Traceability of Locally Hamiltonian and Locally Traceable Graphs

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If $\mathcal{P}$ is a given graph property, we say that a graph G is *locally* $\mathcal{P}$ if $\langle N(v) \rangle$ has property $\mathcal{P}$ for every $v \in V(G)$ where $\langle N(v) \rangle$ is the induced graph on the open neighbourhood of the vertex v. Pareek and Skupień (C. M. Pareek and Z. Skupień, On the smallest non-Hamiltonian locally Hamiltonian graph, *J. Univ. Kuwait (Sci.)*, 10:9 - 17, 1983) posed the following two questions.

Question 1 Is 9 the smallest order of a connected nontraceable locally traceable graph?

Question 2 Is 14 the smallest order of a connected nontraceable locally hamiltonian graph?

We answer the second question in the affirmative, but show that the correct number for the first question is 10. We develop a technique to construct connected locally hamiltonian and locally traceable graphs that are not traceable. We use this technique to construct such graphs with various prescribed properties.

Keywords: locally traceable, locally hamiltonian, nonhamiltonian, nontraceable

1 Introduction

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The order $n(G)$ and size $e(G)$ of G are the cardinalities of $V(G)$ and $E(G)$, respectively. If $X \subseteq V(G)$ then $\langle X \rangle$ denotes the subgraph of G induced by X. If $v \in V(G)$ then $N(v)$ denotes the open neighbourhood of v in G and $N[v]$ denotes the closed neighbourhood of v in G. A graph G is hamiltonian if it contains a cycle C that contains all the vertices in $V(G)$ and G is traceable if it contains a path P that contains all the vertices in $V(G)$. A graph G is traceable from u to v, where $u, v \in V(G)$, if there is a Hamilton path with u and v as end vertices. We define $comp(G)$ to be the number of components of a graph G. Let $P = p_1 \dots p_i$ and $Q = q_1 \dots q_j$ be two paths in G. Then the concatenation of the two paths $p_1 \dots p_i q_1 \dots q_j$ is denoted by PQ. For undefined concepts we refer the reader to [3].

A graph G is called *locally hamiltonian* if $\langle N(v) \rangle$ is hamiltonian for every vertex v in G. The notion of local hamiltonicity was introduced by Skupień [11] in 1965. A graph G is called *locally connected*

if $\langle N(v) \rangle$ is connected for every vertex v in G. The notion of local connectedness was introduced by Chartrand and Pippert [5] in 1974 and has been extensively studied - see for example [11, 6, 4, 7, 10]. A graph G is called *locally traceable* if $\langle N(v) \rangle$ is traceable for every vertex v in G. In general, if $\mathcal{P}$ is a given graph property, we say that G is *locally $\mathcal{P}$* if $\langle N(v) \rangle$ has property $\mathcal{P}$ for every $v \in V(G)$.

We are particularly interested in the traceability properties of locally connected, locally traceable and locally hamiltonian graphs. We abbreviate locally connected to LC, locally traceable to LT and locally hamiltonian to LH.

Another local property that is often studied in connection with hamiltonicity is the property of being claw-free (i.e. not having $K_{1,3}$ as induced subgraph). Note that a graph G is claw-free if and only if $\alpha(\langle N(v) \rangle) \leq 2$ for every $v \in V(G)$ where $\alpha(\langle N(v) \rangle)$ is the independence number of $\langle N(v) \rangle$. A well-known example where local properties provide valuable information regarding hamiltonicity is the following theorem, proved by Oberly and Sumner [8] in 1979.

Theorem 1.1 [8] *If G is a connected LC claw-free graph, then G is hamiltonian.*

Note that this implies that a connected LC claw-free graph is also traceable. The requirement that G be claw-free cannot be dropped, even if LC is replaced with LH. In fact, we show that the difference between the number of vertices and the length of a longest path in a connected LH graph may be arbitrarily large.
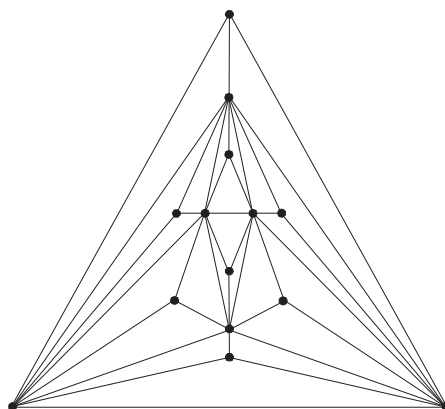
Fig. 1: A connected nontraceable LH graph of order 14.

In 1983 Pareek and Skupień [10] considered the traceability of LH and LT graphs. Figure 1 depicts a connected nontraceable LH graph of order 14. It was presented in 1972 as an example of a maximal planar nontraceable graph of smallest order by Goodey [7], who also proved that every maximal planar graph of order less than 14 is traceable. It is easily seen that every maximal planar graph is LH. Pareek and Skupień [10] posed the following two questions.

Question 1 Is 9 the smallest order of a connected nontraceable LT graph?

Question 2 Is 14 the smallest order of a connected nontraceable LH graph?

In Section 2 we show that the correct value in Question 1 is 10 and we present the 6 connected non-

traceable LT graphs of order 10 that were found by means of a computer search. We also show that the maximum vertex degree of nontraceable LT graphs is at least 6. We develop a technique that we call edge identification to construct nontraceable LT graphs, and use this technique to show that there are planar connected nontraceable LT graphs of all orders greater than 9. We show, moreover, that for every $n \geq 10$ there exists a connected nontraceable LT graph with maximum degree 7.

In Section 3 we answer the second question in the affirmative by proving that there is no connected nontraceable LH graph of order less than 14. Using a technique called triangle identification we show that there are planar connected nontraceable LH graphs of every order greater than 13. We also show that there exist connected nontraceable LH graphs with minimum vertex degree k for all $k \geq 3$.

2   Nontraceable LT graphs

We begin this section by defining a construction that will be extensively used in what follows.

Construction 2.1 (*Edge identification*) *Let G_1 and G_2 be two LT graphs such that $E(G_i)$ contains an edge $u_i v_i$ so that there is a Hamilton path in $\langle N(u_i) \rangle$ that ends at v_i and a Hamilton path in $\langle N(v_i) \rangle$ that ends at u_i, $i = 1, 2$. Now create a larger graph G by identifying the edges $u_1 v_1$ and $u_2 v_2$ to a single edge uv (*see Figure 2*).*
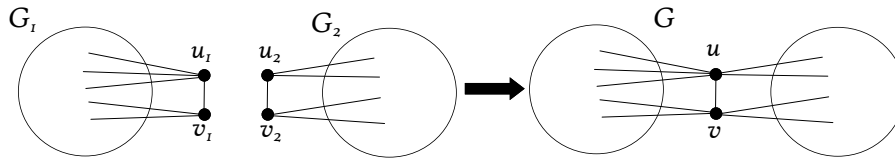
Fig. 2: The edge identification procedure.

Theorem 2.2 *Let G_1 and G_2 be two LT graphs that satisfy the conditions of Construction 2.1. If G_1 and G_2 are combined by means of edge identification to create a graph G, then G is LT. If G is traceable, then both G_1 and G_2 are traceable.*

Proof: Let $u_i v_i \in E(G_i)$, $i = 1, 2$ be the two edges used in Construction 2.1 to form the edge uv in $E(G)$.

First suppose $w \in V(G) - \{u, v\}$. Since the neighbourhood of w is restricted to vertices that are either all in G_1 or all in G_2, $\langle N_G(w) \rangle$ is traceable.

Now suppose w is one of u and v, say u. Let $Q_1 v_1$ be a Hamilton path in $\langle N_{G_1}(u_1) \rangle$ and let $v_2 Q_2$ be a Hamilton path in $\langle N_{G_2}(u_2) \rangle$, where Q_1 and Q_2 are paths in G_1 and G_2, respectively. Then $Q_1 v Q_2$ is a Hamilton path in $\langle N_G(u) \rangle$. Using a similar argument, we can also find a Hamilton path in $\langle N_G(v) \rangle$. Hence G is LT.

Now assume P is a Hamilton path in G. If uv is an edge of P, then P is of the form illustrated in Figure 3 (a). If uv is not an edge of P, then P is of the form illustrated in Figure 3 (b) or (c). In all three cases it is easy to find Hamilton paths in G_1 and G_2. □
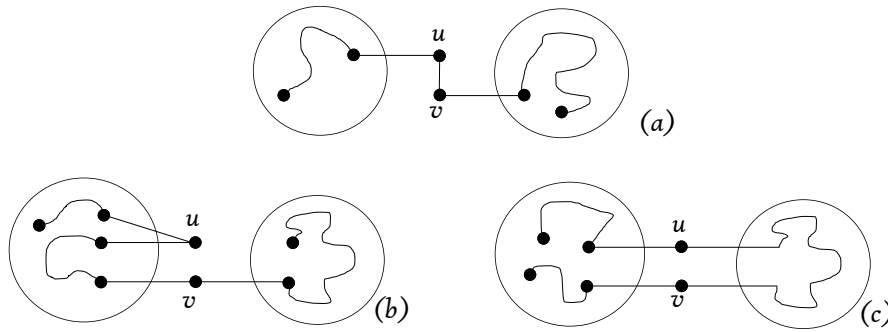
Fig. 3: Edge identification preserves LT and nontraceable properties.

The following observation will be useful for selecting suitable edges to use in edge identification.

Observation 2.3 *Let v be a vertex of degree two in a LT graph. Then any edge incident with v is suitable for use in edge identification.*

This can easily be seen by noting that if $N(v) = \{u, w\}$, the edge uw is the Hamilton path of $\langle N(v)\rangle$, and since $d_{\langle N(u)\rangle}(v) = 1$, any Hamilton path of $\langle N(u)\rangle$ starts at v. In particular, if a LT graph G is combined with K_3 by means of edge identification to create a graph H, then the vertex $v \in V(K_3)$ that is not incident with the edge used in the procedure, will have degree two. Hence any one of its incident edges will still be suitable for use in edge identification.

The first property of a connected nontraceable LT graph G we will investigate, is a lower bound for $\Delta(G)$. Theorem 2.4 will be useful in this regard. The graphs M_3, M_4 and M_5 referred to in the theorem are shown in Figure 4.
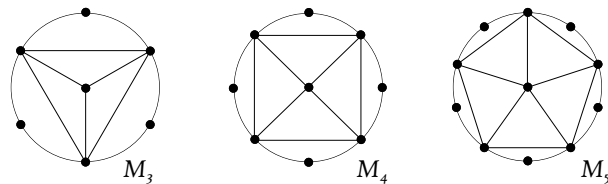

Fig. 4: The graphs M_3, M_4 and M_5.

A graph G is *fully cycle extendable* if every vertex in G lies on a 3-cycle and, for every nonhamiltonian cycle C of G, there exists a cycle C' in G that contains all the vertices of C plus a single new vertex.

Theorem 2.4 [1] *Suppose G is a connected LT graph with $n(G) \geq 3$ and $\Delta(G) \leq 5$. Then G is fully cycle extendable if and only if $G \notin \{M_3, M_4, M_5\}$.*

We are now ready to present the following theorem.

Theorem 2.5 *If G is a connected nontraceable LT graph, then $\Delta(G) \geq 6$, and this bound is sharp.*

Proof: Since the graphs M_3, M_4 and M_5 in Figure 4 are traceable, it follows from Theorem 2.4 that $\Delta(G) \geq 6$ (a fully cycle extendable graph is hamiltonian, and therefore traceable). Four copies of the graph M_3 can be combined using edge identification to create the graph in Figure 5 with maximum degree 6. It is easy to see that this graph is nontraceable. Hence the bound is sharp. □
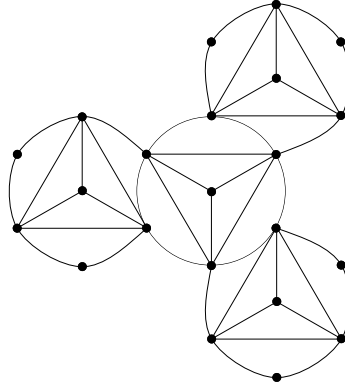
Fig. 5: A connected nontraceable LT graph with maximum vertex degree 6.

Next we answer the first question posed by Pareek and Skupień.

Theorem 2.6 *If G is a connected nontraceable LT graph, then $n(G) \geq 10$.*

Proof: By Theorem 2.5, G has a vertex w of degree k at least 6. Let $v_1 v_2 \ldots v_k$ be a Hamilton path of $\langle N(w) \rangle$, and let $X = \langle V(G) - N[w] \rangle$.

We make the following observations:

(i) $\langle N[w] \rangle$ is traceable from v_i to v_{i+1} (indices taken modulo k).

(ii) $\langle N[w] \rangle$ is traceable from v_1 and v_k to any vertex in $N[w]$.

(iii) Since $\langle N[w] \rangle$ is hamiltonian and G is nontraceable, $n(X) \geq 2$.

(iv) Each component of X has at least two neighbours in $N(w)$.

(v) If $comp(X) \geq 2$, then X has at least three neighbours in $N(w)$.

Suppose $n(G) < 10$. Then it follows from Theorem 2.5 and (iii) above that $\Delta(G) = 6$, $n(X) = 2$ and $n(G) = 9$. Let $V(X) = \{x_1, x_2\}$. Since G is nontraceable, x_1 and x_2 are nonadjacent. Then by (ii) and (iv), no vertex in X can be adjacent to either v_1 or v_6. If x_1, say, is adjacent to both v_i and v_{i+1} (indices modulo 6), then $G - x_2$ is hamiltonian, and therefore G is traceable. If x_1 is adjacent to v_i and x_2 is adjacent to v_{i+1} (indices modulo 6), then by (i) G is traceable. Hence by (iv) and (v) we have a contradiction. □

A computer search of graphs of order 10 resulted in the 6 nontraceable LT graphs shown in Figure 6. The search was done by constructing all possible graphs of order 10 with maximum vertex degree of either 6 or 7. The graphs were then tested for local traceability and traceability. Finally, graphs that were isomorphic to each other were eliminated from the list of graphs that were found. Since the search space is relatively small, it was feasible to do the search in Visual Basic in MicroSoft Excel. Note that all the graphs in Figure 6 have maximum vertex degree 7. It is reasonably straightforward, although tedious, to prove analytically that every connected nontraceable LT graph of order 10 has maximum vertex degree 7.
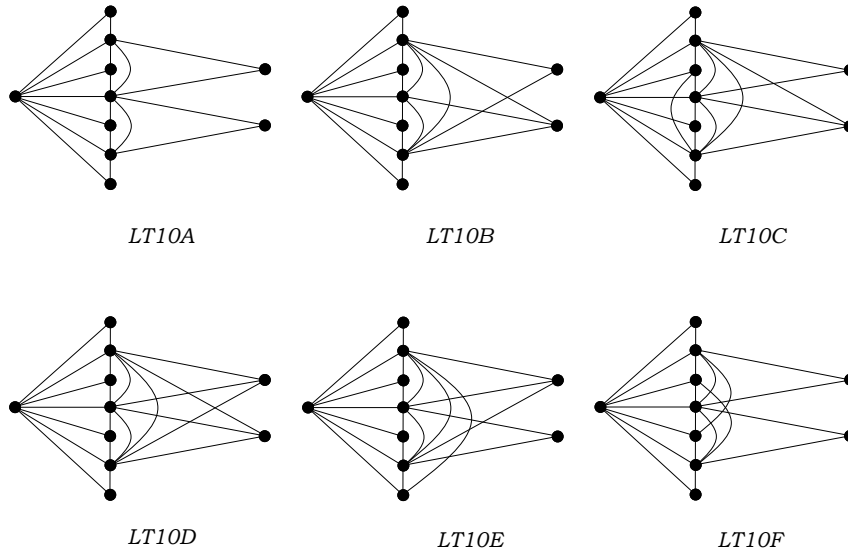

LT10A   *LT10B*   *LT10C*

LT10D   *LT10E*   *LT10F*

Fig. 6: The nontraceable LT graphs of order 10.

Observation 2.7 *If two planar LT graphs G_1 and G_2 are combined using edge identification to create graph G, then G is planar.*

Theorem 2.8 *For any $k \geq 10$ there exists a connected planar nontraceable LT graph G that has order k and $\Delta(G) = 7$.*

Proof: Let G_0 be the graph LT10A, depicted in Figure 6 and redrawn as the first graph in Figure 7. For each $i \geq 1$, let G_i be the graph obtained by combining G_{i-1} with a K_3 by means of edge identification, starting with the edge v_1v_2, and after that each time using the edge between the vertices of degree two and three of the last added triangle, as shown in Figure 7. It follows from repeated application of Observation 2.7, that for $k \geq 10$, the graph G_{k-10} is a connected planar nontraceable LT graph of order k and it is clear from Figure 7 that it has maximum degree 7.

□
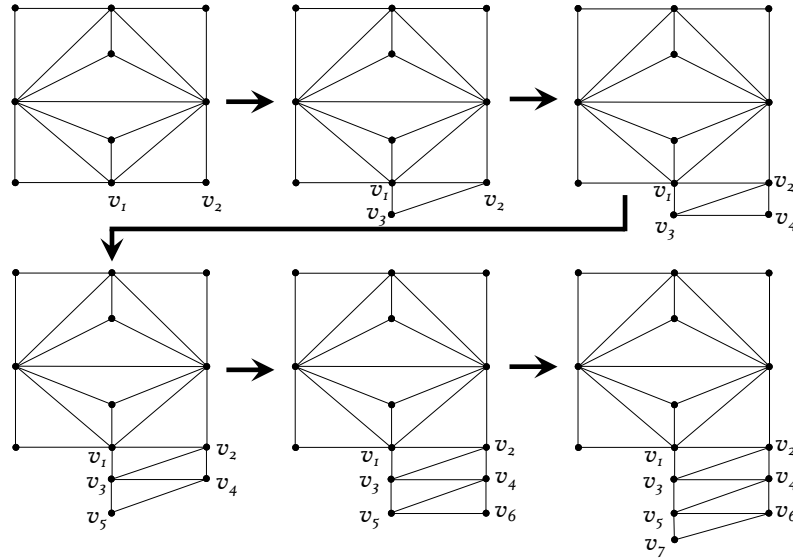
Fig. 7: Constructing nontraceable LT graphs with $\Delta(G) = 7$.

Note that the same procedure can be implemented using the graph in Figure 5 to create planar nontraceable LT graphs of any order greater than or equal to 22 with maximum vertex degree 6.

3 Nontraceable LH graphs

We begin this section by addressing Pareek and Skupień's second question.

As mentioned earlier, the graph in Figure 1 is a connected nontraceable LH graph of order 14. Thus it remains to prove that every LH graph of order less than 14 is traceable.

Pareek [9] claimed that every connected nonhamiltonian LH graph has maximum degree 8. However, as explained in [12], there are gaps in his proof. We were unable to fill those gaps, so we regard Pareek's claim as unproved. However, the following weaker result is valid. It is partly proved in [9] and a complete proof is given in [1].

Theorem 3.1 *If G is a connected nonhamiltonian LH graph, then $\Delta(G) \geq 7$.*

Lemma 3.2 *Let G be an LH graph and let $v \in V(G)$. Then $\alpha(\langle N(v)\rangle) \leq d(v)/2$ and G is 3-connected.*

Note that if w is any vertex in an LH graph, then $\langle N[w]\rangle$ contains a wheel as a subgraph. The following two results concerning wheels will be used extensively throughout the proof of our main result in this section.

Lemma 3.3 *Let W be a wheel of order $d+1$, $d \geq 3$ with centre vertex w and let C be the cycle $v_1 \ldots v_d v_1$ in $N(w)$. Then W has a Hamilton path between v_i and v_j, for every pair i, j with $1 \leq i < j \leq d$.*

Moreover every edge of C lies on some Hamilton path between v_i and v_j except for the edge v_iv_j (when $j = i + 1$).
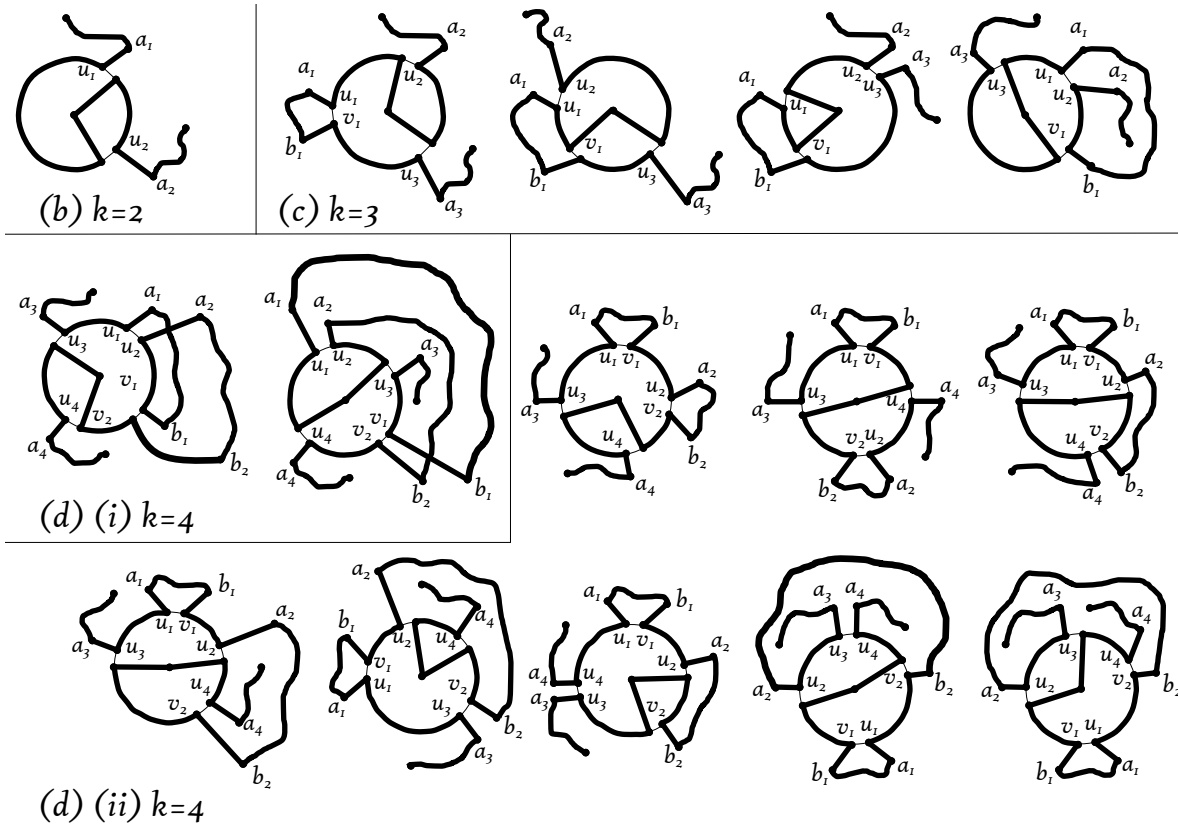

Fig. 8: The Hamilton paths referred to in Observation 3.4.

Figure 8 illustrates the Hamilton paths in Observation 3.4 for the cases (b), (c) and (d).

We define a k-path cover of a graph G to be a set of k disjoint paths that contain all the vertices in G.

Observation 3.4 *Suppose a graph G contains a wheel W with centre vertex w and let C be the cycle $v_1 \ldots v_d v_1$ in $\langle N(w) \rangle$. Suppose $G - V(W)$ has a k-path cover $Q_1, \ldots, Q_k$. Let a_i, b_i be the end-vertices of Q_i, $i = 1 \ldots, k$. (If Q_i is a singleton, then $a_i = b_i$.) Then the following hold.*

(a) *If $k = 1$ and a_1 has a neighbour in C, then G is traceable.*

(b) *If $k = 2$ and C contains a pair of distinct vertices $\{u_1, u_2\}$ such that $u_i \in N(a_i)$, $i = 1, 2$, then G is traceable.*

(c) *Suppose $k = 3$ and C contains two distinct pairs of distinct vertices $\{u_1, v_1\}$ and $\{u_2, u_3\}$ such that $u_i \in N(a_i)$ for $i = 1, 2, 3$ and $v_1 \in N(b_1)$. Then G is traceable if the set $\{u_1, v_1, u_2, u_3\}$ contains two consecutive vertices of C.*

(d) *Suppose $k = 4$ and C contains three distinct pairs of distinct vertices $\{u_1, v_1\}$, $\{u_2, v_2,\}$ and $\{u_3, u_4\}$ such that $u_i \in N(a_i)$ for $i = 1, 2, 3, 4$ and $v_i \in N(b_i)$ for $i = 1, 2$. Then G is traceable if either of the following hold.*

    (i) *The vertices u_2 and v_2 are the respective successors of u_1 and v_1 on C.*

    (ii) *The vertices u_1 and v_1 are consecutive vertices of C and the set $\{u_2, v_2, u_3, u_4\}$ contains a pair of consecutive vertices of C.*

Note that by "distinct pairs of distinct vertices" we mean that the two vertices in a given pair are distinct and any two given pairs have at most one vertex in common.

Lemma 3.3 implies the following.

Corollary 3.5 *If G is a connected nontraceable LH graph, then $\Delta(G) \leq n - 4$.*

Lemma 3.6 *Suppose G is a connected LH graph. For any $w \in V(G)$, let $C = v_1 v_2 \ldots v_d v_1$ be a Hamilton cycle in $\langle N(w) \rangle$ and let $X = G - N(w)$. Let S be the union of any s components of X. Then the following hold.*

    (i) *If for some $v_i \in N(w)$, v_i has at least one neighbour in each component of S, then $|N_C(v_i) \cap N_C(V(S))| \geq s + 1$ and $|N_C(V(S))| \geq s + 2$.*

    (ii) *If $s \in \{2, 3\}$, then $|N_C(V(S))| \geq s + 2$.*

Proof:

    (i) Since $\langle N_S(v_i) \cup \{w\} \rangle$ has at least $s + 1$ components, and since $\langle N(v_i) \rangle$ is hamiltonian, $\langle N(v_i) - \{w\} \rangle$ has a Hamilton path P with initial and terminal vertices on C. Since the maximal subpaths of P that intersect each component of S are preceded and followed by vertices on C, $|N_C(v_i) \cap N_C(V(S))| \geq s + 1$, and since $v_i \in N_C(V(S))$, the result follows.

    (ii) Suppose $|N_C(V(S))| \leq s + 1$. Since G is 3-connected, each component of S has at least 3 neighbours on C, and so, if $s \in \{2, 3\}$, it follows from the pigeonhole principle that there is some vertex v_i on C that has a neighbour in each component of S. The result follows from (i).

$\square$

The following observation will be used extensively in the proof of our main result in this section.

Observation 3.7 *If H is a connected graph of order $n \leq 5$, then one of the following holds.*

(a) *H is hamiltonian.*

(b) *H is nonhamiltonian but traceable and H has a Hamilton path Q with end-vertices a, b such that $d(a) \leq 1$, $d(b) \leq 2$ if $n \leq 4$ and $d(a) \leq 2$ if $n = 5$.*

(c) *H is nontraceable and has a 2-path cover Q_1, Q_2, such that Q_i has an end-vertex a_i of degree 1 for $i = 1, 2$, and all the end-vertices of Q_1 and Q_2 are independent.*

(d) *$H = K_{1,4}$.*

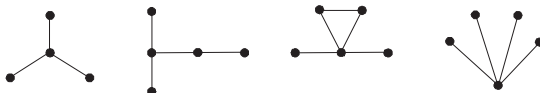Figure 9 shows the connected nontraceable graphs of order $n \leq 5$.

Fig. 9: The connected nontraceable graphs of order $n \leq 5$.

Theorem 3.8 *Suppose G is a connected LH graph of order $n \leq 13$. Then G is traceable.*

Proof: Suppose to the contrary that G is a connected nontraceable LH graph of $n \leq 13$. Let w be a vertex in G of degree $d = \Delta(G)$, let $C = v_1 \ldots v_d v_1$ be a Hamilton cycle in $\langle N(w) \rangle$ and $X = G - N[w]$. By Theorem 3.1 and Corollary 3.5, $\Delta(G) \in \{7, 8, 9\}$.

Suppose $\Delta(G) = 9$. Then $|V(X)| \leq 3$. If $E(X) \neq \emptyset$, then since G is 3-connected, it follows from Observation 3.4(a) and (b) that D is traceable. If $E(X) = \emptyset$, it follows from Lemma 3.6(ii), that X has at least two consecutive neighbours on C. Hence, since G is 3-connected, Observation 3.4(c) implies that G is traceable. We may therefore assume $\Delta(G) \in \{7, 8\}$.

Now let $Q_1, \ldots, Q_k$ be a minimum path cover of X and let a_i, b_i be the end-vertices of Q_i, $i = 1 \ldots, k$. (If Q has only one vertex, then $a_i = b_i$.) Since $Q_1, \ldots, Q_k$ is a minimum path cover of X, $a_i a_j, b_i b_j, a_i b_j \notin E(G)$ for $i \neq j$.

Claim 1: *If $v_i \in C$, then v_i is adjacent to at most 2 components of X.*

Proof of Claim 1: By Lemma 3.2, v_i is adjacent to at most $\frac{\Delta(G)}{2} - 1$ components in X, and since $\Delta(G) \in \{7, 8\}$, we need only consider the case where $\Delta(G) = 8$ and some $v_i \in C$ is adjacent to exactly three components in X. Hence if $k = 3$ then $V(X) = \{a_1, a_2, a_3\}$ or $V(X) = \{a_1, a_2, a_3, b_3\}$, otherwise $k = 4$ and $V(X) = \{a_1, a_2, a_3, a_4\}$. Without loss of generality we may assume $\{a_1, a_2, a_3\} \subset N(v_1)$. Since $\Delta(G) = 8$, it follows from Lemma 3.6(i) that v_1 has exactly 4 neighbours on C. Since $\{a_1, a_2, a_3, w\}$ is an independent set in $\langle N(v_1) \rangle$, and since $\langle N(v_1) \rangle$ is Hamiltonian, there exists an a_i and a_j in $N(v_1)$, $a_i \neq a_j$, such that $a_i \in N(v_8)$ and $a_j \in N(v_2)$. But, since G is 3-connected, this contradicts Observation 3.4(c) if $k = 3$ and it contradicts Observation 3.4(d)(ii) if $k = 4$.

We now consider the k-path cover $Q_1, ..., Q_k$ of X. There are five cases to consider.

Case $k = 1$.
Since G is 3-connected, it follows from Observation 3.7(a) and (b) that an end-vertex of Q_1 has a neighbour on C. Hence by Observation 3.4, G is traceable.

Case $k = 2$.
Since G is 3-connected, it follows from Observation 3.7(a), (b) and (c) that there are two distinct vertices

u_1 and u_2 on C such that u_i is adjacent to an end-vertex of Q_i, $i = 1, 2$. Hence, by Observation 3.4, G is traceable.

Case $k = 3$.

If X is a star $K_{1,4}$, and x its central vertex, then $\alpha(\langle N(x) \rangle) = 4$, which contradicts Lemma 3.2, since in this case $\Delta(G) = 7$. Hence X has either 2 or 3 components and each component of X has at most 4 vertices and at least one component is a singleton. Thus we may assume that $Q_1 = \{a_1\}$ and that a_1 has three distinct neighbours on C. Moreover, by Observation 3.7(a), (b), (c) and the fact that G is 3-connected, we may assume that either each of a_2 and a_3 has at least two neighbours on C or a_2 has at least three neighbours on C and a_3 has at least one neighbour on C. If a neighbour of a_3 (or b_3) is the successor or predecessor of a neighbour of a_2 (or b_2) on C, it follows from Observation 3.4(c) that G is traceable. Also if two of the neighbours of a_1 are consecutive on C, Observation 3.4(c) implies that G is traceable.

It remains to consider the case where no neighbour of a_i is a successor or predecessor of a neighbour of a_j (or b_j) on C for $i \neq j$, and if $a_i = b_i$, a_i has no consecutive neighbours on C.

If $\Delta(G) = 7$ we may therefore assume that $N(a_1) = \{v_1, v_3, v_5\}$ and since $\langle N(a_1) \rangle$ is hamiltonian, $v_1 v_3, v_1 v_5, v_3 v_5 \in E(G)$. Since the set $\{a_2, a_3\}$ has at least four neighbours in $\{v_1, v_3, v_5\}$, at least one of v_i, $i = 1, 3, 5$, will be of degree 8, a contradiction.

Hence $\Delta(G) = 8$ and $n(V(X)) \leq 4$ and $V(X) = \{a_1, a_2, a_3\}$ or $V(X) = \{a_1, a_2, a_3, b_3\}$. But now $|N_C(V(X))| = 4$, contradicting Lemma 3.6(ii).

Case $k = 4$.

If $n(X) = 4$, $V(X) = \{a_1, a_2, a_3, a_4\}$ and if $n(X) = 5$, $V(X) = \{a_1, a_2, a_3, a_4, b_4\}$. Observe also that since $\delta(G) \geq 3$, there are at least 12 edges between $V(C)$ and $V(X)$. We make the following claims.

Claim 2: *If a_k is an isolated vertex in X, and if $v_i \in N(a_k)$, then $v_{i-1} \notin N(a_k)$ and $v_{i+1} \notin N(a_k)$. If $n(X) = 5$ and $v_i \in N(a_4)$, then $v_{i-1} \notin N(b_4)$ and $v_{i+1} \notin N(b_4)$.*

Proof of Claim 2: First suppose $V(X) = \{a_1, a_2, a_3, a_4\}$.

Suppose to the contrary that $\{v_1, v_2\} \subseteq N(a_1)$. By Claim 1 and since G is 3-connected, there are at least seven edges between the $d - 2$ vertices in $C - \{v_1, v_2\}$ and $V(X) - \{a_1\}$. By Observation 3.4 (d)(ii), no two consecutive vertices on the path $v_3 v_4 \ldots v_d$ have neighbours in $V(X) - \{a_1\}$. Hence at most $\lceil \frac{d-2}{2} \rceil$ vertices on the path $v_3 v_4 \ldots v_d$ are neighbours of $X - a_1$. Since $d \in \{7, 8\}$, no more than three such vertices exist. But then one of these vertices will have at least three neighbours in $V(X)$, contradicting Claim 1.

Now suppose $V(X) = \{a_1, a_2, a_3, a_4, b_4\}$. Note that in this case $\Delta(G) = 7$.

If $v_1 \in N(a_4)$ and $v_2 \in N(b_4)$, the argument above will be directly applicable. So assume without loss of generality that $\{v_1, v_2\} \subseteq N(a_1)$. If $N(a_2) \cap \{v_1, v_2\} = \emptyset$, then by Observation 3.4(d)(ii), $N(a_2) = \{v_3, v_5, v_7\}$. Hence, again by Observation 3.4(d)(ii), $N_C(\{a_3, a_4, b_4\}) \subseteq \{v_3, v_5, v_7\}$. But then each of v_3, v_5 and v_7 will have neighbours in three components of X, contrary to Claim 1.

If $\{v_1, v_2\} \subset N(a_2)$, then by Claim 1 and Observation 3.4(d)(ii), $N(\{a_3, a_4, b_4\}) = \{v_4, v_5, v_6\}$. But since $\delta(G) \geq 3$, this again contradicts Claim 1. Therefore a_2, and by symmetry, a_3, each has exactly one neighbour in $\{v_1, v_2\}$. Hence by Observation 3.4(d)(ii) $N(a_2, a_3) = \{v_1, v_2, v_4, v_6\}$. This implies that no vertex in $V(C)$ is adjacent to a_4 or b_4 contradicting the fact that G is 3-connected.

Claim 3: $\Delta(G) = 8$ *and* $X = \{a_1, a_2, a_3, a_4\}$.

Proof of Claim 3: Suppose $\Delta(G) = 7$. By Claim 1 and since each component of X has at least three distinct neighbours in $V(C)$, we may assume without loss of generality that v_1 has neighbours in two components of X. Suppose v_1 is adjacent to a_i and a_j where $i, j \neq 4$. Then by Claim 2, $\{a_i, a_j\} \cap N(\{v_2, v_7\}) = \emptyset$. If $n(X) = 5$ and, say $j = 4$, then Claim 2 implies that $\{a_i, b_4\} \cap N(\{v_2, v_7\}) = \emptyset$. By Lemma 3.6(i), v_1 has at least three neighbours in $V(C)$ other than v_2 and v_7, and since v_1 is also adjacent to w, $d(v_1) \geq 8$, a contradiction.

Claim 4: *If* $v_i \in N(a_1) \cap N(a_2)$, *then there exists a* $v_j \neq v_i$ *such that* $v_j \in N(a_1) \cap N(a_2)$.

Proof of Claim 4: Suppose $\{a_1, a_2\} = N_X(v_1)$. By Claim 2, $\{v_2, v_8\} \cap \{N(a_1) \cup N(a_2)\} = \emptyset$. By Lemma 3.6(i) and since $\Delta(G) = 8$, v_1 has exactly three neighbours other than v_2 and v_8 in $V(C)$. Since $\langle N(v_1) \rangle$ is hamiltonian, one of these three neighbours must be adjacent to both a_1 and a_2.

Claim 5: $d(a_i) = 3$ *for all* $a_i \in X$.

Proof of Claim 5: Suppose to the contrary that $d(a_1) > 3$. Then by Claim 2, $d(a_1) = 4$ and we may assume without loss of generality that $N(a_1) = \{v_1, v_3, v_5, v_7\}$. By Observation 3.4(d)(i) at most one of $\{v_2, v_4, v_6, v_8\}$ is in $N(a_i)$, $a_i \neq a_1$. Since $\delta(G) \geq 3$ this implies that each $a_i \neq a_1$ is adjacent to at least two vertices in $N(a_1)$, contradicting Claim 1.

We can now proceed with the main proof of the theorem.

By Claim 5 there are 12 edges between $V(C)$ and $V(X)$. Hence by Claim 1 we may assume without loss of generality that $N_X(v_1) = \{a_1, a_2\}$. By Claims 2 and 5 we may also assume that either $N(a_1) = \{v_1, v_3, v_5\}$ or $N(a_1) = \{v_1, v_3, v_6\}$. By Claim 4, $|N(a_1) \cap N(a_2)| \geq 2$. Hence, by Claim 1 we may assume that for at least one of a_3 and a_4, say a_4 has no neighbour in $N(a_1)$. Furthermore, by Observation 3.4(d)(i), no two neighbours of a_4 are both successors (or both predecessors) of neighbours of a_1 on C. Also, by Claim 2, no two neighbours of a_4 are consecutive vertices on C. But then $d(a_4) < 3$, a contradiction.

Case $k = 5$.

In this case $X = \{a_1, a_2, a_3, a_4, a_5\}$ and $\Delta(G) = 7$. Since $d(a_i) \geq 3$ there are at least 15 edges between $V(C)$ and $V(X)$. But then some v_i on C will be adjacent to at least three components in X contradicting Claim 1. $\qquad \square$

The following construction will be used to find nontraceable LH graphs with certain prescribed properties.

Construction 3.9 *(Triangle identification) For* $i = 1, 2$, *let* G_i *be a LH graph that contains a triangle* X_i *such that for each vertex* $x \in V(X_i)$, *there is a Hamilton cycle of* $\langle N(x) \rangle$ *that contains the edge* $X_i - x$. *Suppose* $V(X_i) = \{u_i, v_i, w_i\}$, $i = 1, 2$. *Now create a larger graph* G *by identifying the vertices* u_i, $i = 1, 2$ *to a single vertex* u, *and similarly the vertices* v_i, $i = 1, 2$ *to* v *and* w_i, $i = 1, 2$ *to* w, *and by retaining all the edges present in the original two graphs (see Figure 10).*

Theorem 3.10 *If two LH graphs* G_1 *and* G_2 *are combined by triangle identification, the resulting graph* G *is LH. If* G *is traceable, then both* G_1 *and* G_2 *are traceable.*

Proof: We use the notation defined in Construction 3.9. First we prove that G is LH. Let X be the triangle of G formed by identifying the vertices of X_1 and X_2 in Construction 3.9. Observe that if $y \in$
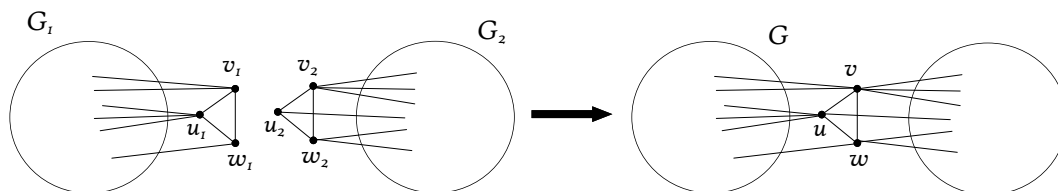
Fig. 10: The triangle identification procedure.

$V(G_1 - X_1)$, then $N_G(y) = N_{G_1}(y)$, except for a possible label change of vertices in $N_{G_1}(y) \cap V(X_1)$ to the corresponding vertices in $V(X)$. Hence if $y \in V(G_1 - X_1)$, then $\langle N_G(y) \rangle$ is hamiltonian. The same is true for $y \in V(G_2 - X_2)$. Now suppose $y \in V(X)$, say $y = u$. Let $v_1 Q_1 w_1 v_1$ and $w_2 Q_2 v_2 w_2$ be Hamilton cycles of $\langle N_{G_1}(u_1) \rangle$ and $\langle N_{G_2}(u_2) \rangle$ respectively. Then $v Q_1 w Q_2 v$ is a Hamilton cycle of $\langle N_G(u) \rangle$. Using a similar argument, we can also find Hamilton cycles for $\langle N_G(v) \rangle$ and $\langle N_G(w) \rangle$.

Now suppose G is traceable. Since only vertices in $V(X)$ have neighbours in both G_1 and G_2, Figure 11 shows the possible patterns that a Hamilton path in G can follow. The Hamilton path in Figure 11(a) uses two edges of X, the ones in Figure 11(b)-(d) use only one edge of X and the ones in Figure 11(e)-(i) do not use any edge of X. In each case it is easily seen that each of G_1 and G_2 has a Hamilton path. □

We are now in a position to use triangle identification to obtain nontraceable LH graphs with specific properties.

Theorem 3.11 *If two planar LH graphs G_1 and G_2 are combined using triangle identification, the resulting graph G is planar.*

Proof: First we show that a separating triangle (a separating triangle is a triangle that does not border a face in a plane representation of the graph) is not suitable for use in triangle identification. Let v_1, v_2 and v_3 be the vertices of a separating triangle in G_1. Since LH graphs are 3-connected, each vertex in the separating triangle has neighbours both inside the triangle and outside the triangle. It follows that in $\langle N(v_1) \rangle$ the edge $v_2 v_3$ is a cut edge and is therefore not part of a Hamilton cycle in $\langle N(v_1) \rangle$. Therefore the triangle is not suitable for triangle identification.

Let X_1 and X_2 be the respective triangles of G_1 and G_2 that were used in the triangle identification procedure of Construction 3.9 to form the triangle X of G. Since G_1 and G_2 are planar, G_1 can be drawn such that the edges of X_1 border the outer face of G_1, and G_2 can be drawn such that the edges of X_2 border an inner face of G_2 in a plane representation. The triangle identification procedure will then essentially draw $G_1 - X_1$ inside X and $G_2 - X_2$ outside X. Hence the resulting graph G is planar. □

Observation 3.12 *There exists a planar LH graph G of order n for every $n \geq 4$ such that $\Delta(G) \leq 6$.*

Proof: Such a graph can be constructed in the following manner: start with K_4 drawn in a plane representation. Attach an additional vertex to the three outer vertices in K_4 to create graph G_5. Keep repeating this procedure (add an additional vertex by connecting it to the three outer vertices in G_i). The procedure
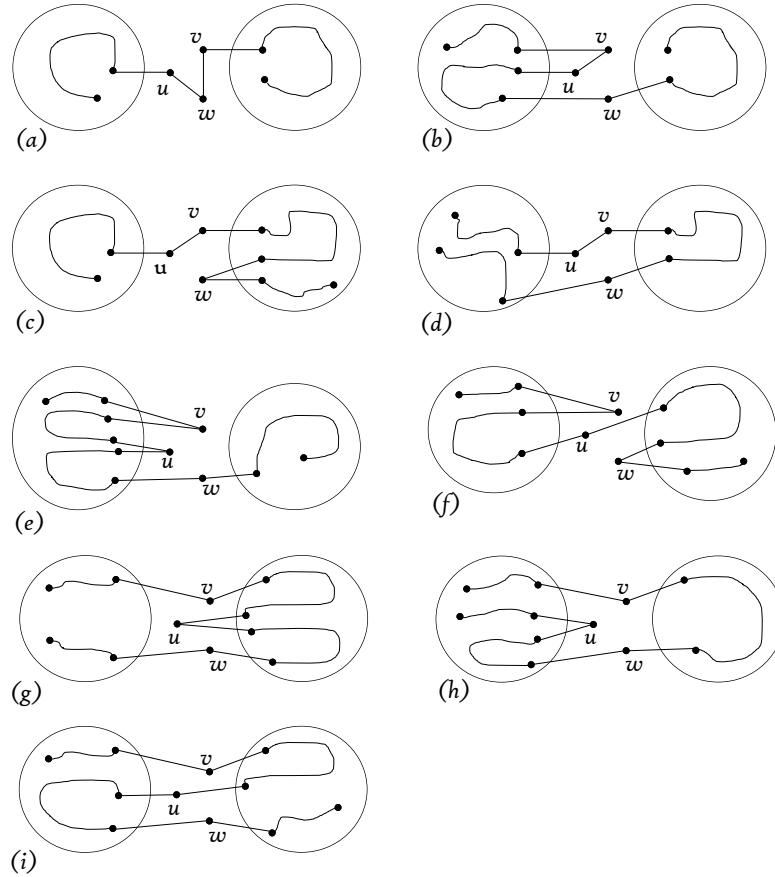
Fig. 11: The possible Hamilton paths through G.

essentially starts off with K_4, which is LH, and in each step uses triangle identification to combine G_i with K_4, so it is clear that the new graph G_{i+1} is also LH. Moreover, by drawing the graph in each step so that edges between the last three vertices added border the outer plane, the maximum vertex degree can be limited to six. See Figure 12. □

Theorem 3.13 *There exists a connected planar nontraceable LH graph of order n with $\Delta(G) \leq 10$ for every $n \geq 14$.*

Proof: First note that the nontraceable LH graph of order 14 in Figure 13 is planar. This is the same graph as shown in Figure 1 redrawn in a more convenient representation. Also note the three vertices of the LH graphs constructed in Observation 3.12 that border the outer plane are suitable for use in triangle identification. Label these three vertices u, v and w having degrees 3,4 and 5, respectively. By identifying u with v_5 in Figure 13, v with v_2, and w with u_5, we get a planar nontraceable LH graph G with maximum
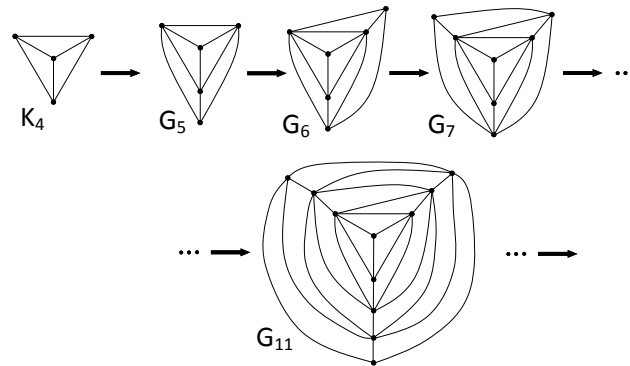
Fig. 12: Constructing a planar LH graph with maximum vertex degree 6.

vertex degree of 10. If we start with an LH graph H from Observation 3.12 of order $k, k \geq 4$, then $n(G) = 11 + k$. □
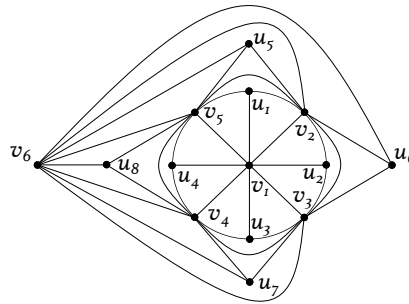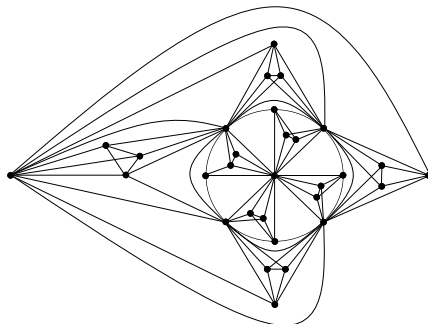

Fig. 13: The order 14 nontraceable LH graph shown in Section 1 in a different representation.

Theorem 3.14 *For any integer $k \geq 3$ there exists a nontraceable LH graph G with $\delta(G) = k$.*

Proof: To construct such a graph we start with the order 14 nontraceable LH graph H shown in Figure 13. Since complete graphs of order greater than 3 are LH, we can construct the graph G by combining multiple copies of K_{k+1} with G by means of triangle identification in such a way that each vertex of H is used at least once in a triangle identification procedure. Since a triangle can be used only once in triangle identification, we must use a new triangle for each step.

To see that a triangle with vertices x_1, x_2 and x_3 in a LH graph G_1 can only be used once in triangle identification to combine G_1 with a LH graph G_2, note that before triangle identification the edge $x_2 x_3$ is part of a Hamilton cycle in $\langle N_{G_1}(x_1) \rangle$. After triangle identification, the edge $x_2 x_3$ is replaced in the Hamilton cycle in $\langle N_G(x_1) \rangle$ by a path with vertices that originated from G_2. The same constraint does not apply to vertices.

Specifically, the triangles formed by edges between the vertices in the following sets in $V(H)$ can be used: $\{v_1, u_1, v_2\}$, $\{v_1, u_2, v_3\}$, $\{v_1, u_3, v_4\}$, $\{v_1, u_4, v_5\}$, $\{v_2, v_3, u_6\}$, $\{v_3, v_4, u_7\}$, $\{v_5, v_6, u_8\}$, and $\{v_5, v_2, u_5\}$. This results in the graph in Figure 14 (in this case K_5 was used for the triangle identification, so the minimum vertex degree is 4). □

Fig. 14: A nontraceable LH graph with minimum vertex degree 4.

Finally we show that the difference between the order n of a LH graph G and the length of a longest path in G can be made arbitrarily large.

Theorem 3.15 *For any natural number $k > 0$ there exists a LH graph G such that the difference between the order n of G and the length of a longest path in G is at least k.*

Proof: Consider the nontraceable LH graph G_0 in Figure 13 and let $U_0 = \{u_1, \ldots, u_8\}$ represent the vertices of degree 3 in G_0. Note that U_0 is an independent set of cardinality 8. Hence, since $|V(G_0) - U_0| = 6$, any path in G_0 omits at least one vertex in U_0. Let G_i, $i = 1, \ldots, m$ be m copies of G_0, and denote the vertex in $V(G_i)$ corresponding to u_j in U_0 by $u_{i,j}$ and the vertex in $V(G_i)$ corresponding to v_j in $V(G_0) - U_0$ by $v_{i,j}$. We construct the graph $\triangle G_k$ by combining the graph $\triangle G_{k-1}$, with G_k using triangle identification in the following way. For $k = 2$, combine G_1 with G_2 by means of Construction 3.9 by identifying the vertices $v_{1,2}$, $v_{1,3}$ and $u_{1,6}$ with $v_{2,5}$, $v_{2,4}$ and $u_{2,8}$, respectively, and retaining the labels $v_{1,2}$, $v_{1,3}$, $u_{1,6}$ for the new vertices to form $\triangle G_2$. In general, combine $\triangle G_{k-1}$ with G_k by indentifying $v_{k-1,2}$, $v_{k-1,3}$ and $u_{k-1,6}$ with $v_{k,5}$, $v_{k,4}$ and $u_{k,8}$, respectively, and retaining the labels $v_{k-1,2}$, $v_{k-1,3}$, and $u_{k-1,6}$ to form $\triangle G_k$, for $k = 3, \ldots, m$. Now let $G = \triangle G_m$. Then $n(G) = 11m+3$. Let U be the set of all vertices in G labeled $u_{q,p}$, where q and p are any subscripts found in $V(G)$. Then $|U| = 7m+1$ and $V(G) - U$, with $|V(G) - U| = 4m+2$, contains all the vertices labeled $v_{q,p}$. Since U_0 is an independent set, it follows from our triangle identification procedure, that U is also an independent set of vertices. Hence a longest path in G omits at least $(7m + 1) - (4m + 2) - 1 = 3m - 2$ vertices in U. Now let $G = \triangle G_{\lceil \frac{k+2}{3} \rceil}$, i.e. G is constructed by means of repeated triangle identification of $\lceil \frac{k+2}{3} \rceil$ copies of G_0. Then the difference between the $n(G)$ and the length of a longest path in G is at least k.

□

Note that in the construction used in the proof of Theorem 3.15, the maximum vertex degree of G is 14

regardless of the difference between the order of G and the length of a longest path in G.

4 Conclusion

Some questions that are still unanswered are the following:

- What is the smallest size of a connected nontraceable LT graph with n vertices? The smallest size that we found is $2n - 1$, but connected LT graphs of size $2n - 3$ exist. However, Astratian and Oksimets [2] showed that LT graphs that are not maximal outerplanar (maximal outerplanar graphs are hamiltonian) have size at least $2n - 2$.

- We found connected nontraceable LT graphs with maximum vertex degree of 6 for all orders greater than or equal to 22, but we do not know what the smallest order for such a graph is.

- What is the minimum value of $\Delta(G)$ if G is a connected nontraceable LH graph? We know the number is at least 7, but the we do not know of any nontraceable LH graph with $\Delta(G) < 8$.

- We have shown that there exist connected nontraceable LH graphs with maximum vertex degree no greater than 10 for all orders greater than 13. This bound is probably not sharp. We think it likely that best possible limit is 9.

Acknowledgements

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