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The double competition multigraph of a digraph

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In this article, we introduce the notion of the double competition multigraph of a digraph. We give characterizations of the double competition multigraphs of arbitrary digraphs, loopless digraphs, reflexive digraphs, and acyclic digraphs in terms of edge clique partitions of the multigraphs.

Keywords: competition graph, competition multigraph, competition-common enemy graph, double competition multigraph, edge clique partition

1 Introduction

The competition graph of a digraph is defined to be the intersection graph of the family of the out-neighborhoods of the vertices of the digraph (see [6] for intersection graphs). A *digraph* D is a pair $(V(D), A(D))$ of a set $V(D)$ of *vertices* and a set $A(D)$ of ordered pairs of vertices, called *arcs*. An arc of the form (v, v) is called a *loop*. For a vertex x in a digraph D, we denote the *out-neighborhood* of x in D by $N^+_D(x)$ and the *in-neighborhood* of x in D by $N^-_D(x)$, i.e., $N^+_D(x) := \{v \in V(D) \mid (x, v) \in A(D)\}$ and $N^-_D(x) := \{v \in V(D) \mid (v, x) \in A(D)\}$. A *graph* G is a pair $(V(G), E(G))$ of a set $V(G)$ of *vertices* and a set $E(G)$ of unordered pairs of vertices, called *edges*. The *competition graph* of a digraph D is the graph which has the same vertex set as D and has an edge between two distinct vertices x and y if and only if $N^+_D(x) \cap N^+_D(y) \neq \emptyset$. R. D. Dutton and R. C. Brigham [3] and F. S. Roberts and J. E. Steif [8] gave characterizations of competition graphs by using edge clique covers of graphs. The notion of competition graphs was introduced by J. E. Cohen [2] in 1968 in connection with a problem in ecology, and several variants and generalizations of competition graphs have been studied.

In 1987, D. D. Scott [11] introduced the notion of double competition graphs as a variant of the notion of competition graphs. The *double competition graph* (or the *competition-common enemy graph* or the *CCE graph*) of a digraph D is the graph which has the same vertex set as D and has an edge between two distinct vertices x and y if and only if both $N^+_D(x) \cap N^+_D(y) \neq \emptyset$ and $N^-_D(x) \cap N^-_D(y) \neq \emptyset$ hold. See [4, 5, 10, 12] for recent results on double competition graphs.

A *multigraph* M is a pair $(V(M), E(M))$ of a set $V(M)$ of *vertices* and a multiset $E(M)$ of unordered pairs of vertices, called *edges*. Note that, in our definition, multigraphs have no loops. We may consider

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a multigraph M as the pair $(V(M), m_M)$ of the vertex set $V(M)$ and the nonnegative integer-valued function $m_M : \binom{V}{2} \to \mathbb{Z}_{\geq 0}$ on the set $\binom{V}{2}$ of all unordered pairs of V where $m_M(\{x, y\})$ is defined to be the number of multiple edges between the vertices x and y in M. The notion of competition multigraphs was introduced by C. A. Anderson, K. F. Jones, J. R. Lundgren, and T. A. McKee [1] in 1990 as a variant of the notion of competition graphs. The *competition multigraph* of a digraph D is the multigraph which has the same vertex set as D and has m_{xy} multiple edges between two distinct vertices x and y, where m_{xy} is the nonnegative integer defined by $m_{xy} = |N_D^+(x) \cap N_D^+(y)|$. See [9, 13] for recent results on competition multigraphs.

In this article, we introduce the notion of the double competition multigraph of a digraph, and we give characterizations of the double competition multigraphs of arbitrary digraphs, loopless digraphs, reflexive digraphs, and acyclic digraphs in terms of edge clique partitions of the multigraphs.

2 Main Results

We define the double competition multigraph of a digraph as follows.

Definition. Let D be a digraph. The *double competition multigraph* of D is the multigraph which has the same vertex set as D and has m_{xy} multiple edges between two distinct vertices x and y, where m_{xy} is the nonnegative integer defined by

$$m_{xy} = |N_D^+(x) \cap N_D^+(y)| \cdot |N_D^-(x) \cap N_D^-(y)|,$$

i.e., the multigraph M defined by $V(M) = V(D)$ and $m_M(\{x, y\}) = m_{xy}$.

Recall that a *clique* of a multigraph M is a set of vertices of M which are pairwise adjacent. We consider the empty set $\emptyset$ as a clique of any multigraph for convenience. A multiset is also called a *family*. An *edge clique partition* of a multigraph M is a family $\mathcal{F}$ of cliques of M such that any two distinct vertices x and y are contained in exactly $m_M(\{x, y\})$ cliques in the family $\mathcal{F}$. For a positive integer n, let $[n]$ denote the set $\{1, 2, \ldots, n\}$.

Theorem 1. *Let M be a multigraph with n vertices. Then, M is the double competition multigraph of an arbitrary digraph if and only if there exist an ordering $(v_1, \ldots, v_n)$ of the vertices of M and a double indexed edge clique partition $\{S_{ij} \mid i, j \in [n]\}$ of M such that the following condition holds:*

(I) *for any $i, j \in [n]$, if $|A_i \cap B_j| \geq 2$, then $A_i \cap B_j = S_{ij}$,*

where A_i and B_j are the sets defined by

$$A_i = S_{i*} \cup T_i^+, \quad S_{i*} := \bigcup_{p \in [n]} S_{ip}, \quad T_i^+ := \{v_b \mid a, b \in [n], v_i \in S_{ab}\}, \tag{1}$$

$$B_j = S_{*j} \cup T_j^-, \quad S_{*j} := \bigcup_{q \in [n]} S_{qj}, \quad T_j^- := \{v_a \mid a, b \in [n], v_j \in S_{ab}\}. \tag{2}$$

Proof: First, we show the only-if part. Let M be the double competition multigraph of an arbitrary digraph D. Let $(v_1, \ldots, v_n)$ be an ordering of the vertices of D. For $i, j \in [n]$, we define

$$S_{ij} := \{v_k \in V(D) \mid (v_i, v_k), (v_k, v_j) \in A(D)\}. \tag{3}$$

Then S_{ij} is a clique of M. Let $\mathcal{F}$ be the family of S_{ij}'s whose size is at least two, i.e.,

$$\mathcal{F} := \{S_{ij} \mid i, j \in [n], |S_{ij}| \geq 2\}. \tag{4}$$

By the definition of a double competition multigraph, $\mathcal{F}$ is an edge clique partition of M.

We show that the condition (I) holds. Fix i and j in $[n]$ and let A_i and B_j be sets as defined in (1) and (2). Since $S_{ij} \subseteq A_i$ and $S_{ij} \subseteq B_j$, it holds that $S_{ij} \subseteq A_i \cap B_j$. Now we assume that $|A_i \cap B_j| \geq 2$ and take any vertex $v_k \in A_i \cap B_j$. There are four cases for v_k arising from the definitions of A_i and B_j as follows: (i) $v_k \in S_{i*} \cap S_{*j}$; (ii) $v_k \in S_{i*} \cap T_j^-$; (iii) $v_k \in T_i^+ \cap S_{*j}$; (iv) $v_k \in T_i^+ \cap T_j^-$. To show $A_i \cap B_j \subseteq S_{ij}$, we will check that $v_k \in S_{ij}$ for each case.

case (i): Since $v_k \in S_{i*}$, there exists $p \in [n]$ such that $v_k \in S_{ip}$. Since $v_k \in S_{*j}$, there exists $q \in [n]$ such that $v_k \in S_{qj}$. By (3), $v_k \in S_{ip}$ implies $(v_i, v_k), (v_k, v_p) \in A(D)$, and $v_k \in S_{qj}$ implies $(v_q, v_k), (v_k, v_j) \in A(D)$. Therefore we have $(v_i, v_k), (v_k, v_j) \in A(D)$, which implies $v_k \in S_{ij}$.

case (ii): Since $v_k \in S_{i*}$, there exists $p \in [n]$ such that $v_k \in S_{ip}$. Since $v_k \in T_j^-$, there exists $b \in [n]$ such that $v_j \in S_{kb}$. By (3), $v_k \in S_{ip}$ implies $(v_i, v_k), (v_k, v_p) \in A(D)$, and $v_j \in S_{kb}$ implies $(v_k, v_j), (v_j, v_b) \in A(D)$. Therefore we have $(v_i, v_k), (v_k, v_j) \in A(D)$, which implies $v_k \in S_{ij}$.

case (iii): Since $v_k \in T_i^+$, there exists $a \in [n]$ such that $v_i \in S_{ak}$. Since $v_k \in S_{*j}$, there exists $q \in [n]$ such that $v_k \in S_{qj}$. By (3), $v_i \in S_{ak}$ implies $(v_a, v_i), (v_i, v_k) \in A(D)$, and $v_k \in S_{qj}$ implies $(v_q, v_k), (v_k, v_j) \in A(D)$. Therefore we have $(v_i, v_k), (v_k, v_j) \in A(D)$, which implies $v_k \in S_{ij}$.

case (iv): Since $v_k \in T_i^+$, there exists $a \in [n]$ such that $v_i \in S_{ak}$. Since $v_k \in T_j^-$, there exists $b \in [n]$ such that $v_j \in S_{kb}$. By (3), $v_i \in S_{ak}$ implies $(v_a, v_i), (v_i, v_k) \in A(D)$, and $v_j \in S_{kb}$ implies $(v_k, v_j), (v_j, v_b) \in A(D)$. Therefore we have $(v_i, v_k), (v_k, v_j) \in A(D)$, which implies $v_k \in S_{ij}$.

Thus we obtain $A_i \cap B_j \subseteq S_{ij}$, and so $A_i \cap B_j = S_{ij}$. Hence the condition (I) holds.

Next, we show the if part. Let M be a multigraph with n vertices, and suppose that there exist an ordering $(v_1, \ldots, v_n)$ of the vertices of M and a double indexed edge clique partition $\mathcal{F} = \{S_{ij} \mid i, j \in [n]\}$ of M such that the condition (I) holds.

We define a digraph D by $V(D) := V(M)$ and

$$A(D) := \bigcup_{i,j \in [n]} \left(\bigcup_{v_k \in S_{ij}} \{(v_i, v_k), (v_k, v_j)\} \right). \tag{5}$$

Let M' denote the double competition multigraph of D. We show that $M = M'$. Since $V(M) = V(M')$, it is enough to show $m_M = m_{M'}$. Take any two distinct vertices v_k and v_l and let $t := m_M(\{v_k, v_l\})$. Since $\mathcal{F}$ is an edge clique partition of M, the vertices v_k and v_l are contained in exactly t cliques $S_{ij} \in \mathcal{F}$. So, for some nonnegative integers r and s with $rs = t$, there are r common in-neighbors $v_{i_1}, \ldots, v_{i_r}$ and s common out-neighbors $v_{j_1}, \ldots, v_{j_s}$ of the vertices v_k and v_l in D. Therefore it follows that $m_{M'}(\{v_k, v_l\}) = |N_D^-(v_k) \cap N_D^-(v_l)| \cdot |N_D^+(v_k) \cap N_D^+(v_l)| \geq rs = t$. Thus $m_M(\{v_k, v_l\}) \leq m_{M'}(\{v_k, v_l\})$. Again, take any two distinct vertices v_k and v_l and let $t' := m_{M'}(\{v_k, v_l\})$. Then, for some nonnegative integers r' and s' with $r's' = t'$, there are r' common in-neighbors $v_{i_1}, \ldots, v_{i_{r'}}$ and s' common out-neighbors $v_{j_1}, \ldots, v_{j_{s'}}$ of the vertices v_k and v_l in D. For each $i \in \{i_1, \ldots, i_{r'}\}$, since $(v_i, v_k), (v_i, v_l) \in A(D)$, it follows that $\{v_k, v_l\} \subseteq A_i$. Similarly, for each $j \in \{j_1, \ldots, j_{s'}\}$, since $(v_k, v_j), (v_l, v_j) \in A(D)$, it follows that $\{v_k, v_l\} \subseteq B_j$. Therefore, $\{v_k, v_l\} \subseteq A_i \cap B_j$ for any $i \in \{i_1, \ldots, i_{r'}\}$ and any $j \in \{j_1, \ldots, j_{s'}\}$. By the condition (I), we have $A_i \cap B_j = S_{ij}$. Therefore $\{v_k, v_l\} \subseteq S_{ij}$ for any $i \in \{i_1, \ldots, i_{r'}\}$ and any $j \in \{j_1, \ldots, j_{s'}\}$ and this implies that $m_M(\{v_k, v_l\}) = |\{S_{i,j} \in \mathcal{F} \mid \{v_k, v_l\} \subseteq S_{i,j}\}| \geq r's' = t'$. Thus $m_M(\{v_k, v_l\}) \geq m_{M'}(\{v_k, v_l\})$. Hence it holds that $m_M(\{v_k, v_l\}) = m_{M'}(\{v_k, v_l\})$ for any two distinct vertices v_k and v_l, that is, $m_M = m_{M'}$, i.e., $M = M'$. So M is the double competition multigraph of D. □

A digraph D is said to be *loopless* if D has no loops, i.e., $(v, v) \notin A(D)$ holds for any $v \in V(D)$.

Theorem 2. *Let M be a multigraph with n vertices. Then, M is the double competition multigraph of a loopless digraph if and only if there exist an ordering $(v_1, \ldots, v_n)$ of the vertices of M and a double indexed edge clique partition $\{S_{ij} \mid i, j \in [n]\}$ of M such that the following conditions hold:*

(I) *for any $i, j \in [n]$, if $|A_i \cap B_j| \geq 2$, then $A_i \cap B_j = S_{ij}$;*

(II) *for any $i, j \in [n]$, $v_i \notin S_{ij}$ and $v_j \notin S_{ij}$,*

where A_i and B_j are the sets defined as (1) *and* (2).

Proof: First, we show the only-if part. Let M be the double competition multigraph of a loopless digraph D. Let $(v_1, \ldots, v_n)$ be an ordering of the vertices of D. Let S_{ij} $(i, j \in [n])$ be the sets defined as (3), and let $\mathcal{F}$ be the family defined as (4). Then S_{ij} is a clique of M, and $\mathcal{F}$ is an edge clique partition of M. Moreover, we can show, as in the proof of Theorem 1, that the condition (I) holds. Now we show that the condition (II) holds. Take any vertex $v_k \in S_{ij}$. Then $(v_i, v_k), (v_k, v_j) \in A(D)$. Since D is loopless, we have $v_i \neq v_k$ and $v_i \neq v_k$. Therefore it follows that $v_i \notin S_{ij}$ and $v_j \notin S_{ij}$. Thus the condition (II) holds.

Next, we show the if part. Let M be a multigraph with n vertices, and suppose that there exists an ordering $(v_1, \ldots, v_n)$ of the vertices of M and a double indexed edge clique partition $\{S_{ij} \mid i, j \in [n]\}$ of M such that the conditions (I) and (II) hold. We define a digraph D by $V(D) := V(M)$ and $A(D)$ given in (5). By the condition (II), it follows from the definition of D that $(v_i, v_i) \notin A(D)$ for any $i \in [n]$. Therefore D is a loopless digraph. Moreover we can show, as in the proof of Theorem 1, that M is the double competition multigraph of D. □

A digraph D is said to be *reflexive* if all the vertices of D have loops, i.e., $(v, v) \in A(D)$ holds for any $v \in V(D)$.

Theorem 3. *Let M be a multigraph with n vertices. Then, M is the double competition multigraph of a reflexive digraph if and only if there exist an ordering $(v_1, \ldots, v_n)$ of the vertices of M and a double indexed edge clique partition $\{S_{ij} \mid i, j \in [n]\}$ of M such that the following conditions hold:*

(I) *for any $i, j \in [n]$, if $|A_i \cap B_j| \geq 2$, then $A_i \cap B_j = S_{ij}$;*

(III) *for any $i \in [n]$, $v_i \in S_{i*} \cup S_{*i}$,*

where A_i, B_j, S_{i*}, and S_{*i} are the sets defined as (1) *and* (2).

Proof: First, we show the only-if part. Let M be the double competition multigraph of a reflexive digraph D. Let $(v_1, \ldots, v_n)$ be an ordering of the vertices of D. Let S_{ij} $(i, j \in [n])$ be the sets defined as (3), and let $\mathcal{F}$ be the family defined as (4). Then S_{ij} is a clique of M, and $\mathcal{F}$ is an edge clique partition of M.

Moreover, we can show, as in the proof of Theorem 1, that the condition (I) holds. Now we show that the condition (III) holds. Since D is reflexive, we have $(v_i, v_i) \in A(D)$ for any $i \in [n]$. Then it follows that there exists $p \in [n]$ such that $v_i \in S_{ip}$ or $v_i \in S_{pi}$. Therefore $v_i \in S_{i*} \cup S_{*i}$. Thus the condition (III) holds.

Next, we show the if part. Let M be a multigraph with n vertices, and suppose that there exist an ordering $(v_1, \dots, v_n)$ of the vertices of M and a double indexed edge clique partition $\mathcal{F} = \{S_{ij} \mid i, j \in [n]\}$ of M such that the conditions (I) and (III) hold. We define a digraph D by $V(D) := V(M)$ and $A(D)$ given in (5). Fix any $i \in [n]$. By the condition (III), there exists $p \in [n]$ such that $v_i \in S_{ip}$ or $v_i \in S_{pi}$. Then it follows from the definition of D that $(v_i, v_i) \in A(D)$. Therefore D is a reflexive digraph. Moreover we can show, as in the proof of Theorem 1, that M is the double competition multigraph of D. □

A digraph D is said to be *acyclic* if D has no directed cycles. An ordering $(v_1, \dots, v_n)$ of the vertices of a digraph D, where n is the number of vertices of D, is called an *acyclic ordering* of D if $(v_i, v_j) \in A(D)$ implies $i < j$. It is well known that a digraph D is acyclic if and only if D has an acyclic ordering.

Theorem 4. *Let M be a multigraph with n vertices. Then, M is the double competition multigraph of an acyclic digraph if and only if there exist an ordering $(v_1, \dots, v_n)$ of the vertices of M and a double indexed edge clique partition $\{S_{ij} \mid i, j \in [n]\}$ of M such that the following conditions hold:*

(I) *for any $i, j \in [n]$, if $|A_i \cap B_j| \geq 2$, then $A_i \cap B_j = S_{ij}$;*

(IV) *for any $i, j, k \in [n]$, $v_k \in S_{ij}$ implies $i < k < j$,*

where A_i and B_j are the sets defined as (1) *and* (2).

Proof: First, we show the only-if part. Let M be the double competition multigraph of an acyclic digraph D. Let $(v_1, \dots, v_n)$ be an acyclic ordering of the vertices of D. Let S_{ij} $(i, j \in [n])$ be the sets defined as (3), and let $\mathcal{F}$ be the family defined as (4). Then S_{ij} is a clique of M, and $\mathcal{F}$ is an edge clique partition of M. Moreover, we can show, as in the proof of Theorem 1, that the condition (I) holds. Now we show that the condition (IV) holds. Suppose that $v_k \in S_{ij}$. Then $(v_i, v_k), (v_k, v_j) \in A(D)$. Since $(v_1, \dots, v_n)$ is an acyclic ordering of D, $(v_i, v_k) \in A(D)$ implies $i < k$ and $(v_k, v_j) \in A(D)$ implies $k < j$ Therefore $i < k < j$. Thus the condition (IV) holds.

Next, we show the if part. Let M be a multigraph with n vertices, and suppose that there exist an ordering $(v_1, \dots, v_n)$ of the vertices of M and a double indexed edge clique partition $\{S_{ij} \mid i, j \in [n]\}$ of M such that the conditions (I) and (IV) hold. We define a digraph D by $V(D) := V(M)$ and $A(D)$ given in (5). By the condition (IV), it follows from the definition of D that $(v_1, \dots, v_n)$ is an acyclic ordering of D. Therefore D is an acyclic digraph. Moreover we can show, as in the proof of Theorem 1, that M is the double competition multigraph of D. □

Remark 5. The condition (I) in Theorems 1, 2, 3, and 4 may be replaced by the following condition:

(I)$'$ for any $i, j \in [n]$, $A_i \cap B_j = S_{ij}$.

Proof: If the condition (I)$'$ is satisfied, then so is the condition (I). Suppose that the condition (I) is satisfied. If $|A_i \cap B_j| \geq 2$, then it follows from the condition (I) that $A_i \cap B_j = S_{ij}$. If $|A_i \cap B_j| = 0$, then $A_i \cap B_j = \emptyset$. Since $S_{ij} \subseteq A_i \cap B_j$, we have $S_{ij} = \emptyset$. Therefore, $A_i \cap B_j = S_{ij}$. If $|A_i \cap B_j| = 1$, then $A_i \cap B_j = \{v_k\}$ for some $k \in [n]$. Since $S_{ij} \subseteq A_i \cap B_j$, we have $S_{ij} = \emptyset$ or $S_{ij} = \{v_k\}$. If $S_{ij} = \{v_k\}$, then $A_i \cap B_j = S_{ij}$. If $S_{ij} = \emptyset$, then we replace $S_{ij} = \emptyset$ by $S_{ij} = \{v_k\}$. Then $\mathcal{F}$ is still an edge clique partition of M, and $A_i \cap B_j = S_{ij}$. Thus the condition (I)$'$ holds. Hence the remark holds. □

At the end of this paper, we mention two corollaries related to the edge clique partition number of a multigraph. Recall that the *edge clique partition number* of a multigraph M is the minimum size of an edge clique partition of M and is denoted by $\theta^*(M)$. As a corollary of Theorem 1, we obtain a necessary condition for multigraphs being the double competition multigraph of a digraph.

Corollary 6. *If a multigraph M with n vertices is the double competition multigraph of a digraph, then $\theta^*(M) \leq n^2$.*

For the double competition multigraphs of acyclic digraphs, we can improve the above upper bound for the edge clique partition numbers of multigraphs.

Corollary 7. *If a multigraph M with n vertices is the double competition multigraph of an acyclic digraph, then $\theta^*(M) \leq \frac{1}{2}(n-2)(n-3)$.*

Proof: Suppoe that a multigraph M with n vertices is the double competition multigraph of an acyclic digraph. Then, by Theorem 4, there exist an ordering $(v_1, \ldots, v_n)$ of the vertices of M and a double indexed edge clique partition $\{S_{ij} \mid i, j \in [n]\}$ of M satisfying the conditions (I) and (IV). It follows from the condition (IV) that, if $j \leq i+1$, then $S_{ij} = \emptyset$. If $j = i+2$, then $S_{ij} = \emptyset$ or $S_{ij} = \{v_{i+1}\}$, which does not cover an edge of M. Therefore, the family $\{S_{ij} \mid i, j \in [n], i+3 \leq j\}$ is an edge clique partition of M. Thus the corollary holds. □

Remark 8. In [7], the authors defined the *double multicompetition number* $dk^*(M)$ of a multigraph M to be the minimum nonnegative integer k such that M together with k new isolated vertices is the double competition multigraph of some acyclic digraph. In this context, Theorem 4 gives a characterization of multigraphs whose double multicompetition number is equal to 0.

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