Discrete Mathematics and Theoretical Computer Science DMTCS vol. **17:2**, 2015, 49–62

On graphs double-critical with respect to the colouring number

Matthias Kriesell[1]* Anders Sune Pedersen[2,3]†

[1] *Department of Mathematics, Ilmenau University of Technology, Germany.*
[2] *Department of Mathematics and Computer Science, University of Southern Denmark, Odense, Denmark.*
[3] *Research Clinic on Gambling Disorders, Aarhus University Hospital, Denmark.*

received 21st Oct. 2012, revised 2nd Jan. 2015, accepted 18th Aug. 2015.

The *colouring number* $\mathrm{col}(G)$ of a graph G is the smallest integer k for which there is an ordering of the vertices of G such that when removing the vertices of G in the specified order no vertex of degree more than $k-1$ in the remaining graph is removed at any step. An edge e of a graph G is said to be *double*-col-*critical* if the colouring number of $G - V(e)$ is at most the colouring number of G minus 2. A connected graph G is said to be *double*-col-*critical* if each edge of G is double-col-critical. We characterise the double-col-critical graphs with colouring number at most 5. In addition, we prove that every 4-col-critical non-complete graph has at most half of its edges being double-col-critical, and that the extremal graphs are precisely the odd wheels on at least six vertices. We observe that for any integer k greater than 4 and any positive number ϵ, there is a k-col-critical graph with the ratio of double-col-critical edges between $1 - \epsilon$ and 1.

Keywords: graph colouring, graph characterizations, degenerate graphs, colouring number, double-critical graphs

1 Introduction

All graphs considered in this paper are assumed to be simple and finite.(i) The cycle on n vertices is denoted by C_n. The complete graph K_n on n vertices is referred to as an n-*clique*. Let G denote a graph. The number of vertices in a largest clique contained in G is denoted by $\omega(G)$. The vertex-connectivity of G is denoted by $\kappa(G)$. The number of vertices and edges in G is denoted by $n(G)$ and $m(G)$, respectively. Given a vertex v in G, $N(v, G)$ denotes the set of vertices in G adjacent to v; $\deg(v, G)$ denotes the cardinality of $N(v, G)$, and it is referred to as the *degree* of v (in G). A vertex of degree 1 is referred to as a *leaf*. The minimum degree and maximum degree of G is denoted $\delta(G)$ and $\Delta(G)$, respectively. Given a subset S of the vertices of G, the subgraph of G induced by the vertices of S is denoted by $G[S]$, and we let $N(S, G)$ denote the set $\cup_{s \in S} N(s, G) \setminus S$. The *square of a graph* G, denoted by G^2, is the graph obtained from G by adding edges between any pair of vertices of G which are at distance

*Email: matthias.kriesell@tu-ilmenau.de
†Email: asp@imada.sdu.dk

(i) The reader is referred to [1] for definitions of graph-theoretic concepts used but not explicitly defined in this paper.

2 in G. Given two graphs H and G, the *complete join* of G and H, denoted by $G + H$, is the graph obtained from two disjoint copies of H and G by joining each vertex of the copy of G to each vertex of the copy of H. The chromatic number of G is denoted by $\chi(G)$, while the list-chromatic number of G is denoted by $\chi_\ell(G)$. Let ψ denote some graph parameter. An edge e of G is said to be *double-ψ-critical* if $\psi(G - V(e)) \leq \psi(G) - 2$. A connected graph G is said to be *double-ψ-critical* if each edge of G is double-ψ-critical. For brevity, we may also refer to double-χ-critical edges and graphs as, simply, *double-critical* edges and graphs, respectively.

The introduction of the concept of double-ψ-critical graphs in [12] was inspired by a special case of the Erdős-Lovász Tihany Conjecture [2], namely the special case which states that the complete graphs are the only double-critical graphs. We refer to this special case of the Erdős-Lovász Tihany Conjecture as the *Double-Critical Graph Conjecture*. The Double-Critical Graph Conjecture is settled in the affirmative for the class of graphs with chromatic number at most 5, but remains unsettled for the class of graphs with chromatic number k, for each value of $k \geq 6$. [4, 11, 16, 17]. Using the computer programs SAGE [15] and `geng` [10], we verified the Double-Critical Graph Conjecture for all graphs on at most 12 vertices (see [13]).

In [12], it was proved that if G is a double-χ_ℓ-critical graph with $\chi_\ell(G) \leq 4$, then G is complete. It is an open problem whether there is a non-complete double-χ_ℓ-critical graph with list-chromatic number at least 5.

The double-κ-critical graphs, which in the literature are referred to as *contraction-critical* graphs (since the vertex-connectivity drops by one after contraction of any edge), are well-understood in the case where κ is 4. Some structural results have been obtained for contraction-critical graphs with vertex-connectivity 5. (See [6, Sec. 4] for references on contraction-critical graphs.)

Bjarne Toft(ii) posed the problem of characterising the double-col-critical graphs. Here col denotes the colouring number which is defined in the paragraph below.

In this paper, we characterise the double-col-critical graphs with colouring number at most 5.

In the remaining part of this section, we define the colouring number and present some fundamental properties of this graph parameter.

The colouring number of a graph. Suppose that we are given a non-empty graph G and an ordering $v_1, \ldots, v_n$ of the vertices of G. Now we may colour the vertices of G in the order $v_1, \ldots, v_n$ such that in the ith step the vertex v_i is assigned the smallest possible positive integer which is not assigned to any neighbour of v_i among $v_1, \ldots, v_{i-1}$. This produces a colouring of G using at most

$$\max_{i\in\{1,\ldots,n\}} \deg(v_i, G[v_1, \ldots, v_i]) + 1$$

colours. Taking the minimum over the set S_n of all permutations of $\{1, \ldots, n\}$, we find that the chromatic number of G is at most

$$\min_{\pi\in\mathrm{S}_n} \left\{ \max_{i\in\{1,\ldots,n\}} \deg(v_{\pi(i)}, G[v_{\pi(1)}, \ldots, v_{\pi(i)}]) \right\} + 1 \tag{1}$$

The number in (1) is called the *colouring number* of G, and it is denoted by $\mathrm{col}(G)$. The colouring number of the empty graph K_0 is defined to be zero. By (1), $\mathrm{col}(G) \leq \Delta(G) + 1$ for any graph G.

(ii) Private communication to the second author from Bjarne Toft, Odense, August 2008.

The colouring number was introduced by Erdős and Hajnal [3], but equivalent concepts were introduced independently by several other authors. It can be shown (see, for instance, [19]) that the colouring number of any non-empty graph G is equal to

$$\max\{\delta(H) \mid H \text{ is an induced subgraph of } G\} + 1 \tag{2}$$

and that the colouring number can be computed in polynomial time [9]. The non-empty graphs with colouring number at most $k+1$ are also said to be k-*degenerate* [8]. Thus, a non-empty graph G is k-degenerate if and only if there is an ordering of the vertices of G such that when removing the vertices of G in the specified order no vertex of degree more than k in the remaining graph is removed at any step.

The colouring number is monotone on subgraphs, that is, if F is a subgraph of a graph G then $\mathrm{col}(F) \leq \mathrm{col}(G)$. For ease of reference, we state the following elementary facts concerning the colouring number of graphs.

Observation 1 *For any graph G,*

(i) $\mathrm{col}(G) = 0$ *if and only if G is the empty graph,*

(ii) $\mathrm{col}(G) = 1$ *if and only if G contains at least one vertex but no edges,*

(iii) $\mathrm{col}(G) = 2$ *if and only if G is forest containing at least one edge, and*

(iv) $\mathrm{col}(G) \geq 3$ *if and only if G contains at least one cycle.*

A graph G is said to be k-col-*critical*, or, simply, col-*critical*, if $\mathrm{col}(G) = k$ and $\mathrm{col}(F) < k$ for every proper subgraph F of G. Similarly, a graph G is said to be k-col-*vertex-critical*, or, simply, col-*vertex-critical*, if $\mathrm{col}(G) = k$ and $\mathrm{col}(F) < k$ for every induced proper subgraph F of G. It is easy to see that every connected r-regular graph is $(r+1)$-col-critical.

Observation 2 *For any* col-*vertex-critical graph G,*

(i) $\mathrm{col}(G) = 0$ *if and only if* $G \simeq K_0$,

(ii) $\mathrm{col}(G) = 1$ *if and only if* $G \simeq K_1$,

(iii) $\mathrm{col}(G) = 2$ *if and only if* $G \simeq K_2$, *and*

(iv) $\mathrm{col}(G) = 3$ *if and only if G is a cycle.*

Observation 3 *For any graph G and any element $x \in E(G) \cup V(G)$, if* $\mathrm{col}(G - x) < \mathrm{col}(G)$ *then* $\mathrm{col}(G - x) = \mathrm{col}(G) - 1$.

Observation 4 *A graph G is* col-*vertex-critical if and only if* $\mathrm{col}(G - v) < \mathrm{col}(G)$ *for every vertex v in G.*

Observation 5 *Given any graph G, there is a* col-*critical subgraph F of G with* $\mathrm{col}(G) = \mathrm{col}(F) = \delta(F) + 1$. *In particular, if G is* col-*critical then* $\mathrm{col}(G) = \delta(G) + 1$.

Proof: Recall that $\mathrm{col}(G) = \max\{\delta(H) \mid H \subseteq G\} + 1$. Among the subgraphs H of G with $\mathrm{col}(G) = \delta(H) + 1$, let F denote a minimal one, that is, $\delta(F') < \delta(F)$ for every proper subgraph F' of F. (This minimum exists since G is finite.) Then F is col-critical with $\mathrm{col}(F) = \delta(F) + 1 = \mathrm{col}(G)$. □

Observation 6 *Given any graph G, there is a* col*-vertex-critical induced subgraph F of G with* $\mathrm{col}(G) = \mathrm{col}(F) = \delta(F) + 1$. *In particular, if G is* col*-vertex-critical then* $\mathrm{col}(G) = \delta(G) + 1$.

Proof: Let F denote a minimal induced subgraph of G with $\mathrm{col}(F) = \mathrm{col}(G)$. This implies $\mathrm{col}(F') < \mathrm{col}(F)$ for any induced proper subgraph F' of F, in particular, F is a col-vertex-critical graph. Suppose $\mathrm{col}(F) > \delta(F) + 1$. Then there is some proper induced subgraph F' of F with $\delta(F') + 1 = \mathrm{col}(F)$, and so $\mathrm{col}(F') \geq \mathrm{col}(F)$, a contradiction. Hence $\mathrm{col}(F) = \delta(F) + 1$. If G is col-vertex-critical, then $F = G$, and the desired result follows. □

The two following results may be of interest in their own right.

Proposition 1 (Pedersen [12]) *For any two non-empty disjoint graphs G_1 and G_2, the colouring number of the complete join $G_1 + G_2$ is at most*

$$\min\{\mathrm{col}(G_1) + n(G_2), \mathrm{col}(G_2) + n(G_1)\} \tag{3}$$

and at least

$$\min\{\mathrm{col}(G_1) + n(J_2), \mathrm{col}(G_2) + n(J_1)\} \tag{4}$$

where, for each $i \in \{1, 2\}$, J_i is any subgraph of G_i with minimum degree equal to $\mathrm{col}(G_i) - 1$.

If, in addition, $\mathrm{col}(G_i) = \delta(G_i) + 1$ *for each $i \in \{1, 2\}$ (in particular, if both G_1 and G_2 are* col-*vertex-critical), then the colouring number of the complete join $G_1 + G_2$ is equal to the minimum in* (3).

A graph G is said to be *decomposable* if there is a partition of $V(G)$ into two (non-empty) sets V_1 and V_2 such that, in G, every vertex of V_1 is adjacent to every vertex of V_2. Given any graph G, we let $V_\delta(G)$ denote the set of vertices of G of minimum degree in G. Clearly, $V_\delta(G)$ is non-empty for any non-empty graph.

Proposition 2 (Pedersen [12]) *Let G denote a decomposable graph. Then G is* col-*critical if and only if the vertex set of G can be partitioned into two sets V_1 and V_2 such that $G = G_1 + G_2$, where $G_i := G[V_i]$ for $i \in \{1, 2\}$, G_1 is regular, and*

(i) *$V(G_2) \setminus V_\delta(G_2)$ is an independent set of G_2, and*

$$\delta(G_1) + n(G_2) = \delta(G_2) + n(G_1)$$

or

(ii) *G_2 is an edgeless graph, and*

$$n(G_1) - \delta(G_1) - n(Q) < n(G_2) < n(G_1) - \delta(G_1)$$

where Q denotes a smallest component of G_1 (in terms of the number of vertices).

Moreover, $\mathrm{col}(G) = \delta(G_1) + n(G_2) + 1$ *in both* (i) *and* (ii).

2 Double-col-critical graphs

The analogue of the Double-Critical Graph Conjecture with χ replaced by col does not hold. For instance, the non-complete graph C_6^2 is 4-regular, 5-col-critical, and double-col-critical. Since C_6^2 is planar, it also follows that it is not even true that every double-col-critical graph with colouring number 5 contains a K_5 minor. (In [5], it was proved that every double-critical graph G with $\chi(G) \leq 7$ at least contains a $K_{\chi(G)}$ minor.) It is easy to see that the square of any cycle of length at least 5 is a double-col-critical graph with colouring number 5.

Observation 7 *Any double-*col*-critical graph is* col*-vertex-critical.*

Proof: Let G denote a double-col-critical graph. If there are no vertices in G, then we are done. Let v denote an arbitrary but fixed vertex of G. If there is no vertex in G adjacent to v, then we are done, since then, by the connectedness of G, G is just the singleton K_1. Let u denote a neighbour of v. By Observation 4, we need to show $\mathrm{col}(G - v) < \mathrm{col}(G)$. The fact that G is double-col-critical implies $\mathrm{col}(G - u - v) \leq \mathrm{col}(G) - 2$. Suppose $\mathrm{col}(G - v) \geq \mathrm{col}(G)$. Then

$$\mathrm{col}((G - v) - u) \leq \mathrm{col}(G) - 2 = \mathrm{col}(G - v) - 2$$

which contradictions Observation 3. This shows $\mathrm{col}(G - v)$ is strictly less than $\mathrm{col}(G)$, as desired. □

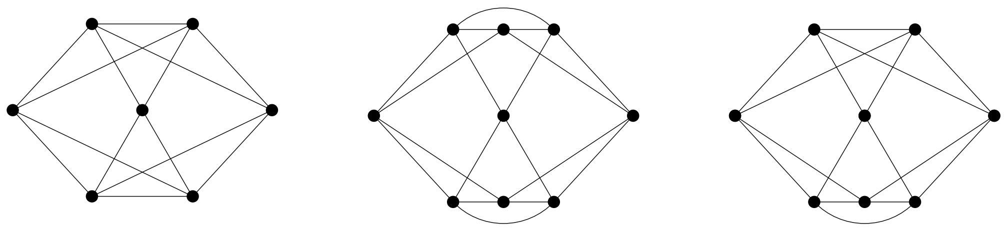
Observation 8 *For each integer $k \in \{0, 1, 2, 3, 4\}$, the only double-*col*-critical graph with colouring number k is the k-clique.*

Proof: Let G denote a double-col-critical graph, and define $k := \mathrm{col}(G)$. Then, by Observation 7, G is also col-vertex-critical, and so, by Observation 6, $\delta(G) = k - 1$. If $k \leq 3$, then the desired result follows immediately from Observation 2. Suppose $k = 4$. Then, for any edge $e \in E(G)$, $\mathrm{col}(G - V(e)) \leq 2$ and so, by Observation 1, $G - V(e)$ is a forest. Fix an edge $xy \in E(G)$. If $G - x - y$ contains no edges, then G is 2-degenerate and so $\mathrm{col}(G) \leq 3$, a contradiction. Let T denote a component of $G - x - y$ with at least one edge, and let u and v denote two leafs of T. Since, as noted above, $\delta(G) = 3$, it follows that both u and v are adjacent to both x and y. If u and v are adjacent in T, then $G[\{u, v, x, y\}] \simeq K_4$, and so, since G is also col-vertex-critical, $G \simeq K_4$. Hence we may assume that u has a neighbour t in $T - v$. Now $G[\{x, y, v\}]$ is a 3-clique in $G - t - u$ and so $\mathrm{col}(G - t - u) \geq \mathrm{col}(G[\{x, y, v\}]) = 3$, a contradiction. This completes the proof. □

It is easy to verify that the graphs Q_1, Q_2, and Q_3 in Figure 1 are double-col-critical and have colouring number 5. None of the graphs Q_1, Q_2, and Q_3 are squares of a cycle. We shall see that Q_1, Q_2, Q_3, and the squares of the cycles of length at least 5 are all the double-col-critical graphs with colouring number 5. First a few preliminary observations.

Observation 9 *If G is a double-*col*-critical graph, then $\delta(G) = \mathrm{col}(G) - 1$ and every pair of adjacent vertices of G has a common neighbour of degree $\delta(G)$ in G.*

Proof: Let G denote a double-col-critical graph. Then, by Observation 7, G is also col-vertex-critical, and so, by Observation 6, $\delta(G) = \mathrm{col}(G) - 1$. Let xy denote an arbitrary edge of G. Now, by the definition of the colouring number, $G - x - y$ has minimum degree at most $(\mathrm{col}(G) - 2) - 1$ which is equal to $\delta(G) - 2$. This means that some vertex of $V(G) \setminus \{x, y\}$, say z, which has degree at least $\delta(G)$ in G has



(a) The graph Q_1. (b) The graph Q_2. (c) The graph Q_3.

Fig. 1: The graphs Q_1, Q_2, and Q_3, depicted above, are the only double-col-critical graphs with colouring number 5 which are not squares of cycles.

degree at most $\delta(G) - 2$ in $G - x - y$. The only way this can happen is if z has degree $\delta(G)$ in G and is adjacent to both x and y in G. This completes the argument. □

Observation 10 *If G is a non-complete double-*col*-critical graph, then G does not contain a clique of order* $\mathrm{col}(G) - 1$.

Proof: Let G denote a non-complete double-col-critical graph. By Observation 7, G is col-vertex-critical, and so, since G is also non-complete, G cannot contain a clique of order more than $\mathrm{col}(G) - 1$. Also, by Observation 7, $\delta(G) = \mathrm{col}(G) - 1$. Suppose that G contains a clique K of order $\mathrm{col}(G) - 1$. Clearly, $G - V(K)$ is not empty. If $G - V(K)$ contains an edge xy, then $\mathrm{col}(G) - 1 = \mathrm{col}(K) \leq \mathrm{col}(G - x - y) \leq \mathrm{col}(G) - 2$, a contradiction. Hence $G - V(K)$ is edgeless, and so, since $\delta(G) = \mathrm{col}(G) - 1 = n(K)$, it follows that each vertex of $V(G) \setminus V(K)$ is adjacent to every vertex of $V(K)$, in particular, G contains a clique of order $\mathrm{col}(G)$, a contradiction. □

Proposition 3 *Every double-*col*-critical graph with colouring number at least* 3 *is* 2*-connected.*

Proof: Let G denote a double-col-critical graph with $\mathrm{col}(G) \geq 3$. Since G is col-vertex-critical, it is connected with $\mathrm{col}(G) = \delta(G) + 1$. Suppose G is not 2-connected, and let x denote a cutvertex of G. Each component of $G - x$ has minimum degree at least $\delta(G) - 1$. Let C denote a component of $G - x$, and let e denote some edge of $G - V(C)$. Then, using the fact that G is double-col-critical, col is monotone, and C is a subgraph of $G - V(e)$, we obtain

$$\mathrm{col}(G) - 1 = (\delta(G) + 1) - 1 \leq \mathrm{col}(C) \leq \mathrm{col}(G - V(e)) = \mathrm{col}(G) - 2$$

a contradiction. This shows that G must be 2-connected. □

2.1 Double-col-critical graphs with colouring number 5.

We shall say that a *k-neighbour* of a vertex x is a neighbour of x of degree k.

Observation 11 *If G is a double-*col*-critical graph with colouring number* 5 *and $ab \in E(G)$, then a or b has degree* 4 *in G.*

Proof: Let G denote a double-col-critical graph with colouring number 5. Then $\delta(G) = 4$. Let ab denote an edge of G. Suppose that both a and b have degree greater than 4 in G. By Observation 9, there is a common 4-neighbour c of a and b. We shall make repeated use of Observation 9 and Observation 10. The latter observation implies that G contains no 4-clique. There is a common 4-neighbour d of a and c. Since $\omega(G) \leq 3$, d is not adjacent to b. This implies that there is a common 4-neighbour e of b and c and e is not identical to d. The vertex e is not adjacent to a. The vertex a has degree at least 5, and so the common 4-neighbour of c and d must be e. We note that $\{a, b, c, d, e\}$ induces a subgraph of G of minimum degree 3. Hence $G - \{a, b, c, d, e\}$ contains no edges. Moreover, $\mathrm{col}(G - \{c, d, e\}) \leq \mathrm{col}(G) - 2$, and so $G - \{c, d, e\}$ contains a vertex f of degree at most 2. Since a and b both have degree at least 3 in $G - \{c, d, e\}$ and $N(c, G) = \{a, b, d, e\}$, it follows that this vertex f must be in the set $V(G) \setminus \{a, b, c, d, e\}$ and that f is adjacent both d and e. The vertex f has degree 4 in G. It also follows from the fact that $G - \{a, b, c, d, e\}$ contains no edges that f must be adjacent to both a and b. Since a has degree at least 5 in G, it follows that a must be adjacent to some vertex $g \in V(G) \setminus \{a, b, c, d, e, f\}$. Then $\mathrm{col}(G - a - g) \leq \mathrm{col}(G) - 2 = 3$. On the other hand, $\{b, c, d, e, f\}$ induce a subgraph of $G - a - g$ of minimum degree 3, a contradiction. This contradiction implies that G contains no two adjacent vertices both of which have degree greater than 4. □ Recently, the first author obtained a characterisation of what he called *minimal critical graphs with minimum degree* 4. It turns out that our double-col-critical graphs with colouring number 5 are such graphs, and so – using the characterisation of minimal critical graphs of minimum degree 4 – we obtain a characterisation of the double-col-critical graphs with colouring number 5.

In the following result, which is the main result of this paper, we let Q_1, Q_2, and Q_3 denote the graphs depicted in Figure 1.

Theorem 1 *A graph is double-*col*-critical with colouring number* 5 *if and only if it is isomorphic to* Q_1*,* Q_2*,* Q_3*, or the square of a cycle of length at least* 5*.*

The graph Q_2 is the dual of the Herschel graph which is the smallest nonhamiltonian polyhedral graph.

In order to prove Theorem 1, we first need to introduce a bit of notation and state the above-mentioned characterisation of minimal critical graphs with minimum degree 4.

For the remaining part of this section we shall be using the following notation. We shall let $\mathcal{C}$ denote the set of simple connected graphs of minimum degree at least 4. An edge e of a graph G in $\mathcal{C}$ is *essential* if the graph $G - e$ obtained from G by deleting e is not in $\mathcal{C}$, and let us call e *critical* if the graph G/e obtained by contracting e and simplifying is not in $\mathcal{C}$. An edge e is essential if and only if e is a bridge or at least one of its endvertices has degree 4; and e is critical if and only if the endvertices of e have a common 4-neighbour or $N(V(e), G)$ consists of three common neighbours of the endvertices of e. We are now interested in the *minimal critical* graphs in $\mathcal{C}$, that is, graphs $G \in \mathcal{C}$ with the property that each edge of G is both essential and critical.

For the description of the minimal critical graphs in $\mathcal{C}$, we shall consider a number of *bricks*, that is, any graph isomorphic to one of the following nine graphs: K_5, $K_{2,2,2}$, K_5^-, $K_{2,2,2}^-$, $K_5^\triangledown$, $K_{2,2,2}^\triangledown$, $K_5^{\triangleright\triangleleft}$, $K_{2,2,2}^{\triangleright\triangleleft}$, or K_3 which are depicted in Figure 2. Each brick comes together with its *vertices of attachment*: For K_5 and $K_{2,2,2}$, this is an arbitrary single vertex, for the other seven bricks these are its vertices of degree less than 4. The remaining vertices of the brick are its *internal vertices*, and the edges connecting two inner vertices are called its *internal edges*. Observe that every brick B has one, two, or three vertices of attachment, and that they are pairwise nonadjacent unless B is the triangle, that is, K_3.

It turns out that the minimal critical graphs from $\mathcal{C}$ are either squares of cycles of length at least 5, or they are the edge disjoint union of bricks, following certain rules. This is made precise in the following

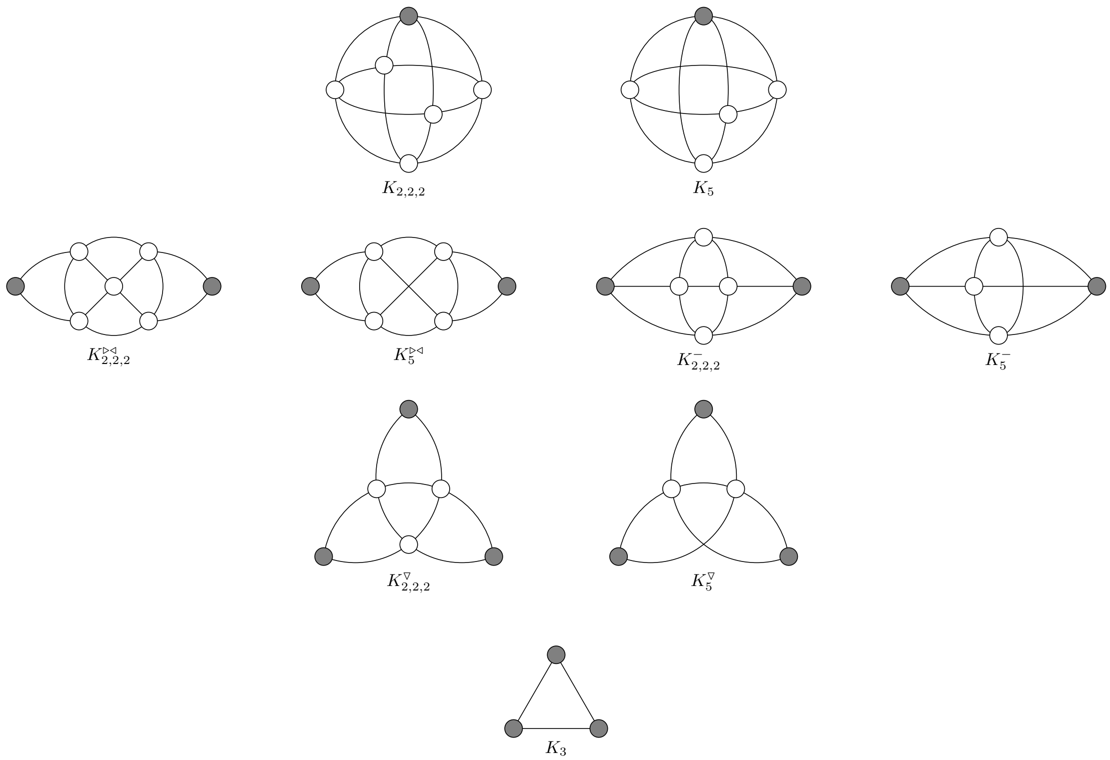


Fig. 2: The nine bricks. Vertices of attachment are displayed solid.

theorem.

Theorem 2 (Kriesell [7]) *A graph is a minimal critical graph in $\mathcal{C}$ if and only if it is the square of a cycle of length at least 5 or arises from a connected multihypergraph H of minimum degree at least 2 with at least one hyperedge and $|V(e)| \in \{1, 2, 3\}$ for all hyperedges e by replacing each hyperedge e by a brick B_e (see Figure 2) such that the vertices of attachment of B_e are those in $V(e)$ and at the same time the only objects of B_e contained in more than one brick, and*

(TB) *the brick B_e is triangular only if each vertex $x \in V(e)$ is incident with precisely one hyperedge f_x different from e and the corresponding brick B_{f_x} is neither K_5, K_5^-, $K_{2,2,2}$, nor $K_{2,2,2}^-$, and, for any other vertex $y \in V(e) \setminus \{x\}$ and hyperedge f_y containing y but distinct from e, we have*

- (i) *$V(f_x) \cap V(f_y) \neq \emptyset$ only if not both of B_{f_x} and B_{f_y} are triangular, and*
- (ii) *$f_x = f_y$ only if B_{f_x} is $K_5^{\triangleright\triangleleft}$ or $K_{2,2,2}^{\triangleright\triangleleft}$.*

Proof of Theorem 1.: In order to prove the desired result, we prove the following equivalent statement.

A graph is double-col-critical with colouring number 5 if and only if it is the square of a cycle of length at least 5 or one of the three graphs obtained by taking the union of two graphs G_1 and G_2 such that $G_i \simeq K_5^{\triangledown}$ or $G_i \simeq K_{2,2,2}^{\triangledown}$ for $i \in \{1,2\}$ and $x \in V(G_1) \cap V(G_2)$ if and only if x has degree 2 in G_1 and degree 2 in G_2.

The 'if'-part of the statement above is straightforward to verify and it is left to the reader.

Let G denote an arbitrary double-col-critical graph with colouring number 5. It follows from the definition of double-col-critical graphs, Observation 6, Observation 7, Observation 9, and Observation 11 that G has the following properties.

(a) G has minimum degree 4;

(b) if x and y are adjacent vertices, then at least one of them has degree 4;

(c) if x and y are adjacent vertices, then they have a common 4-neighbour; and

(d) if x and y are adjacent in G, then $G - x - y$ has no induced subgraph of minimum degree at least 3.

By (a), G is in $\mathcal{C}$. By (b), every edge of G is essential, and, by (c), every edge of G is critical. Thus, G is minimal critical in $\mathcal{C}$, and so Theorem 2 applies. Suppose that G is not the square of a cycle. Then G has a representation by a multihypergraph H as described in Theorem 2.

If e is a 1-hyperedge then the unique attachment vertex of the corresponding brick B_e in G is a cutvertex of G, a contradiction to Proposition 3.

Suppose that there exists a 2-hyperedge e with $V(e) = \{u,v\}$. If B_e is $K_5^{\triangleright\triangleleft}$ or $K_{2,2,2}^{\triangleright\triangleleft}$ then $B_e - V(e)$ is an induced subgraph of G of minimum degree 3, and since the vertex $u \in V(e)$ has only two neighbours in that subgraph, there is an edge in $G - (V(B_e) \setminus V(e))$, contradicting (d). If, otherwise, B_e is K_5^- or $K_{2,2,2}^-$ then, by Theorem 2, B_e is an induced subgraph of G of minimum degree 3. By (a), there is a vertex in $V(G) \setminus V(B_e)$, and it has a neighbour in $V(G) \setminus V(B_e)$. This contradicts (d). Hence there are only 3-hyperedges in H.

Suppose that H contains a 3-hyperedge e for which the corresponding brick B_e is triangular. It follows from (a) and (d) that some vertex $q \in V(G) \setminus V(B_e)$ is adjacent to at least two vertices in $V(B_e)$. The vertex q is not adjacent to all three vertices of $V(B_e)$, since otherwise $G[V(B_e) \cup \{q\}]$ would induce a 4-clique in G which contradicts Observation 10. Let x and y denote the neighbours of q in $V(B_e)$. By Theorem 2 (TB), x is incident to exactly one hyperedge f_x different from e. Similarly, y is incident to exactly one hyperedge f_y different from e. If $f_x = f_y$, then, by Theorem 2 (TB.ii), B_{f_x} is $K_5^{\triangleright\triangleleft}$ or $K_{2,2,2}^{\triangleright\triangleleft}$, in particular, f_x is a 2-hyperedge, a contradiction. Hence $f_x \neq f_y$ and so, since $q \in V(f_x) \cap V(f_y)$, it follows from Theorem 2 (TB.i) that not both B_{f_x} and B_{f_y} are triangular bricks. The fact that f_x and f_y are distinct and $q \in V(f_x) \cap V(f_y)$ implies that q is an attachment vertex of both B_{f_x} and B_{f_y}. Since q is adjacent to x and both q and x are attachment vertices, it follows that B_{f_x} must be triangular. Similarly, B_{f_y} must be triangular, and so we have obtained a contradiction. This shows that each hyperedge in H is of the type $K_5^{\triangledown}$ or $K_{2,2,2}^{\triangledown}$.

Let e denote an arbitrary 3-hyperedge of H. If there are two vertices $x, y \in V(e)$ of degree exceeding 4 in G then $G - (V(B_e) \setminus \{x,y\})$ has minimum degree at least 3, contradicting (d) applied to any internal edge of B_e. Therefore, if there is a vertex $x \in V(e)$ of degree exceeding 4 in G, then the two vertices $y, z \in V(e) \setminus \{x\}$ are incident with precisely one further 3-hyperedge f_y and f_z, respectively, both distinct from e. If $f_y \neq f_z$ then one may argue as above that $G - V(B_e - x)$ has minimum degree at least 3, contradicting (d) applied to any internal edge of B_e. Hence $f_y = f_z =: f$. Let w be the vertex in $V(f) \setminus \{y, z\}$. If $w \neq x$ then $\{w, x\}$ forms a 2-separator, and otherwise $w = x$ is a cutvertex as x has degree exceeding 4. In either case, $G - (V(B_e - x) \cup V(B_f))$ has minimum degree at least 3, again contradicting (d).

Hence all vertices of attachment have degree 4 in G. Let e denote a 3-hyperedge in H, and let x, y, and z denote the vertices of $V(e)$. Again let f_x, f_y, f_z denote the unique 3-hyperedge distinct from e incident with x, y, z, respectively. If they are pairwise distinct then $G - V(B_e)$ has minimum degree at least 3, contradiction to (d). If $f := f_y = f_z \neq f_x$ then let w be the vertex in $V(f)$ distinct from y, z. As $f_x \neq f$, we have $w \neq x$, so that $G - (V(e) \cup V(f))$ has minimum degree at least 3, contradicting (d), unless there is a vertex in $V(f_x)$ adjacent to both x and w; in this latter case, the unique 3-hyperedge distinct from f incident with w must be f_x, and so the vertex u in $V(f_x) \setminus \{w, x\}$ is a cutvertex of G, a contradiction to Proposition 3. Hence $f_x = f_y = f_z$, and the desired statement follows. □ Given our

success in characterising the double-col-critical graphs with colouring number 5, we venture to ask for a characterisation of the double-col-critical graphs with colouring number 6. If G is a double-col-critical graph, then $G + K_k$ is a double-col-critical graph with $\mathrm{col}(G + K_k) = \mathrm{col}(G) + k$ (see Proposition 4). This implies that the graphs $Q_1 + K_1$, $Q_2 + K_1$, $Q_3 + K_1$, and $C + K_1$, where C is the square of any cycle of length at least 5, are all double-col-critical graphs with colouring number 6. These are not the only double-col-critical graphs with colouring number 6; the icosahedral graph is yet another double-col-critical graph with colouring number 6. This latter fact was also observed by Stiebitz [18, p. 323], although in a somewhat different setting. Using the computer programs SAGE [15] and `geng` [10], we determined all the double-col-critical graphs with colouring number 6 and at most 10 vertices; there are 116 such graphs and they are available at the second authors homepage [14].

The standard 6-regular toroidal graphs obtained from the toroidal grids by adding all diagonals in the same direction have colouring number 7 and are double-col-critical.

Complete joins of double-col-critical graphs. In [5], it was observed that if G is the complete join $G_1 + G_2$, then G is double-critical if and only if both G_1 and G_2 are double-critical. Next we prove that the 'if'-part of the analogous statement for double-col-critical graphs is true. The 'only if'-part is not true, as follows from considering the double-col-critical graph C_6^2: We have $C_6^2 \simeq C_4 + \overline{K_2}$ but neither C_4 nor $\overline{K_2}$ is double-col-critical.

Proposition 4 *If G_1 and G_2 are two disjoint double-*col*-critical graphs, then the complete join $G_1 + G_2$ is also double-*col*-critical with*

$$\mathrm{col}(G_1 + G_2) = \min\{\mathrm{col}(G_1) + n(G_2), \mathrm{col}(G_2) + n(G_1)\}$$

Proof: Let G_1 and G_2 denote two disjoint double-col-critical graphs. Then, by Observation 7, both G_1 and G_2 are col-vertex-critical, and so, by Proposition 1,

$$\mathrm{col}(G_1 + G_2) = \min\{\mathrm{col}(G_1) + n(G_2), \mathrm{col}(G_2) + n(G_1)\}$$

We need to prove that $\mathrm{col}((G_1 + G_2) - x - y) \leq \mathrm{col}(G_1 + G_2) - 2$ for every edge $e = xy \in E(G)$; by symmetry, it suffices to consider (1) $x, y \in V(G_1)$ and (2) $x \in V(G_1)$ and $y \in V(G_2)$. Suppose

$x, y \in V(G_1)$. Then

$$\begin{aligned}\operatorname{col}((G_1 + G_2) - x - y) &= \operatorname{col}((G_1 - x - y) + G_2)\\ &\le \min\{\operatorname{col}(G_1 - x - y) + n(G_2), \operatorname{col}(G_2) + n(G_1 - x - y)\}\\ &\le \min\{\operatorname{col}(G_1) - 2 + n(G_2), \operatorname{col}(G_2) + n(G_1) - 2\}\\ &= \min\{\operatorname{col}(G_1) + n(G_2), \operatorname{col}(G_2) + n(G_1)\} - 2\end{aligned}$$

where we applied Proposition 1 and the fact that G_1 is double-col-critical. A similar argument applies in case (2). We omit the details. □ Proposition 4 and the fact that both C_6^2 and K_t are double-col-critical

immediately implies the following result, which, in particular, shows that, for each integer $k \geq 6$, there is a non-regular double-col-critical graph with colouring number k.

Corollary 1 *For any positive integer t, the graph $G_t := C_6^2 + K_t$ is a double-col-critical graph with $\operatorname{col}(G_t) = t + 5$, $\delta(G_t) = n(G_t) - 2$ and $\Delta(G_t) = n(G_t) - 1$.*

3 Double-col-critical edges

In [5], Kawarabayashi, the second author, and Toft initiated the study of the number of double-critical edges in graphs. In this section, we study the number of double-col-critical edges in graphs. Kawarabayashi, the second author, and Toft proved the following theorem for which we shall prove an analogue for the colouring number.

The complete join $C_n + K_1$ of a cycle C_n and a single vertex is referred to as a *wheel*, and it is denoted W_n. If n is odd, we refer to W_n as an *odd wheel*.

Theorem 3 (Kawarabayashi, Pedersen & Toft [5]) *If G denotes a 4-critical non-complete graph, then G contains at most $m(G)/2$ double-critical edges. Moreover, G contains precisely $m(G)/2$ double-critical edges if and only if G is an odd wheel of order at least 6.*

The following result is just a slight reformulation of Theorem 3.

Corollary 2 *If G denotes a 4-chromatic graph with no 4-clique, then G contains at most $m(G)/2$ double-critical edges. Moreover, G contains precisely $m(G)/2$ double-critical edges if and only if G is an odd wheel of order at least 6.*

Proof: Let G denote a 4-chromatic graph with no 4-clique. If $e = xy$ is a double-critical edge in G, then e is a critical edge of G and x is a critical vertex of G. We remove non-critical elements from G until we are left with a 4-critical subgraph G'. At no point did we remove an endvertex of a double-critical edge. Thus, the number of double-critical edges in G is equal to the number of double-critical edges in G'. Clearly, G' is a non-complete graph, and so, by Theorem 3, the number of double-critical edges in G' is at most $m(G')/2$ which is at most $m(G)/2$. The second part of the corollary now follows easily. □

The following result — which is an analogue of Theorem 3 with the chromatic number replaced by the colouring number — extends Observation 8.

Proposition 5 *If G denotes a 4-col-critical non-complete graph, then G contains at most $m(G)/2$ double-col-critical edges. Moreover, G contains precisely $m(G)/2$ double-col-critical edges if and only if G is a wheel of order at least 6.*

Lemma 1 *If e and f are two double-*col*-critical edges in a* 4-col*-critical non-complete graph, then e and f are incident.*

Proof: Let G denote a col-critical graph with $\mathrm{col}(G) = 4$. Then, by Observation 5, $\delta(G) = 3$. We must have $\omega(G) \leq 3$, since G is a 4-col-critical non-complete graph.

Suppose e is an arbitrary double-col-critical edge in G. Then $\mathrm{col}(G - V(e)) = 2$ which, by Observation 1 (iii), means that $G - V(e)$ is a forest containing at least one edge and, since $\delta(G) = 3$, $\delta(G - V(e)) \geq 1$ and each leaf in $G - V(e)$ is adjacent to both endvertices of e in G. Let u and v denote two leafs of $G - V(e)$. Now, if G contains some double-col-critical edge f which is not incident to e, then $G - V(f)$ contains no cycles, since $\mathrm{col}(G - V(f)) = 2$, and so f is incident to both u and v, which implies $G[\{u, v\} \cup V(e)] \simeq K_4$, a contradiction. This means that any two double-col-critical edges of G are incident, and the proof is complete. □

Proof of Proposition 5: Let G denote a 4-col-critical non-complete graph. Then $n(G) \geq 5$ and, by Observation 5, $\delta(G) = 3$ which implies $m(G) \geq \lceil n(G) \cdot \delta(G)/2 \rceil \geq 8$. By Lemma 1, we only have to consider two cases: (i) G contains three incident double-col-critical edges xy, yz, and xz, or (ii) there is a vertex $v \in V(G)$ such that every double-col-critical edge of G is incident to v. If (i) holds, then, since $m(G) \geq 8$, the desired statement follows. Suppose (ii) holds. Then the number of double-col-critical edges in G is at most $\deg(v, G)$. We may assume that there is at least one double-col-critical edge, say, vw in G. Suppose $G - v$ is disconnected. Then, since $\delta(G) = 3$, each component of $G - v$ has minimum degree at least 2, and so, in particular, some component of $G - v - w$ has minimum degree at least 2. This, however, contradicts the fact that $G - v - w$ is a forest. Hence $G - v$ is connected. By Observation 3, $\mathrm{col}(G - v) \geq 3$ and so, by Observation 1 (iii), $G - v$ contains a cycle. Hence $G - v$ is a connected graph with at least one cycle, and so $m(G - v) \geq n(G - v)$. Thus,

$$m(G) = \deg(v, G) + m(G - v) \geq \deg(v, G) + (n(G) - 1) \geq 2\deg(v, G)$$

which implies that the number of double-col-critical edges is at most $m(G)/2$ and that the number of double-col-critical edges is equal to $m(G)/2$ only if $\deg(v, G) = n(G) - 1$ and $G - v$ is a cycle.

Conversely, if G is a wheel on at least five vertices, then it is easy to see that exactly $m(G)/2$ edges of G are double-col-critical. This completes the proof. □ Let k denote some integer greater than 3. Let D_k denote the $2k$-cycle with vertices labelled cyclically $v_0 v_1 \dots v_k u_{k-1} u_{k-2} \dots u_1$. Let F_k denote the graph

$$D_k^2 - u_1 v_1 - u_{k-1} v_{k-1} + v_1 v_{k-1} + u_1 u_{k-1}$$

Figure 3 depicts a drawing of F_5.

Observation 12 *For every integer k greater than 3, the graph F_k, as defined as above, is a* 5-col*-critical graph with colouring number 5 in which all edges except $v_1 v_{k-1}$ and $u_1 u_{k-1}$ are double-*col*-critical.*

Proposition 6 *For each integer p greater than 4 and positive real number ϵ, there is a p-*col*-critical graph G with the ratio of double-*col*-critical edges between $1 - \epsilon$ and 1.*

Proof: If $p = 5$, the desired result follows directly from Observation 12 by letting k tend to infinity. Let p denote an integer greater than 5 and ϵ a positive real number. Let k denote an integer a lot greater than p, and let G denote the graph obtained by taking the complete join of F_k and $\overline{K_{p-5}}$. Then, by

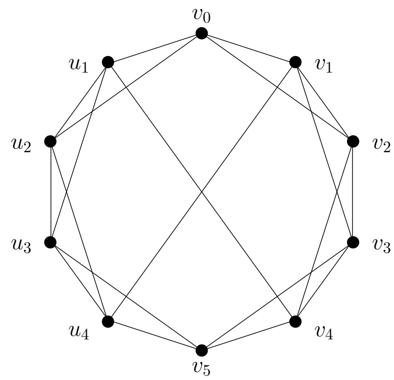


Fig. 3: The graph F_5 has colouring number 5 and all edges of F_5, except u_1v_4 and v_1u_4, are double-col-critical.

Proposition 2 (ii), G is p-col-critical, and, since k is a lot greater than p, all but the edges v_1v_{k-1} and u_1u_{k-1} are double-col-critical in G. By letting k tend to infinity the ratio of double-col-critical edges in G will from a certain point onwards be between $1 - \epsilon$ and 1. This completes the argument. □

Proposition 6 means that there is no result corresponding to Proposition 5 for colouring numbers greater than 4.

4 Concluding remarks

By a theorem of Mozhan [11] and, independently, Stiebitz [16], K_5 is the only double-critical graph with chromatic number 5, that is, K_5 is the only 5-chromatic graph with 100% double-critical edges, but we do not know whether there are non-complete 5-chromatic graphs with the percentage of double-critical edges arbitrarily close to a 100. In [5], Kawarabayashi, the second author, and Toft conjectured that if G is a 5-critical non-complete graph, then G contains at most $(2 + \frac{1}{3n(G)-5})\frac{m(G)}{3}$ double-critical edges.

As we have seen, the story is a bit different for the colouring number. By Proposition 6 for $p = 5$, there are non-complete graphs with colouring number 5 with the percentage of double-col-critical edges arbitrarily close to a 100. This only makes Theorem 1 all the more interesting. By Theorem 1, we are able to distinguish between graphs with colouring number 5 having 99.99% double-col-critical edges and those that have a 100% double-col-critical edges.

The problem of obtaining a concise structural description of the double-col-critical graphs with colouring number $k \geq 6$ remains open. Given any graph G, it can be decided in polynomial time whether or not G is double-col-critical, since the colouring number itself can be computed in polynomial time. Nevertheless, given the structural complexity of the double-col-critical graphs with colouring number 6 mentioned on page 58, it seems likely that even the problem of obtaining a concise structural description of the double-col-critical graphs with colouring number 6 is non-trivial.

Acknowledgements

We thank Bjarne Toft for posing the problem of characterising the double-col-critical graphs and his many insightful comments on colouring and degeneracy of graphs.

References

[1] J. A. Bondy and U. S. R. Murty. *Graph Theory*, volume 244 of *Graduate Texts in Mathematics*. Springer, New York, 2008.

[2] P. Erdős. Problem 2. In *Theory of Graphs (Proc. Colloq., Tihany, 1966)*, page 361. Academic Press, New York, 1968.

[3] P. Erdős and A. Hajnal. On chromatic number of graphs and set-systems. *Acta Math. Acad. Sci. Hungar*, 17: 61–99, 1966.

[4] T. R. Jensen and B. Toft. *Graph Coloring Problems*. Wiley-Interscience Series in Discrete Mathematics and Optimization. John Wiley & Sons Inc., New York, 1995. A Wiley-Interscience Publication.

[5] K. Kawarabayashi, A. S. Pedersen, and B. Toft. Double-critical graphs and complete minors. *Electron. J. Combin.*, 17(1):Research Paper 87, 27 pp., 2010.

[6] M. Kriesell. A survey on contractible edges in graphs of a prescribed vertex connectivity. *Graphs Combin.*, 18 (1):1–30, 2002.

[7] M. Kriesell. Nonseparating K_4-subdivisions in graphs of minimum degree at least 4. Technical Report arXiv:1101.5278, Jan 2011. IMADA-preprint, 25 pages.

[8] D. R. Lick and A. T. White. k-degenerate graphs. *Canad. J. Math.*, 22:1082–1096, 1970.

[9] D. W. Matula and L. L. Beck. Smallest-last ordering and clustering and graph coloring algorithms. *J. Assoc. Comput. Mach.*, 30(3):417–427, 1983.

[10] B. D. McKay. `geng` - a program for generating graphs, 2008. URL `http://cs.anu.edu.au/~bdm/nauty/`. Version 2.4.

[11] N. N. Mozhan. Twice critical graphs with chromatic number five. *Metody Diskret. Analiz.*, (46):50–59, 73, 1987.

[12] A. S. Pedersen. *Contributions to the Theory of Colourings, Graph Minors, and Independent Sets*. PhD thesis, University of Southern Denmark, 2010.

[13] A. S. Pedersen, September 2012. URL `http://www.imada.sdu.dk/~asp/noSmallNonCompleteDC.pdf`.

[14] A. S. Pedersen, October 2012. URL `http://www.imada.sdu.dk/~asp/double-col-critical.php`.

[15] W. A. Stein et al. *Sage Mathematics Software (Version 4.6.2)*. The Sage Development Team, 2012. `http://www.sagemath.org`.

[16] M. Stiebitz. K_5 is the only double-critical 5-chromatic graph. *Discrete Math.*, 64(1):91–93, 1987.

[17] M. Stiebitz. On k-critical n-chromatic graphs. In *Combinatorics (Eger, 1987)*, volume 52 of *Colloq. Math. Soc. János Bolyai*, pages 509–514. North-Holland, Amsterdam, 1988.

[18] M. Stiebitz. Decomposing graphs under degree constraints. *J. Graph Theory*, 23(3):321–324, 1996.

[19] B. Toft. Colouring, stable sets and perfect graphs. In *Handbook of Combinatorics, Vol. 1, 2*, pages 233–288. Elsevier, Amsterdam, 1995.