

Connectivity of Fibonacci cubes, Lucas cubes, and generalized cubes

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If f is a binary word and d a positive integer, then the generalized Fibonacci cube $Q_d(f)$ is the graph obtained from the d -cube Q_d by removing all the vertices that contain f as a factor, while the generalized Lucas cube $Q_d(\bar{f})$ is the graph obtained from Q_d by removing all the vertices that have a circulation containing f as a factor. The Fibonacci cube Γ_d and the Lucas cube Λ_d are the graphs $Q_d(11)$ and $Q_d(\bar{11})$, respectively. It is proved that the connectivity and the edge-connectivity of Γ_d as well as of Λ_d are equal to $\lfloor \frac{d+2}{3} \rfloor$. Connected generalized Lucas cubes and generalized Fibonacci cubes are proved to be 2-connected. It is asked whether the connectivity equals minimum degree also for all generalized Fibonacci/Lucas cubes. It was checked by computer that the answer is positive for all f and all $d \leq 9$.

Keywords: Fibonacci cube; Lucas cube; generalized Fibonacci cube; generalized Lucas cube; connectivity; combinatorics on words

1 Introduction

Fibonacci cubes [4] and Lucas cubes [16] form hypercube-like classes of graphs that have found several applications and were extensively studied so far, see the recent survey [9]. The topics studied include different metric aspects [1, 10, 11, 13], a number of computer science issues [3, 18, 20, 21], applications in chemistry [23, 24, 25], and a variety of additional topics [2, 15, 17]. It is hence quite surprising, that, to the best of our knowledge, the connectivity of these cubes has not yet been established. This is even more surprising after recalling that Fibonacci cubes were originally introduced as a model for interconnection networks. Actually, in the seminal paper on the Fibonacci cubes it was stated without a proof that if $d \geq 3$, then $\lfloor \frac{d}{8} \rfloor \leq \kappa(\Gamma_d) \leq \kappa'(\Gamma_d) \leq \lfloor \frac{d-2}{3} \rfloor$ [4, Theorem 3]. Moreover, an exact value for the connectivity of

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Fibonacci cubes (in a more general framework) was asserted in [5, Theorem 2, point 3], but no proof was provided and the stated result also does not appear to be correct (at least for Fibonacci cubes).

In this paper we fill this gap by determining the vertex- and the edge-connectivity of Fibonacci cubes and of Lucas cubes, see Section 2. In the subsequent section we prove that the generalized Fibonacci cubes are always 2-connected while in Section 4 we characterize connected generalized Lucas cubes. In the final section we ask whether the connectivity of all generalized Fibonacci/Lucas cubes equals the minimum degree. Using a computer, the answer turned out to be positive for all strings f and all dimensions $d \leq 9$.

In the rest of this section we recall the basic concepts needed and notation used in this paper. The d -cube Q_d , $d \geq 0$, is the graph whose vertices are the binary words (alias strings) of length d , two vertices are adjacent if they differ in exactly one bit. In particular, $Q_0 = K_1$ and $Q_1 = K_2$. The vertex deleted d -cube, that is, the graph obtained from Q_d by removing one of its vertices, will be denoted by Q_d^- . The *Fibonacci cube* Γ_d , $d \geq 0$, is the graph obtained from Q_d by removing all vertices that contain two consecutive 1s, while the *Lucas cube* Λ_d , $d \geq 0$, is obtained from Γ_d by further removing the vertices that start and end with 1. We will use the concatenation notation, in particular, for a string u , the notation $u = 1u'$ means that u starts with 1, and 1^d denotes the string consisting of d 1s. As usual, for a graph G , its (vertex-)connectivity and edge-connectivity will be denoted by $\kappa(G)$ and $\kappa'(G)$, respectively, and its minimum degree with $\delta(G)$. We will write $G \cong H$ to denote that G and H are isomorphic graphs. Finally, the Fibonacci numbers are defined by $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$, $n \geq 2$.

2 Fibonacci cubes and Lucas cubes

To determine the connectivity of Fibonacci cubes, we first recall the *fundamental decomposition* of Γ_d . If $d \geq 1$, then the vertex set of Γ_d naturally partitions into the sets

$$A_d = \{b_1 \dots b_d \mid b_1 = 1\} \quad \text{and} \quad B_d = \{b_1 \dots b_d \mid b_1 = 0\}.$$

Since a string of A_d , $d \geq 2$, necessarily starts with 10, the set A_d induces a subgraph of Γ_d isomorphic to Γ_{d-2} . Similarly, B_d induces a subgraph of Γ_d isomorphic to Γ_{d-1} . Moreover, each vertex $1u$ of A_d has exactly one neighbor in B_d , namely the vertex $0u$. In other words, the edges between A_d and B_d form a matching from A_d to B_d .

Recall (cf. [9]) that $|V(\Gamma_d)| = F_{d+2}$. From [12] (cf. Corollary 3.3 and the last remark in Section 5) we also recall:

Lemma 2.1 *If $d \geq 1$, then $\delta(\Gamma_d) = \delta(\Lambda_d) = \lfloor \frac{d+2}{3} \rfloor$.*

Our first main result now reads as follows.

Theorem 2.2 *If $d \geq 1$, then*

$$\kappa(\Gamma_d) = \kappa'(\Gamma_d) = \left\lfloor \frac{d+2}{3} \right\rfloor.$$

Proof: The result can be checked for $d \leq 5$ by inspection. Suppose now that the result is true for $d \leq 3k+2$, $k \geq 1$. Continuing by induction we are going to prove the result for $d = 3k+3$, $d = 3k+4$, and $d = 3k+5$.

Let $d = 3k+3$. Then by the fundamental decomposition, Γ_{3k+3} decomposes into the sets A_{3k+3} and B_{3k+3} , and there is a matching from A_{3k+3} to B_{3k+3} . Let $X_{3k+1} \cong \Gamma_{3k+1}$ and $X_{3k+2} \cong \Gamma_{3k+2}$ be the subgraphs of Γ_{3k+3} induced on A_{3k+3} and B_{3k+3} , respectively. By the induction hypothesis, $\kappa(X_{3k+1}) =$

$\kappa(X_{3k+2}) = k + 1$. We claim that $\kappa(\Gamma_{3k+3}) \geq k + 1$ and suppose on the contrary that Γ_{3k+3} contains a separating set S with $|S| = k$. If $S \subseteq A_{3k+3}$, then $X_{3k+1} \setminus S$ is connected and hence Γ_{3k+3} is connected. Similarly Γ_{3k+3} is connected if $S \subseteq B_{3k+3}$. So necessarily some vertices of S lie in A_{3k+3} and some in B_{3k+3} . But then both $X_{3k+1} \setminus S$ and $X_{3k+2} \setminus S$ are connected. Moreover, as $k = |S| < |A_{3k+3}| = F_{3k+3}$, there exists an edge that connects a vertex of $X_{3k+1} \setminus S$ with a vertex of $X_{3k+2} \setminus S$. We have thus proved that $\kappa(\Gamma_{3k+3}) \geq k + 1$ holds. By Lemma 2.1 we conclude that $\kappa(\Gamma_{3k+3}) = k + 1$.

Let $d = 3k + 4$. We need to show that $\kappa(\Gamma_{3k+4}) = k + 2$ and using Lemma 2.1 we only need to prove that $\kappa(\Gamma_{3k+4}) \geq k + 2$. Let $X_{3k+2} \cong \Gamma_{3k+2}$ and $X_{3k+3} \cong \Gamma_{3k+3}$ be the subgraphs of Γ_{3k+4} induced by the fundamental decomposition A_{3k+4} and B_{3k+4} , respectively. By the induction hypothesis and the already proved case where $d = 3k + 3$, $\kappa(X_{3k+2}) = k + 1$ and $\kappa(X_{3k+3}) = k + 1$. Suppose that Γ_{3k+4} contains a separating set S with $|S| = k + 1$. If $S \subseteq A_{3k+4}$, then since any vertex of $A_{3k+4} \setminus S$ has a neighbor in B_{3k+4} , the graph $\Gamma_{3k+4} \setminus S$ is connected. Similarly, if both A_{3k+4} and B_{3k+4} contain some vertices of S , then $X_{3k+2} \setminus S$ and $X_{3k+3} \setminus S$ are both connected and so is $\Gamma_{3k+4} \setminus S$. Assume finally that $S \subseteq B_{3k+4}$ and consider the fundamental decomposition of $X_{3k+3} \cong \Gamma_{3k+3}$. It decomposes into $Y \cong \Gamma_{3k+1}$ and $Z \cong \Gamma_{3k+2}$ that are (by induction) both of connectivity $k + 1$. Since S disconnects X_{3k+3} and $|S| = k + 1$, we infer that $S \subseteq V(Z)$, that is, every vertex of S , considered as a vertex of Z , begins with 0. It follows that any vertex of S is considered as a vertex of X_{3k+3} and starts with 00. Now, the subgraph of X_{3k+3} induced by the vertices starting with 01 is connected. Moreover, since there are $|V(\Gamma_{3k+1})| = F_{3k+3} > k + 1$ independent edges between the vertices of X_{3k+3} starting with 010 and the vertices starting with 000, there is a vertex of X_{3k+3} starting with 01 that has a neighbor starting with 00 in the graph $\Gamma_{3k+4} \setminus S$. This vertex has in turn a neighbor in X_{3k+2} . Recalling that $A_{3k+4} \cap S = \emptyset$ and using the fact that any vertex of $X_{3k+2} \setminus S$ has a neighbor in X_{3k+3} , we conclude that $\Gamma_{3k+4} \setminus S$ is connected.

The last case to consider is $d = 3k + 5$. Let A_{3k+5} , B_{3k+5} , $X_{3k+3} \cong \Gamma_{3k+3}$, and $X_{3k+4} \cong \Gamma_{3k+4}$ have the same meaning as in the previous cases. By the already proved previous cases, $\kappa(X_{3k+3}) = k + 1$ and $\kappa(X_{3k+4}) = k + 2$. We need to show that $\kappa(\Gamma_{3k+5}) \geq k + 2$ and assume that Γ_{3k+5} contains a separating set S with $|S| = k + 1$. But now S cannot lie completely in X_{3k+4} (as $\kappa(X_{3k+4}) = k + 2$), while in the other cases we can argue again as we did in the first paragraph to conclude that S cannot be a separating set.

Hence we have proved that $\kappa(\Gamma_d) = \delta(\Gamma_d)$. As for any graph G , the inequalities $\kappa(G) \leq \kappa'(G) \leq \delta(G)$ hold, the result follows. \square

Lucas cubes also admit a *fundamental decomposition* as follows. The vertex set of Λ_d , $d \geq 1$, partitions into the sets

$$A_d = \{b_1 \dots b_d \mid b_1 = 1\} \quad \text{and} \quad B_d = \{b_1 \dots b_d \mid b_1 = 0\}.$$

Since a string of A_d , $d \geq 3$, necessarily starts with 10 and ends with 0, the set A_d induces a subgraph of Λ_d isomorphic to Γ_{d-3} . Similarly, B_d induces a subgraph of Λ_d isomorphic to Γ_{d-1} . Moreover, each vertex $1u$ of A_d has exactly one neighbor in B_d , namely the vertex $0u$. Thus the edges between A_d and B_d form a matching from A_d to B_d .

For the Lucas cubes we have a result parallel to Theorem 2.2 with a single exception: $\kappa(\Lambda_4) = 1$ and $\kappa'(\Lambda_4) = 2$.

Theorem 2.3 *If $d \geq 1$, $d \neq 4$, then*

$$\kappa(\Lambda_d) = \kappa'(\Lambda_d) = \left\lfloor \frac{d+2}{3} \right\rfloor.$$

Proof: We proceed similarly as in the proof of Theorem 2.2, however not all arguments will be parallel. Moreover, in the proof we will apply Theorem 2.2. First, we have checked the result for $d \leq 8$ by computer. Assuming that the result is true for $d \leq 3k + 2$; $k \geq 2$, we are going to prove it for $d = 3k + 3$, $d = 3k + 4$, and $d = 3k + 5$.

Let $d = 3k + 3$. By the fundamental decomposition, Λ_{3k+3} is decomposed into the sets A_{3k+3} and B_{3k+3} , and there is a matching from A_{3k+3} to B_{3k+3} . Let $X_{3k} \cong \Gamma_{3k}$ and $X_{3k+2} \cong \Gamma_{3k+2}$ be the subgraphs of Γ_{3k+3} induced on A_{3k+3} and B_{3k+3} , respectively. By Theorem 2.2, $\kappa(X_{3k}) = k$ and $\kappa(X_{3k+2}) = k + 1$. Then as in the proof of Theorem 2.2 we infer that Λ_{3k+3} does not contain a separating set of size k . Hence $\kappa(\Lambda_{3k+3}) \geq k + 1$ and by Lemma 2.1, $\kappa(\Lambda_{3k+3}) = k + 1$.

Let $d = 3k + 4$. We need to show that $\kappa(\Lambda_{3k+4}) = k + 2$. By Lemma 2.1 we only need to prove that $\kappa(\Lambda_{3k+4}) \geq k + 2$. Let $X_{3k+1} \cong \Gamma_{3k+1}$ and $X_{3k+3} \cong \Gamma_{3k+3}$ be the subgraphs of Λ_{3k+4} induced on the fundamental decomposition, A_{3k+4} and B_{3k+4} , respectively. By Theorem 2.2, $\kappa(X_{3k+1}) = k + 1$ and $\kappa(X_{3k+3}) = k + 1$. Suppose that Λ_{3k+4} contains a separating set S with $|S| = k + 1$. If $S \subseteq A_{3k+4}$, then since any vertex in $A_{3k+4} \setminus S$ has a neighbor in B_{3k+4} , the graph $\Lambda_{3k+4} \setminus S$ is connected. Similarly, if both A_{3k+4} and B_{3k+4} contain some vertices of S , then $X_{3k+1} \setminus S$ and $X_{3k+3} \setminus S$ are both connected and so is $\Lambda_{3k+4} \setminus S$. Assume finally that $S \subseteq B_{3k+4}$ and consider the decomposition of X_{3k+3} into the subgraphs Y and Z , whose vertices start with 01 and 00, respectively where each vertex in $V(Y)$ has a neighbor in $V(Z)$. Then $Y \cong \Gamma_{3k+1}$ and $Z \cong \Gamma_{3k+2}$ and they are both of connectivity $k + 1$ by Theorem 2.2. If $S \subseteq V(Y)$, then as any vertex in $V(Y \setminus S)$ has a neighbor in $V(Z)$, $\Lambda_{3k+4} \setminus S$ is connected. If both Y and Z contain some vertices of S , then $Y \setminus S$ and $Z \setminus S$ are both connected. Considering that $|S| < |V(Y)| < |V(Z)|$, we conclude that $X_{3k+3} \setminus S$ is connected and so is $\Lambda_{3k+4} \setminus S$. Assume $S \subseteq V(Z)$. Decompose Z into the subgraphs C and D , whose vertices end with 0 and 1, respectively. Then $C \cong \Gamma_{3k+1}$ and $D \cong \Gamma_{3k}$ that are of connectivity $k + 1$ and k , respectively by Theorem 2.2. Also decompose Y into the subgraphs E and H , whose vertices end with 0 and 1, respectively. Then $E \cong \Gamma_{3k}$ and $H \cong \Gamma_{3k-1}$ are of connectivity k by Theorem 2.2. Note that every vertex in $V(Y)$ has a neighbor in $V(Z)$. Also each vertex in $V(D)$ has a neighbor in $V(C)$ and each vertex in $V(H)$ has a neighbor in $V(E)$. As $|S| < |V(E)|$, there is a vertex $x \in V(C \setminus S)$ which has a neighbor $x' \in V(E) \subseteq V(Y)$. Also x has a neighbor x'' in A_{3k+4} . Considering that X_{3k+1} and Y are both connected, $V(X_{3k+1}) \cup V(Y) \cup V(C \setminus S)$ is connected. To show that $\Lambda_{3k+4} \setminus S$ with its vertex set $A_{3k+4} \cup V(Y) \cup V(C \setminus S) \cup V(D \setminus S)$ is connected, we only need to show that all the vertices in $V(D \setminus S)$ are connected to some vertex in $V(Y) \cup V(C \setminus S)$. As $|S| < |V(H)|$, there is a vertex y in $V(D \setminus S)$ which has a neighbor in $V(H) \subseteq V(Y)$. As long as $|S \cap V(C)| \geq 2$, we have $|S \cap V(D)| \leq k - 1$ and hence $D \setminus S$ is connected. Therefore in this case all the vertices in $V(D \setminus S)$ are connected to y and hence to the vertices in $V(Y) \cup V(C \setminus S)$. Note that any vertex in $V(D)$ has a neighbor in $V(C)$. Therefore if $S \cap V(C) = \emptyset$, then $\Lambda_{3k+4} \setminus S$ is connected. Assume $|S \cap V(C)| = 1$ and $D \setminus S$ is disconnected. Choose a vertex u in $D \setminus S$. If all the neighbors of u are in S , then $|S| = k + 1 \geq \delta(\Lambda_{3k+4})$ and hence $k + 1 \geq \kappa(\Lambda_{3k+4})$. If u has a neighbor u' in $V(C \setminus S)$, then it is connected all the vertices in $V(Y) \cup V(C \setminus S)$. If not, then u has a neighbor v in $V(D \setminus S)$. Considering that $|S \cap V(C)| = 1$ and hence $S \cap V(C) = \{u'\}$, v has a neighbor v' in $V(C \setminus S)$. Therefore v and hence u is connected to all the vertices in $V(Y) \cup V(C \setminus S)$.

Finally let $d = 3k + 5$, and let A_{3k+5} , B_{3k+5} , $X_{3k+2} \cong \Gamma_{3k+2}$, and $X_{3k+4} \cong \Gamma_{3k+4}$ have the same meaning as before. By the already proved previous cases, $\kappa(X_{3k+2}) = k + 1$ and $\kappa(X_{3k+4}) = k + 2$. We can now proceed as in the last part of the proof of Theorem 2.2. \square

3 Generalized Fibonacci cubes

Fibonacci cubes and Lucas cubes were recently extended to generalized Fibonacci cubes [6] and to generalized Lucas cubes [7] as follows. If f is an arbitrary binary word and d is a positive integer, then the *generalized Fibonacci cube* $Q_d(f)$ is the graph obtained from Q_d by removing all the vertices that contain f as a factor. We should point out that earlier the term generalized Fibonacci cubes was used in [14, 22] for the special classes $Q_d(1^n)$. Similarly, the *generalized Lucas cube* $Q_d(\overline{f})$ is the graph obtained from Q_d by removing all the vertices that have a circulation containing f as a factor. Using these notations, $\Gamma_d \cong Q_d(11)$ and $\Lambda_d \cong Q_d(\overline{11})$.

For a binary string $b = b_1 \dots b_d$, let \bar{b} be the binary complement of b and let $b^R = b_d \dots b_1$ be the reverse of b . The following basic result helps to significantly reduce the number of cases to be considered.

Lemma 3.1 [6, Lemmas 2.2 and 2.3] *If f be a binary string and $d \geq 1$, then $Q_d(f) \cong Q_d(\bar{f}) \cong Q_d(f^R)$.*

It was observed in [8, p.2] that every generalized Fibonacci cube is connected. The case $Q_d(10) \cong P_{d+1}$ is not interesting and we have treated the case $Q_d(11) = \Gamma_d$ in the previous section. For any other forbidden string f we have the following general result.

Theorem 3.2 *If f is a binary string with $|f| \geq 3$, and $d \geq 3$, then $Q_d(f)$ is 2-connected.*

Proof: Due to Lemma 3.1 we may assume throughout the proof that f begins with 0. In addition, we may also assume that $d > |f|$, where $|f|$ denotes the cardinality of f . Indeed, if $d = |f|$, then $Q_d(f) \cong Q_d^-$, while if $d < |f|$, then $Q_d(f) \cong Q_d$. Since $d \geq 3$, both Q_d^- and Q_d are 2-connected.

Let now u be an arbitrary vertex of $Q_d(f)$ containing at least one 0. Let $P_1(u)$ be the path in Q_d between u and 1^d that is obtained by changing from left to right one by one the bits 0 of u . This path lies completely in $Q_d(f)$. Indeed, if f would be a substring of a vertex w that lies on $P_1(u)$, then since f starts with 0, only the bits left of this 0 would be changed. This would in turn imply that f would already be a substring of $u \in V(Q_d(f))$. We now distinguish the following cases.

Case 1: $f = 0f'0$.

Subcase 1.1: $u = x0y0z$, $|x|, |y|, |z| \geq 0$, $|x| + |y| + |z| = d - 2$.

Let $P_2(u)$ be the path between u and 1^d that is obtained by changing from right to left one by one the bits 0 of u . Since f ends with 0, an argument parallel to the above implies that $P_2(u)$ lies completely in $Q_d(f)$. By the construction, $P_1(u)$ and $P_2(u)$ are different, internally disjoint paths.

Subcase 1.2: $u = 1^r 01^s$, $r, s \geq 0$, $r + s = d - 1$.

Since $d \geq 4$, we may assume by the symmetry that $s \geq 2$. Let now $P_2(u)$ be the path $u \rightarrow 1^r 001^{s-1} \rightarrow 1^{r+1} 01^{s-1} \rightarrow 1^d$. Note that $P_2(u)$ lies in $Q_d(f)$ and that it is internally disjoint from $P_1(u)$; the latter path in this case is $u \rightarrow 1^d$.

In any of the two subcases, for any vertex u of $Q_d(f)$ there exist two different, internally disjoint $u, 1^d$ -paths. It follows that if w is any fixed vertex of $Q_d(f)$, then any vertex of $Q_d(f) - w$ is connected with a path to 1^d . Consequently, w is not a cut-vertex and hence $Q_d(f)$ is 2-connected.

We still need to consider the forbidden strings f that finish with 1, and first consider the strings that start with two zeros.

Case 2: $f = 00f'1$.

Suppose first that u contains at least two zeros. Then set $P_2(u)$ be the path in Q_d between u and 1^d that is obtained by changing from left to right, but starting with the second bit of 0, one by one the bits 0 of u , and finally by changing the first bit of 0. It is again straightforward to see that $P_2(u)$ is a path of $Q_d(f)$. Suppose next that $u = 1^r 01^s$, $r, s \geq 0$, $r + s = d - 1$, and assume without loss of generality that $s \geq 2$. Now let $P_2(u)$ be the path $u \rightarrow 1^r 0101^{s-2} \rightarrow 1^{r+2} 01^{s-2} \rightarrow 1^d$. So in any case we have constructed two different, internally disjoint $u, 1^d$ -paths in $Q_d(f)$ which in turn implies that $Q_d(f)$ is 2-connected.

The last case to consider is when $f = 01f'1$. In the subcase when f is of the form $01f''11$, Lemma 3.1 implies that $Q_d(01f''11) \cong Q_d(00f'''01)$. Since this situation was already treated in Case 2, it remains to consider the following:

Case 3: $f = 01f'01$.

Let $P'_2(u)$ be the path in Q_d between u and 0^d obtained from u by changing from right to left one by one the bits 1 of u . We observe, having in mind that f ends with 1, that $P'_2(u)$ lies in $Q_d(f)$. If $u = 0^d$, then $P'_2(u)$ consists of a single vertex, hence $V(P'_2(u)) \cap V(P_1(u)) = \{u\}$. Otherwise, considering the last coordinate of u which is equal 1, we infer that also $V(P'_2(u)) \cap V(P_1(u)) = \{u\}$. Let in addition $P''_2(u)$ be the path in Q_d between 0^d and 1^d obtained by changing from right to left one by one the bits 0 of 0^d . Considering the first coordinate of the vertices we find that $V(P''_2(u)) \cap V(P_1(u)) = \{1^d\}$. The concatenation of $P'_2(u)$ and $P''_2(u)$ is a walk between u and 1^d . Let $P_2(u)$ be a $u, 1^d$ -path contained in the concatenation. Then by the above, $V(P_2(u)) \cap V(P_1(u)) = \{u, 1^d\}$. Hence we can again conclude that $Q_d(f)$ is 2-connected. \square

Theorem 3.2 is best possible in the following sense.

Corollary 3.3 *If $d \geq 3$ and $f \in \{001, 010\}$, then $\kappa(Q_d(f)) = \kappa'(Q_d(f)) = 2$.*

Proof: By Theorem 3.2, $\kappa(Q_d(001)) \geq 2$ and $\kappa(Q_d(010)) \geq 2$. On the other hand, the only neighbors of 0^d in $Q_d(001)$ are 10^{d-1} and 010^{d-2} , while the only neighbors of 0^d in $Q_d(010)$ are 10^{d-1} and $0^{d-1}1$. Hence $\delta(Q_d(001)) = \delta(Q_d(010)) = 2$ and the result follows. \square

4 Generalized Lucas cubes

It is not difficult to observe that every generalized Fibonacci cube is connected and, as proved in the previous section, it is also 2-connected. On the other hand, the connectedness of generalized Lucas cubes is not straightforward. In this section we characterize connected generalized Lucas cubes. Again, the next result helps to reduce the number of cases to be considered.

Lemma 4.1 [7, Lemma 1] *If f is a binary string and $d \geq 1$, then $Q_d(\overline{f}) \cong Q_d(\overline{\overline{f}}) \cong Q_d(\overline{f^R})$.*

Before presenting the result(s) of this section, we need to fix some additional notation. For $n \in \mathbb{N}$ we will use $[n]$ to denote the set $\{1, \dots, n\}$. The *weight* $w(f)$ of a binary string f is the number of 1s in f . For $d \geq 1$ and $1 \leq i \leq d$, let e_i be the binary string of length d with 1 in the i -th position and 0 elsewhere. By a *block* of a binary string we mean a substring consisting of the same bit, maximal with respect to inclusion.

Lemma 4.2 *Let f be a binary string with $w(f) \in [|f| - 1]$, and d an integer such that $d \geq |f| \geq 2$. Then the vertices 0^d and 1^d of $Q_d(\overline{f})$ are in the same connected component if and only if $w(f) \in [|f| - 1] \setminus \{1, |f| - 1\}$.*

Proof: Since $Q_d(\overline{f}) \cong Q_d(\overline{\overline{f}})$ holds for any $d \geq 1$ by Lemma 4.1(i), we may without loss of generality assume that $w(f) \leq |f|/2$. If $w(f) = 1$, then it is clear that the vertex 0^d is an isolated vertex of $Q_d(\overline{f})$. Hence assume in the rest that $1 < w(f) \leq |f|/2$. Then since $w(f) \geq 2$ and $w(f) \leq |f|/2$ we infer that $|f| \geq 4$. We now distinguish the following cases.

Case 1: f has two blocks.

We may without loss of generality assume that $f = 1^i 0^{k-i}$. Let $v = 0101 \dots 10$ or $v = 0101 \dots 01$ depending on whether d is odd or even. Note that $v \in Q_d(\overline{f})$. Consider the following path from 0^d to v which we obtain by changing from left to right the bits at even coordinates one by one:

$$0^d \rightarrow 010^{d-2} \rightarrow 01010^{d-4} \rightarrow \dots \rightarrow v.$$

Each vertex of this path does not have a circulation which contains 11 as a substring and hence it does not have a circulation which contains f as a substring either. In other words, this path lies completely in $Q_d(\overline{f})$. Next consider the path from v to 1^d which we obtain by changing from left to right the 0 bits to 1 bits one by one. Certainly each vertex of this path does not have a circulation which contains 00 as a substring and hence it does not have a circulation which contains f as a substring either. Hence also this path lies in $Q_d(\overline{f})$. Therefore by concatenating the above two paths we obtain a path from 0^d to 1^d in $Q_d(\overline{f})$.

Case 2: f has three blocks.

We distinguish two subcases. Assume first that $f = 0^k 1^\ell 0^m$. Then $\ell \geq 2$ and $k, m \geq 1$. If $|f| = 4$, then $f = 0110$ is the unique forbidden string. Because $Q_d(\overline{0110}) \cong Q_d(\overline{1001})$ and $w(0110) = w(1001)$ we may consider $f = 1001$ instead. Consider now the path

$$0^d \rightarrow 10^{d-1} \rightarrow 110^{d-2} \rightarrow \dots \rightarrow 1^{d-3} 0^3 \rightarrow 1^{d-3} 010 \rightarrow 1^{d-3} 011 \rightarrow 1^d$$

to see that there is a $0^d, 1^d$ -path in $Q_d(\overline{1001})$. If $|f| \geq 5$, then we can use the $0^d, 1^d$ -path constructed in Case 1.

In the second subcase let $f = 1^p 0^q 1^r$, where $q \geq 2$ and $p, r \geq 1$. Then first construct the path

$$0^d \rightarrow 10^{d-1} \rightarrow 110^{d-2} \rightarrow \dots \rightarrow 1^{d-\ell-1} 0^{\ell+1} \rightarrow 1^{d-\ell-1} 010^{\ell-1} = v.$$

Since $d - q - 1 \geq 1$ we can change in v all 0 bits one by one (in any order) to complete the above path from 0^d to 1^d .

Case 3: f has more than three blocks.

Consider the following path $0^d \rightarrow 10^{d-1} \rightarrow \dots \rightarrow 1^d$ obtained by changing from left to right the 0 bits to 1 bits one by one. Each vertex in this path has only two blocks and hence it does not have a circulation which contains f as a substring. \square

We can now characterize the connected generalized Lucas cubes as follows.

Theorem 4.3 *Let f be a binary string and d an integer such that $d \geq |f| \geq 2$. Then $Q_d(\overline{f})$ is connected if and only if $w(f) \in [|f|] \setminus \{1, |f| - 1\}$. Moreover, if $w(f) \in \{1, |f| - 1\}$, then $Q_d(\overline{f})$ consists of an isolated vertex and a connected component containing all the other vertices.*

Proof: Suppose first that $w(f) = 0$, that is, $f = 0^k$. Let u be an arbitrary vertex of $Q_d(\overline{0^k})$. Then changing one by one the 0 bits of u to 1, we stay in $Q_d(\overline{0^k})$ and reach the vertex 1^d . Hence every vertex is connected with a path to 1^d and so $Q_d(\overline{0^k})$ is connected. The case when $w(f) = 1$ was done in [7, Proposition 10]. Hence, having in mind Lemma 4.1(i), we can assume in the rest that $2 \leq w(f) \leq |f| - 2$.

Using Lemma 4.1 again we may without loss of generality assume that f starts with 0. For a string u let $b(u)$ denote the length of its longest block of 1's in a circular manner. For instance, $b(101110) = 3$ and $b(1100111011) = 4$. We are going to show that each vertex in $Q_d(\overline{f})$ is connected to 1^d by a path. Take any string $v \in Q_d(\overline{f})$. We distinguish the following two cases.

Case 1: $b(v) < b(f)$.

Consider a path from v to 0^d we obtain by changing one by one the bits 1 to 0. For any vertex u in this path we have $b(u) \leq b(v) < b(f)$ and hence it does not have a circulation which contains f as a substring. We have therefore shown that any vertex of $Q_d(\overline{f})$ is connected with a path to 1^d . By Lemma 4.2 this case is done.

Case 2: $b(v) \geq b(f)$.

Let $v_i = 1, \dots, v_{i+b(v)-1} = 1$ be a substring of v of length $b(v)$ consisting of only 1s. Set $v' = v + e_{i+b(v)}$. We claim that $v' \in Q_d(\overline{f})$. Suppose on the contrary that it contains a copy \tilde{f} of f as a (circular) substring. Then, as \tilde{f} starts with 0, \tilde{f} is contained in $v_{i+b(v)+1}v_{i+b(v)+2} \dots v_{i+b(v)-1}$. But this means that \tilde{f} is contained in a circulation of v as a substring also, a contradiction. Hence the claim is proved. We now proceed by changing one by one the bits 0 that appear after the position $i + b(v)$. Again, all the obtained vertices lie in $Q_d(\overline{f})$. Indeed, if at some point the bit v_j , $j \geq i + b(v) + 1$, was changed, and if the obtained word would contain a copy \tilde{f} of f , then, using the fact that $b(f) \leq b(v)$, \tilde{f} would be contained in $v_{j+1}v_{j+2} \dots v_{i+b(v)-1}$. So again v would contain f , the final contradiction. \square

5 Concluding remarks

Theorem 4.3 thus asserts that for all practical purposes, each generalized cube can be considered as connected (by neglecting an isolated vertex). Confronting Theorem 4.3 with Theorem 3.2, a question arises whether all connected generalized Lucas cubes are actually 2-connected. More generally, we pose the following question:

Question 5.1 *Is it true that $\kappa(Q_d(\overline{f})) = \delta(Q_d(\overline{f}))$ holds for all f and d , except for $Q_4(\overline{11})$?*

The same kind of question can be asked for generalized Fibonacci cubes.

Question 5.2 *Is it true that $\kappa(Q_d(f)) = \delta(Q_d(f))$ for all f and d ?*

The answers to the above questions are likely to be positive. In both cases, using computer, the connectivity was confirmed to be equal to the minimum degree for all forbidden strings f and for dimensions

$4 \leq d \leq 9$. The computations were performed using the Sage program [19]. The same program was used for the computations to verify the base cases of Theorem 2.3.

If the answers to the above questions are indeed positive, an approach different from the one that we used for Fibonacci cubes and for Lucas cubes (that is, using their fundamental decompositions) will be needed in order to prove the corresponding theorems.

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