertices of distance 2 on the cycle. Seongmin Ok and Thomas Perrett (private communication) have also shown that C_p^2 does not have the PH-property when p is large.

On the positive side, Seongmin Ok and Thomas Perrett (private communication) have informed us that they have modified the method of Theorem 3 to obtain an infinite class of 4-regular graphs with the PH-property, namely each graph which can be obtained from a cycle C_p, $p \geq 3$ by replacing each vertex by two vertices and replacing each edge by the four edges joining the corresponding pairs of vertices.

We do not know of graphs of girth at least 5 and with the PH-property.

Open Problem 4 *Are there infinitely many graphs of girth at least 5 and with the PH-property?*

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