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Symmetric Bipartite Graphs and Graphs with Loops

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We show that if the two parts of a finite bipartite graph have the same degree sequence, then there is a bipartite graph, with the same degree sequences, which is symmetric, in that it has an involutive graph automorphism that interchanges its two parts. To prove this, we study the relationship between symmetric bipartite graphs and graphs with loops.

Keywords: bipartite graphs, degree sequence, symmetry

1 Introduction

We say that a finite sequence $\underline{d}$ of nonnegative integers is *bipartite graphic* if $\underline{d}$ can be realized as the degree sequence of both parts of a bipartite simple graph. For example, Figure 1 gives two realizations of the sequence $(2, 2, 1, 1)$. Notice that the realization on the left is symmetric, while the one on the right is not, where by *symmetric* we use the following natural definition.

Fig. 1:

Definition 1 *We say that a bipartite graph G is* symmetric *if there is an involutive graph automorphism of G that interchanges its two parts.*

We will establish the following result.

Theorem 1 *If a sequence $\underline{d}$ is bipartite graphic, then there is a realization of $\underline{d}$ that is symmetric.*

We are not aware of any similar result in the literature. Our proof of Theorem 1 relies on an observation connecting symmetric bipartite graphs with graphs-with-loops.

Definition 2 *By a* graph-with-loops *we mean a graph, without multiple edges, in which there is at most one loop at each vertex.*

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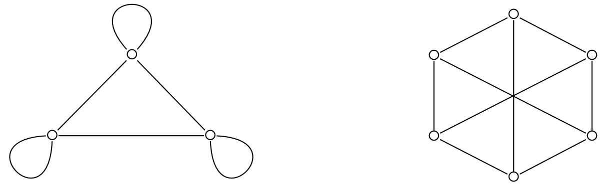

Fig. 2: The complete graph-with-loops on three vertices, and its bipartite double-cover.

Given a symmetric bipartite graph G with involution σ, the quotient graph G/σ has as its vertex set the set of (unordered) pairs $\bar{v} := \{v, \sigma(v)\}$, for vertices v of G, and there is an edge in G/σ between $\bar{v}$ and $\bar{w}$ if there is an edge in G between v and w or between v and $\sigma(w)$. So the quotient graph G/σ is a graph-with-loops. To see how this process can be reversed, recall that for every simple graph G, there is a natural associated bipartite simple graph $\hat{G}$ called the *bipartite double-cover* of G. The graph $\hat{G}$ is the *tensor product* $G \times K_2$ of G with the connected graph K_2 with 2 vertices; the vertex set of $G \times K_2$ is the Cartesian product of the vertices of G and K_2, there are edges in $G \times K_2$ between $(a, 0)$ and $(b, 1)$ and between $(a, 1)$ and $(b, 0)$ if and only if there is an edge in G between a and b; see [HIK11]. By construction, $\hat{G}$ is bipartite and symmetric; the automorphism is the map $\sigma : (a, x) \mapsto (a, 1 - x)$, and G is the quotient graph $\hat{G}/\sigma$. When G is a graph-with-loops, the above construction again produces a symmetric bipartite graph; each loop in G produces just one edge in $G \times K_2$. Figure 2 shows the construction for the complete graph-with-loops G on three vertices.

Note that graphs-with-loops are a special family of multigraphs [Die10] and that for multigraphs, the *degree* of a vertex is usually taken to be the number of edges incident to the vertex, with loops counted twice. For our purposes, a different definition of degree is more appropriate. We introduce the following definition.

Definition 3 *For a graph-with-loops, the* reduced degree *of a vertex is taken to be the number of edges incident to the vertex, with loops counted once.*

So, for example, in the complete graph-with-loops G on three vertices, shown on the left in Figure 2, the vertices each have reduced degree three. The vertices in the tensor product $\hat{G} = G \times K_2$ also have the same degrees as the reduced degrees of the corresponding vertices of G. In general, if $\underline{d}$ is the sequence of reduced degrees of the vertices of a graph-with-loops G, then $\hat{G}$ is a symmetric bipartite graph whose parts have degree sequences $\underline{d}$.

We will employ the following Erdős–Gallai type result; the proof is given in the final section.

Theorem 2 *Let $\underline{d} = (d_1, \dots, d_n)$ be a sequence of nonnegative integers in decreasing order. Then $\underline{d}$ is the sequence of reduced degrees of the vertices of a graph-with-loops if and only if for each integer k with $1 \le k \le n$,*

$$\sum_{i=1}^{k} d_i \le k^2 + \sum_{i=k+1}^{n} \min\{k, d_i\}. \tag{1}$$

Note that here, and elsewhere in this paper, the term "decreasing" is taken in the non-strict sense; in other words, non-increasing. Theorem 2 has the following application.

Theorem 3 *A sequence $\underline{d} = (d_1, \ldots, d_n)$ of nonnegative integers in decreasing order is the sequence of reduced degrees of the vertices of a graph-with-loops if and only if $\underline{d}$ is bipartite graphic.*

Proof: If $\underline{d}$ is the sequence of reduced degrees of the vertices of a graph-with-loops G, then the bipartite double-cover $\hat{G}$ of G has parts with degree sequences $\underline{d}$. Conversely, if $\underline{d}$ is bipartite graphic, then by the Gale–Ryser Theorem [Gal57, Rys57], for each k with $1 \le k \le n$,

$$\sum_{i=1}^{k} d_i \le \sum_{i=1}^{n} \min\{k, d_i\} \le k^2 + \sum_{i=k+1}^{n} \min\{k, d_i\},$$

and so by Theorem 2, $\underline{d}$ is the sequence of reduced degrees of the vertices of a graph-with-loops. □

We can now prove our main result.

Proof of Theorem 1: If $\underline{d}$ is bipartite graphic, then by Theorem 3, $\underline{d}$ is the sequence of reduced degrees of the vertices of a graph-with-loops G. Then the bipartite double-cover $\hat{G}$ of G is symmetric and its parts have degree sequences $\underline{d}$. □

2 Some Remarks

Remark 1 *From Theorem 2 and Theorem 3, condition* (1) *gives an Erdős–Gallai type condition for a sequence to be bipartite graphic, which is analogous to the Gale–Ryser condition.*

Remark 2 *It is clear from the discussion in Section 1 that if a sequence $(d_1, \ldots, d_n)$ is graphic, then by adding a loop at each vertex, $(d_1 + 1, d_2 + 1, \ldots, d_n + 1)$ is the sequence of reduced degrees of the vertices of a graph-with-loops, and so by Theorem 3, the sequence $(d_1 + 1, d_2 + 1, \ldots, d_n + 1)$ is bipartite graphic. Note that the converse is not true; for example, $(4, 4, 2, 2)$ is bipartite graphic, while $(3, 3, 1, 1)$ is not graphic.*

Remark 3 *There are several results in the literature of the following kind: if $\underline{\mathrm{d}}$ is graphic, and if $\underline{\mathrm{d}}'$ is obtained from $\underline{\mathrm{d}}$ using a particular construction, then $\underline{\mathrm{d}}'$ is also graphic. The Kleitman–Wang Theorem is of this kind [KW73]. Another useful result is implicit in Choudum's proof [Cho86] of the Erdős–Gallai Theorem: If a decreasing sequence $\underline{\mathrm{d}} = (d_1, \ldots, d_n)$ of positive integers is graphic, then so is the sequence $\underline{\mathrm{d}}'$ obtained by reducing both d_1 and d_n by one. Analogously, our proof of Theorem 2, which is modelled on Choudum's proof, also establishes the following result: If a decreasing sequence $\underline{\mathrm{d}} = (d_1, \ldots, d_n)$ of positive integers is bipartite graphic, then so is the sequence $\underline{\mathrm{d}}'$ obtained by reducing both d_1 and d_n by one.*

Remark 4 *The authors have pursued related ideas in [CM, CMN, CMN14]. In particular, an application of Theorem 3 is given in [CMN14]. For criteria for sequences to be realized by multigraphs, see [MV09]. For further background on graphic sequences, see [BHJW12, TT08, TV03, Yin09, YL05].*

3 Proof of Theorem 2

The following proof mimics Choudum's proof of the Erdős–Gallai Theorem [Cho86].

For the proof of necessity, consider the set S comprised of the first k vertices. The left hand side of (1) is the number of half-edges incident to S, with each loop counting as one. On the right hand side, k^2 is the number of half-edges in the complete graph-with-loops on S, again with each loop counting as one, while $\sum_{i=k+1}^{n} \min\{k, d_i\}$ is the maximum number of edges that could join vertices in S to vertices outside S. So (1) is obvious.

Conversely, suppose that $\underline{d} = (d_1, \dots, d_n)$ verifies (1) and consider the sequence $\underline{d}\,'$ obtained by reducing both d_1 and d_n by 1. Let $\underline{d}\,''$ denote the sequence obtained by reordering $\underline{d}\,'$ so as to be decreasing. Suppose that $\underline{d}\,''$ satisfies (1) and hence by the inductive hypothesis, there is a graph-with-loops G' that realizes $\underline{d}\,'$. We will show how $\underline{d}$ can be realized. Let the vertices of G' be labelled $v_1, \dots, v_n$. If there is no edge in G' connecting v_1 to v_n, then add one; this gives a graph-with-loops G that realizes $\underline{d}$. Similarly, if there is neither a loop at v_1 nor at v_n, just add loops at both v_1 and v_n. So it remains to treat the case where there is an edge in G' connecting v_1 to v_n, and at least one of the vertices v_1, v_n has a loop.

Now, for the moment, let us assume there is a loop in G' at v_1. Applying the hypothesis to $\underline{d}$, using $k = 1$ gives

$$d_1 \le 1 + \sum_{i=2}^{n} \min\{1, d_i\} \le n,$$

and so $d_1 - 2 < n - 1$. Now in G', the degree of v_1 is $d_1 - 1$ and so apart from the loop at v_1, there are a further $d_1 - 2$ edges incident to v_1. So in G', there is some vertex $v_i \ne v_1$, for which there is no edge from v_1 to v_i. Note that $d'_i > d'_n$. If there is a loop in G' at v_n, or if there is no loop at v_i nor at v_n, then there is a vertex v_j such that there is an edge in G' from v_i to v_j, but there is no edge from v_j to v_n. Now remove the edge $v_i v_j$, and put in edges from v_1 to v_i, and from v_j to v_n, as in Figure 3. This gives a graph-with-loops G that realizes $\underline{d}$. If there is no loop in G' at v_n, but there is a loop at v_i, remove the loop at v_i, add the edge $v_1 v_i$ and add a loop at v_n, as in Figure 4.

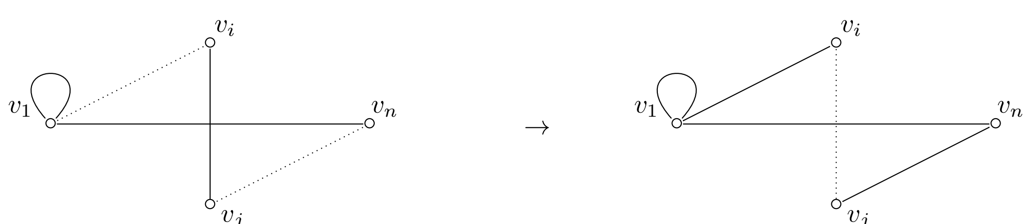

Fig. 3:

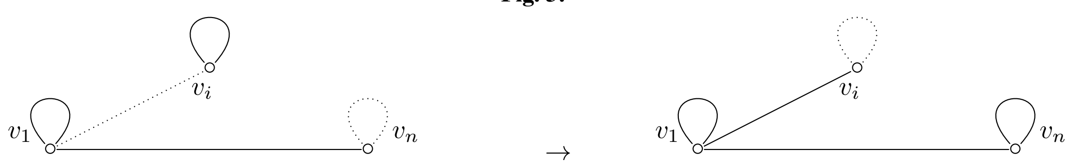

Fig. 4:

Finally, assume there is no loop in G' at v_1, but there is a loop in G' at v_n. So, apart from the loop, there are a further $d_n - 2$ edges incident to v_n. Since $d_1 \ge d_n$, we have $d_1 - 1 > d_n - 2$, and so there is a vertex v_i such that there is an edge in G' from v_1 to v_i, but there is no edge from v_i to v_n. Note that $d'_i > d'_n$, so as there is a loop in G' at v_n, there is a vertex v_j such that there is an edge in G' from v_i to v_j, but there is no edge from v_j to v_n. Now remove the loop at v_n and the edge $v_i v_j$, and put edges $v_j v_n$ and $v_i v_n$ and add a loop at v_1, as in Figure 5. This gives a graph-with-loops G that realizes $\underline{d}$.

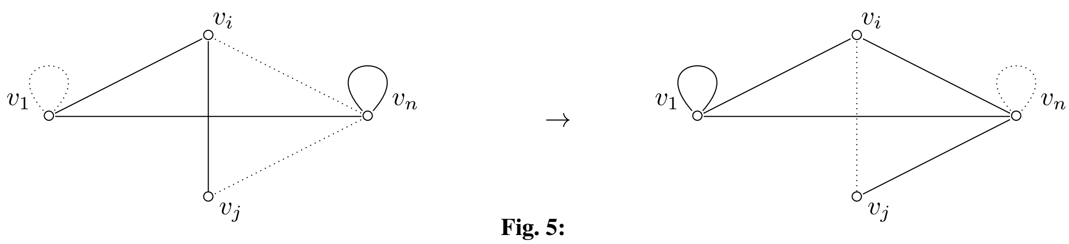

Fig. 5:

It remains to show that $\underline{d}''$ satisfies (1). Define m as follows: if the d_i are all equal, put $m = n-1$, otherwise, define m by the condition that $d_1 = \cdots = d_m$ and $d_m > d_{m+1}$. We have $d''_i = d_i$ for all $i \neq m, n$, while $d''_m = d_m - 1$ and $d''_n = d_n - 1$. Consider condition (1) for $\underline{d}''$:

$$\sum_{i=1}^{k} d''_i \leq k^2 + \sum_{i=k+1}^{n} \min\{k, d''_i\}. \tag{2}$$

For $m \leq k < n$, we have $\sum_{i=1}^{k} d''_i = \sum_{i=1}^{k} d_i - 1$, while $\sum_{i=k+1}^{n} \min\{k, d''_i\} \geq \sum_{i=k+1}^{n} \min\{k, d_i\} - 1$, and so (2) holds. For $k = n$, $\sum_{i=1}^{k} d''_i = \sum_{i=1}^{k} d_i - 2 < k^2$, and so (2) again holds. For $k < m$, first note that if $d_k \leq k$, then $\sum_{i=1}^{k} d''_i = \sum_{i=1}^{k} d_i \leq k^2 \leq k^2 + \sum_{i=k+1}^{n} \min\{k, d''_i\}$. So it remains to deal with the case where $k < m$ and $d_k > k$. We have

$$\sum_{i=1}^{k} d''_i = \sum_{i=1}^{k} d_i \leq k^2 + \sum_{i=k+1}^{n} \min\{k, d_i\}.$$

Notice that as $d_i = d''_i$ except for $i = m, n$, we have $\min\{k, d''_i\} = \min\{k, d_i\}$ except possibly for $i = m, n$. In fact, as $k < m$, we have $d_m = d_k > k$ and $d''_m = d_m - 1 \geq k$ and so $\min\{k, d_m\} = k = \min\{k, d''_m\}$. Hence $\sum_{i=k+1}^{n} \min\{k, d''_i\} \geq \sum_{i=k+1}^{n} \min\{k, d_i\} - 1$. Thus, in order to establish (2), it suffices to show that $\sum_{i=1}^{k} d_i < k^2 + \sum_{i=k+1}^{n} \min\{k, d_i\}$. Suppose instead that $\sum_{i=1}^{k} d_i \geq k^2 + \sum_{i=k+1}^{n} \min\{k, d_i\}$. We have

$$kd_m = \sum_{i=1}^{k} d_i \geq k^2 + \sum_{i=k+1}^{n} \min\{k, d_i\}$$

and so

$$d_m \geq k + \frac{1}{k} \sum_{i=k+1}^{n} \min\{k, d_i\}.$$

Then

$$\sum_{i=1}^{k+1} d_i = (k+1)d_m \geq k(k+1) + \frac{k+1}{k} \sum_{i=k+1}^{n} \min\{k, d_i\}.$$

We have $d_{k+1} = d_m > k$ and so $\min\{k, d_{k+1}\} = k$. Note that $\sum_{i=k+2}^{n} \min\{k, d_i\} \neq 0$ as $k + 2 \leq n$, since $k < m \leq n-1$. So

$$\sum_{i=1}^{k+1} d_i \geq k(k+1) + (k+1) + \frac{k+1}{k} \sum_{i=k+2}^{n} \min\{k, d_i\} > (k+1)^2 + \sum_{i=k+2}^{n} \min\{k, d_i\},$$

contradicting (1). Hence $\underline{d}''$ satisfies (2), as claimed. This completes the proof. □

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