Vertex-Coloring Edge-Weighting of Bipartite Graphs with Two Edge Weights[*]

Hongliang Lu[†]

School of Mathematics and Statistics, Xi'an Jiaotong University, Xi'an, PR China

received 28th Feb. 2015, revised 29th Oct. 2015, accepted 9th Dec. 2015.

Let G be a graph and $\mathcal{S}$ be a subset of Z. A vertex-coloring $\mathcal{S}$-edge-weighting of G is an assignment of weights by the elements of $\mathcal{S}$ to each edge of G so that adjacent vertices have different sums of incident edges weights.

It was proved that every 3-connected bipartite graph admits a vertex-coloring $\mathcal{S}$-edge-weighting for $\mathcal{S} = \{1, 2\}$ (H. Lu, Q. Yu and C. Zhang, Vertex-coloring 2-edge-weighting of graphs, European J. Combin., **32** (2011), 22-27). In this paper, we show that every 2-connected and 3-edge-connected bipartite graph admits a vertex-coloring $\mathcal{S}$-edge-weighting for $\mathcal{S} \in \{\{0, 1\}, \{1, 2\}\}$. These bounds we obtain are tight, since there exists a family of infinite bipartite graphs which are 2-connected and do not admit vertex-coloring $\mathcal{S}$-edge-weightings for $\mathcal{S} \in \{\{0, 1\}, \{1, 2\}\}$.

Keywords: edge-weighting, vertex-coloring, 2-connected, bipartite graph

1 Introduction

In this paper, we consider only finite, undirected and simple connected graphs. For a vertex v of graph $G = (V, E)$, $N_G(v)$ denotes the set of vertices which are adjacent to v and $d_G(v) = |N_G(v)|$ is called the *degree* of vertex v. Let $\delta(G)$ and $\triangle(G)$ denote the minimum degree and maximum degree of graph G, respectively. For $v \in V(G)$ and $r \in Z^+$, let $N_G^r(v) = \{u \in N(v) \mid d_G(u) = r\}$. If $v \in V(G)$ and $e \in E(G)$, we use $v \sim e$ to denote that v is an end-vertex of e. For two disjoint subsets S, T of $V(G)$, let $E_G(S, T)$ denote the subset of edges of $E(G)$ with one end in S and other end in T and let $e_G(S, T) = |E_G(S, T)|$. Let $G = (U, W, E)$ denote a bipartite graph with bipartition (U, W) and edge set E.

Let $\mathcal{S}$ be a subset of Z. An $\mathcal{S}$*-edge-weighting* of a graph G is an assignment $w : E(G) \to S$. An $\mathcal{S}$-edge-weighting w of a graph G induces a coloring of the vertices of G, where the color of vertex v, denoted by $c(v)$, is $\sum_{e \sim v} w(e)$. An $\mathcal{S}$-edge-weighting of a graph G is a *vertex-coloring* if for every edge $e = uv$, $c(u) \neq c(v)$ and we say that G admits a *vertex-coloring* $\mathcal{S}$*-edge-weighting*. If $\mathcal{S} = \{1, 2, \ldots, k\}$, then a vertex-coloring $\mathcal{S}$-edge-weighting of a graph G is usually called a *vertex-coloring* k*-edge-weighting*.

For vertex-coloring edge-weighting, Karoński et al. (2004) posed the following conjecture:

[*]This work was supported by the National Natural Science Foundation of China No. 11471257 and the Fundamental Research Funds for the Central Universities.

[†]Corresponding email: luhongliang215@sina.com

Conjecture 1.1 *Every graph without isolated edges admits a vertex-coloring* 3*-edge-weighting.*

This conjecture is still wide open. Karoński et al. (2004) showed that Conjecture 1.1 is true for 3-colorable graphs. Recently, Kalkowski et al. (2010) showed that every graph without isolated edges admits a vertex-coloring 5-edge-weighting. This result is an improvement on the previous bounds on k established by Addario-Berry et al. (2007), Addario-Berry et al. (2008), and Wang and Yu (2008), who obtained the bounds $k = 30, k = 16$, and $k = 13$, respectively.

Many graphs actually admit a vertex-coloring 2-edge-weighting (in fact, experiments suggest (see Addario-Berry et al. (2008)) that almost all graphs admit a vertex-coloring 2-edge-weighting), however it is not known which ones do not. Khatirinejad et al. (2012) explored the problem of classifying those graphs which admit a vertex-coloring 2-edge-weighting. Chang et al. (2011) had made some progress in determining which classes of graphs admit vertex-coloring 2-edge-weighting, and proved that there exists a family of infinite bipartite graphs (e.g., the generalized θ-graphs) which are 2-connected and admit a vertex-coloring 3-edge-weighting but not vertex-coloring 2-edge-weightings. Lu et al. (2011) showed that every 3-connected bipartite graph admits a vertex-coloring 2-edge-weighting.

We write

$$\mathscr{G}_{12} = \{G \mid G \text{ admits a vertex-coloring } \{1,2\}\text{-edge-weighting}\};$$
$$\mathscr{G}_{01} = \{G \mid G \text{ admits a vertex-coloring } \{0,1\}\text{-edge-weighting}\};$$
$$\mathscr{G}^*_{12} = \{G \mid G \text{ is bipartite and admits a vertex-coloring } \{1,2\}\text{-edge-weighting}\};$$
$$\mathscr{G}^*_{01} = \{G \mid G \text{ is bipartite and admits a vertex-coloring } \{0,1\}\text{-edge-weighting}\}.$$

Dudek and Wajc (2011) showed that determining whether a graph belongs to $\mathscr{G}_{12}$ or $\mathscr{G}_{01}$ is NP-complete. Moreover, they showed that $\mathscr{G}_{12} \neq \mathscr{G}_{01}$. The counterexamples constructed by Dudek and Wajc (2011) are non-bipartite.

Now we construct a bipartite graph, which admits a vertex-coloring 2-edge-weighting but not vertex-coloring $\{0,1\}$-edge-weightings. Let C_6 be a cycle of length six and Γ be a graph obtained by connecting an isolated vertex to one of the vertices of C_6. Take two disjoint copies of Γ. Connect two vertices of degree one of the two copies and this gives a connected bipartite graph G. It is easy to prove that G admits a vertex-coloring 2-edge-weighting but not vertex-coloring $\{0,1\}$-edge-weighting. Hence $\mathscr{G}^*_{01} \neq \mathscr{G}^*_{12}$. Next we would like to propose the following problem.

Problem 1 *Determining whether a graph $G \in \mathscr{G}^*_{12}$ or $G \in \mathscr{G}^*_{01}$ is polynomial?*

In this paper, we characterize bipartite graphs which admit a vertex-coloring $\mathcal{S}$-edge-weighting for $\mathcal{S} \in \{\{0,1\},\{1,2\}\}$, and obtain the following result.

Theorem 1.2 *Let G be a 3-edge-connected bipartite graph $G = (U, W, E)$ with minimum degree $\delta(G)$. If G contains a vertex u of degree $\delta(G)$ such that $G - u$ is connected, then G admits a vertex-coloring $\mathcal{S}$-edge-weighting for $\mathcal{S} \in \{\{0,1\},\{1,2\}\}$.*

By Theorem 1.2, it is easy to obtain the following result, which improves and extends the result obtained by Lu et al. (2011).

Theorem 1.3 *Every 2-connected and 3-edge-connected bipartite graph $G = (U, W, E)$ admits a vertex-coloring $\mathcal{S}$-edge-weighting for $\mathcal{S} \in \{\{0,1\},\{1,2\}\}$.*

So far, all known counterexamples of bipartite graphs, which do not have vertex-coloring $\{0,1\}$-edge-weightings or vertex-coloring $\{1,2\}$-edge-weightings are graphs with minimum degree 2. So we would like to propose the following problem.

Problem 2 *Does every bipartite graph with $\delta(G) \geq 3$ admit a vertex-coloring $\mathcal{S}$-edge-weighting, where $\mathcal{S} \in \{\{0,1\},\{1,2\}\}$.*

A *factor* of a graph G is a spanning subgraph. For a graph G, there is a close relationship between 2-edge-weighting and graph factors. Namely, a 2-edge-weighting problem is equivalent to finding special factors of graphs (see Addario-Berry et al. (2007) and Addario-Berry et al. (2008)). So to find factors with pre-specified degree is an important part of edge-weighting.

Let $g, f : V(G) \to Z$ be two integer-valued functions such that $g(v) \leq f(v)$ and $g(v) \equiv f(v) \pmod 2$ for all $v \in V(G)$. A factor F of G is called (g,f)*-parity factor* if $g(v) \leq d_F(v) \leq f(v)$ and $d_F(v) \equiv f(v) \pmod 2$ for all $v \in V(G)$. For $X \subseteq V(G)$, we write $g(X) = \sum_{x \in X} g(x)$ and $f(X)$ is defined similarly. For (g,f)-parity factors, Lovász obtained a sufficient and necessary condition.

Theorem 1.4 (Lovász (1972)) *A graph G contains a (g,f)-parity factor if and only if for any two disjoint subsets S and T of $V(G)$, it follows that*

$$\eta(S,T) = f(S) - g(T) + \sum_{x \in T} d_{G-S}(x) - \tau(S,T) \geq 0,$$

where $\tau(S,T)$ denotes the number of components C, called g-odd components of $G - S - T$ such that $g(V(C)) + e_G(V(C),T) \equiv 1 \pmod 2$.

In the proof of the main theorems, we also need the following three lemmas.

Theorem 1.5 (Chang et al. (2011)) *Every non-trivial connected bipartite graph $G = (A,B,E)$ with $|A|$ even, admits a vertex-coloring 2-edge-weighting w such that $c(u)$ is odd for $u \in A$ and $c(v)$ is even for $v \in B$.*

Theorem 1.6 (Chang et al. (2011)) *Let $r \geq 3$ be an integer. Every r-regular bipartite graph G admits a vertex-coloring 2-edge-weighting.*

Theorem 1.7 (Khatirinejad et al. (2012)) *Every r-regular graph G admits a vertex-coloring 2-edge-weighting if and only if it admits a vertex-coloring $\{0,1\}$-edge-weighting.*

2 Proof of Theorem 1.2

Corollary 2.1 *Every non-trivial connected bipartite graph $G = (A,B,E)$ with $|A|$ even admits a vertex-coloring $\{0,1\}$-edge-weighting.*

Proof: By Theorem 1.5, G admits a vertex-coloring 2-edge-weighting w such that $c(u)$ is odd for $u \in A$ and $c(v)$ is even for $v \in B$. Let $w'(e) = 0$ if $w(e) = 2$ and $w'(e) = 1$ if $w(e) = 1$. Then w' is a vertex-coloring $\{0,1\}$-edge-weighting of graph G. □

For completing the proof of Theorem 1.2, we need the following two technical lemmas.

Lemma 2.2 *Let G be a bipartite graph with bipartition (U, W), where $|U| \equiv |W| \equiv 1 \pmod 2$. Let $\delta(G) = \delta$ and $u \in U$ such that $d_G(u) = \delta$. If one of the following two conditions holds, then G contains a factor F such that $d_F(u) = \delta$, $d_F(x) \equiv \delta + 1 \pmod 2$ for all $x \in U - u$, $d_F(y) \equiv \delta \pmod 2$ for all $y \in W$ and $d_F(y) \leq \delta - 2$ for all $y \in N_G^{\delta}(u)$.*

(i) *$\delta(G) \geq 4$, G is 3-edge-connected and $G - u$ is connected.*

(ii) *$\delta(G) = 3$, G is 3-edge-connected and $|N_G^{\delta}(u)| \leq 2$.*

Proof: Let M be an integer such that $M \geq \triangle(G)$ and $M \equiv \delta \pmod 2$. Let $m \in \{0, -1\}$ such that $m \equiv \delta \pmod 2$. Let $g, f : V(G) \to Z$ such that

$$g(x) = \begin{cases} \delta & \text{if } x = u, \\ m - 1 & \text{if } x \in U - u, \\ m & \text{if } x \in W, \end{cases}$$

and

$$f(x) = \begin{cases} M + 1 & \text{if } x \in U - u, \\ M & \text{if } x \in (W \cup \{u\}) - N_G^{\delta}(u), \\ \delta - 2 & \text{if } x \in N_G^{\delta}(u). \end{cases}$$

By definition, we have $g(v) \equiv f(v) \pmod 2$ for all $v \in V(G)$. It is sufficient for us to show that G contains a (g, f)-parity factor. Indirectly, suppose that G contains no (g, f)-parity factors. By Theorem 1.4, there exist two disjoint subsets S and T such that

$$\eta(S, T) = f(S) - g(T) + \sum_{x \in T} d_{G-S}(x) - \tau(S, T) < 0,$$

where $\tau(S, T)$ denotes the number of g-odd components of $G - S - T$. Since $f(V(G))$ is even, by parity, we have

$$\eta(S, T) = f(S) - g(T) + \sum_{x \in T} d_{G-S}(x) - \tau(S, T) \leq -2. \tag{1}$$

Hence $S \cup T \neq \emptyset$. We choose S and T such that $S \cup T$ is minimal. Let $A = V(G) - S - T$.

Claim 1. $T \subseteq \{u\}$.

Otherwise, let $v \in T - u$ and $T' = T - v$. We have

$$\begin{aligned}
\eta(S,T') &= f(S) - g(T') + \sum_{x \in T'} d_{G-S}(x) - \tau(S,T') \\
&= f(S) - (g(T) - g(v)) + \left(\sum_{x \in T} d_{G-S}(x) - d_{G-S}(v) \right) - \tau(S,T') \\
&\le f(S) - g(T) + \left(\sum_{x \in T} d_{G-S}(x) - d_{G-S}(v) \right) - (\tau(S,T) - e_G(v,A)) + g(v) \\
&\le f(S) - g(T) + \sum_{x \in T} d_{G-S}(x) - \tau(S,T) + (g(v) - d_{G-S}(v) + e_G(v,A)) \\
&= \eta(S,T) + (g(v) - d_{G-S}(v) + e_G(v,A)) \\
&\le \eta(S,T) - (d_{G-S}(v) - e_G(v,A)) \\
&\le \eta(S,T) \le -2,
\end{aligned}$$

contradicting the choice of S and T.

Claim 2. $S \subseteq N_G^{\delta}(u)$.

Otherwise, suppose that $S - N_G^{\delta}(u) \neq \emptyset$ and let $v \in S - N_G^{\delta}(u)$. Let $S' = S - v$. We have

$$\begin{aligned}
\eta(S',T) &= f(S') - g(T) + \sum_{x \in T} d_{G-S'}(x) - \tau(S',T) \\
&= (f(S) - f(v)) - g(T) + \left(\sum_{x \in T} d_{G-S}(x) + e_G(v,T) \right) - \tau(S',T) \\
&\le f(S) - g(T) + \sum_{x \in T} d_{G-S}(x) - (\tau(S,T) - e_G(v,A)) - f(v) + e_G(v,T) \\
&= f(S) - g(T) + \sum_{x \in T} d_{G-S}(x) - \tau(S,T) + (e_G(v,T) + e_G(v,A) - f(v)) \\
&\le \eta(S,T) + (d_G(v) - f(v)) \\
&\le \eta(S,T) \le -2,
\end{aligned}$$

contradicting the choice of S and T again.

We write $\tau(S,T) = \tau$. By Claims 1 and 2, we have

$$\begin{aligned}
\eta(S,T) &= f(S) - g(T) + \sum_{x \in T} d_{G-S}(x) - \tau \\
&= (\delta - 2)|S| - \delta|T| + |T|(\delta - |S|) - \tau \quad \text{(by Claims 1 and 2)} \\
&= (\delta - 2 - |T|)|S| - \tau \quad \text{(by Claim 1)} \\
&\le -2,
\end{aligned}$$

i.e.,

$$(\delta-2-|T|)|S|+2\le\tau, \tag{2}$$

which implies $\tau\ge 2$ since $|T|\le 1$.

Since $G-u$ is connected, we may see that $S\ne\emptyset$. Note that G is 3-edge-connected, by Claims 1 and 2, we have

$$\begin{aligned}3\tau &\le e_G(A,S\cup T)\\ &= e_G(A,S)+e_G(A,T)\\ &\le (\delta-|T|)|S|+|T|(\delta-|S|)\quad\text{(by Claims 1 and 2)}\\ &= (\delta-2|T|)|S|+|T|\delta,\end{aligned}$$

i.e.,

$$3\tau\le(\delta-2|T|)|S|+|T|\delta. \tag{3}$$

Combining (2) and (3), we may see that

$$\delta|T|\ge(2\delta-|T|-6)|S|+6. \tag{4}$$

If $\delta\ge 4$, then we have

$$\begin{aligned}\delta &\ge \delta|T|\quad\text{(since } |T|\le 1)\\ &\ge (2\delta-|T|-6)|S|+6\quad\text{(since } |S|\ge 1)\\ &\ge 2\delta-|T|\\ &\ge 2\delta-1,\end{aligned}$$

a contradiction. So we may assume that $\delta=3$. Note that $|S|\le|N_G^\delta(u)|\le 2$. By (4), we have

$$3=\delta\ge\delta|T|\ge-|T||S|+6\ge 4, \tag{5}$$

a contradiction again.

This completes the proof. □

Lemma 2.3 *Let G be a bipartite graph with bipartition (U,W), where $|U|\equiv|W|\equiv 1\pmod 2$. Let $\delta(G)=\delta$ and $u\in U$ such that $d_G(u)=\delta$. If one of the following two conditions holds, then G contains a factor F such that $d_F(u)=0$, $d_F(x)\equiv 1\pmod 2$ for all $x\in U-u$, $d_F(y)\equiv 0\pmod 2$ for all $x\in W$ and $d_F(y)\ge 2$ for all $y\in N_G(u)$.*
(i) $\delta(G)\ge 4$, G is 3-edge-connected and $G-u$ is connected.
(ii) $\delta(G)=3$, G is 3-edge-connected and there exists a vertex $v\in N_G(u)$ such that $d_G(v)>3$.

Proof: Let M be an even integer such that $M\ge\triangle(G)$. Let $g,f:V(G)\to Z$ such that

$$g(x)=\begin{cases}0 & \text{if } x\in(\{u\}\cup W)-N_G(u),\\ 2 & \text{if } x\in N_G(u),\\ -1 & \text{if } x\in U-u,\end{cases}$$

and

$$f(x) = \begin{cases} M & \text{if } x \in W \\ 0 & \text{if } x = u, \\ M+1 & \text{if } x \in U - u. \end{cases}$$

Clearly, $g(v) \equiv f(v) \pmod 2$ for all $v \in V(G)$ and $g(V(G))$ is even. It is also sufficient for us to show that G contains a (g, f)-parity factor.

Indirectly, suppose that G contains no (g, f)-parity factors. By Theorem 1.4, there exist two disjoint subsets S and T such that

$$\eta(S,T) = f(S) - g(T) + \sum_{x \in T} d_{G-S}(x) - \tau(S,T) \leq -2, \tag{6}$$

where $\tau(S,T)$ denotes the number of g-odd components of $G - S - T$. We choose S and T such that $S \cup T$ is minimal. Let $A = V(G) - S - T$.

Claim 1. $S \subseteq \{u\}$.

Otherwise, suppose that there exists a vertex $v \in S - u$. Let $S' = S - v$. Then we have

$$\begin{aligned} \eta(S',T) &= f(S') - g(T) + \sum_{y \in T} d_{G-S'}(y) - \tau(S',T) \\ &= (f(S) - f(v)) - g(T) + \left(\sum_{y \in T} d_{G-S}(y) + e_G(v,T) \right) - \tau(S',T) \\ &\leq f(S) - g(T) + \sum_{y \in T} d_{G-S}(y) - f(v) + e_G(v,T) - (\tau(S,T) - e_G(v,A)) \\ &= \eta(S,T) - (f(v) - e_G(v,T) - e_G(v,A)) \\ &\leq \eta(S,T) - (f(v) - d_G(v)) \\ &\leq \eta(S,T) \leq -2, \end{aligned}$$

contradicting the the choice of $S \cup T$.

Claim 2. $T \subseteq N_G(u)$.

Otherwise, suppose that $T - N_G(u) \neq \emptyset$. Let $x \in T - N_G(u)$ and let $T' = T - x$. Then we have

$$\begin{aligned}
\eta(S,T') &= f(S) - g(T') + \sum_{y\in T'} d_{G-S}(y) - \tau(S,T') \\
&= f(S) - (g(T) - g(x)) + \left(\sum_{y\in T} d_{G-S}(y) - d_{G-S}(x)\right) - \tau(S,T') \\
&\le f(S) - (g(T) - g(x)) + \left(\sum_{y\in T} d_{G-S}(y) - d_{G-S}(x)\right) - (\tau(S,T) - e_G(x,A)) \\
&= f(S) - g(T) + \sum_{y\in T} d_{G-S}(y) - \tau(S,T) - (d_{G-S}(x) - e_G(x,A)) + g(x) \\
&= \eta(S,T) - (d_{G-S}(x) - e_G(x,A)) + g(x) \\
&\le \eta(S,T) \le -2,
\end{aligned}$$

contradicting the choice of $S \cup T$.

By Claims 1 and 2, we may see that $f(S) = 0$ and $g(T) = 2|T|$. For simplicity, we write $\tau(S,T) = \tau$. By (6), we see that

$$\tau \ge \sum_{x\in T}(d_G(x) - |S|) - 2|T| + 2, \tag{7}$$

which implies

$$\tau \ge \sum_{x\in T}(\delta - 1) - 2|T| + 2 \ge 2. \tag{8}$$

Note that $G - u$ is connected, so we have $|T| \ge 1$. Since G is 3-edge-connected, we have

$$\begin{aligned}
3\tau &\le \sum_{x\in T}(d_G(x) - |S|) + (\delta - |T|)|S| \\
&= \sum_{x\in T} d_G(x) + (\delta - 2|T|)|S|,
\end{aligned}$$

i.e.,

$$3\tau \le \sum_{x\in T} d_G(x) + (\delta - 2|T|)|S|. \tag{9}$$

Inequalities (7) and (9) implies

$$\begin{aligned}
2\sum_{x\in T} d_G(x) + 6 &\le |S||T| + 6|T| + \delta|S| \quad \text{(since } |S| \le 1 \text{ and } |T| \ge 1) \\
&\le 7|T| + \delta,
\end{aligned}$$

i.e.,

$$7|T| \geq 2\sum_{x\in T} d_G(x) + 6 - \delta. \tag{10}$$

If $\delta \geq 4$, by (10), it follows

$$7|T| \geq 6 + \delta(2|T| - 1) \geq 8|T| + 2,$$

a contradiction. So we may assume that $\delta = 3$. By condition (ii), $\sum_{x\in T} d_G(x) \geq 3|T| + 1$. Combining (10),

$$\begin{aligned}7|T| &\geq 2\sum_{x\in T} d_G(x) + 6 - \delta \\ &\geq 2(3|T| + 1) + 3,\end{aligned}$$

which implies $|T| \geq 5$, a contradiction since $|T| \leq |N_G(u)| \leq 3$.

This completes the proof. □

Proof of Theorem 1.2: By Theorem 1.5 and Corollary 2.1, we can assume that both $|A|$ and $|B|$ are odd.

Firstly, we consider $\mathcal{S} = \{0, 1\}$. If G is 3-regular, by Theorem 1.6, then G admits a vertex-coloring 2-edge-weighting. By Theorem 1.7, G also admits a vertex-coloring $\{0, 1\}$-edge-weighting. So we can assume that $\delta(G) \geq 3$ and G is not 3-regular. If $\delta(G) = 3$, since G is 3-edge-connected, then $G - x$ is connected for every vertex x of G with degree three. Hence there exists a vertex v with degree three such that $N_G(v)$ contains a vertex with degree at least four. Let

$$u^* = \begin{cases} u & \text{if } \delta \geq 4, \\ v & \text{if } \delta = 3. \end{cases}$$

Without loss generality, we may assume that $u^* \in U$ and so it is a vertex satisfying the conditions of Lemma 2.3. Hence by Lemma 2.3, G contains a factor F, which satisfies the following three conditions.

(i) $d_F(u^*) = 0$;

(ii) $d_F(x) \equiv 1 \pmod 2$ for all $x \in U - u^*$;

(iii) $d_F(y) \equiv 0 \pmod 2$ for all $x \in W$ and $d_F(y) \geq 2$ for all $y \in N_G(u^*)$.

Clearly, $d_F(x) \neq d_F(y)$ for all $xy \in E(G)$. We assign weight 1 for each edge of $E(F)$ and weight 0 for each edge of $E(G) - E(F)$. Then we obtain a vertex-coloring $\{0, 1\}$-edge-weighting of graph G.

Secondly, we show that G admits a vertex-coloring 2-edge-weighting. By Theorem 1.6, we may assume that G is not 3-regular. If $\delta = 3$, since G is 3-edge-connected, then G contains a vertex v' such that $d_G(v') = 3$, $G - v'$ is connected and $|N_G^{\delta}(v')| \leq 2$. Let

$$u^* = \begin{cases} u & \text{if } \delta \geq 4, \\ v' & \text{if } \delta = 3. \end{cases}$$

Then u^* is a vertex satisfying the conditions of Lemma 2.2. Hence by Lemma 2.2, G contains a factor F such that

(i) $d_F(u^*) = \delta$;
(ii) $d_F(x) \equiv \delta \pmod 2$ for all $x \in W$ and $d_F(x) \le \delta - 2$ for all $x \in N_G^{\delta}(u^*)$;
(iii) $d_F(y) \equiv \delta + 1 \pmod 2$ for all $y \in U - u^*$.

Let $w : E(G) \to \{1, 2\}$ be a 2-edge-weighting such that $w(e) = 1$ for each $e \in E(F)$ and $w(e') = 2$ for each $e' \in E(G) - E(F)$. Clearly, $c(u^*) = \delta$. If $y \in N_G^{\delta}(u^*)$, since there exists an edge $e \sim y$ such that $e \notin E(F)$, then $c(y) = \sum_{e \sim y} w(e) > \delta$. If $y \in N_G(u^*) - N_G^{\delta}(u^*)$, then $c(y) \ge d_G(y) > \delta$. Hence $c(y) \ne c(u^*)$ for all $y \in N_G(u^*)$. For each $xy \in E(G)$, where $x \in U - u^*$ and $y \in W$, by the choice of F, we have $c(x) \equiv \delta + 1 \pmod 2$ and $c(y) \equiv \delta \pmod 2$. Hence w is a vertex-coloring $\{1, 2\}$-edge-weighting of the graph G.

This completes the proof. □

Corollary 2.4 *Let G be a 3-edge-connected bipartite graph. If $3 \le \delta(G) \le 5$, then G admits a vertex-coloring $\mathcal{S}$-edge-weighting for $\mathcal{S} \in \{\{0, 1\}, \{1, 2\}\}$.*

Proof: Since $3 \le \delta \le 5$ and G is 3-edge-connected, then for every vertex v of degree δ, $G - v$ is connected. By Lemma 2.2 and Theorem 1.2, with the same proof, G admits a vertex-coloring $\mathcal{S}$-edge-weighting for $\mathcal{S} \in \{\{0, 1\}, \{1, 2\}\}$. □

3 Conclusions

In this paper, we prove that every 2-connected and 3-edge-connected bipartite graph admits a vertex-coloring $\mathcal{S}$-edge-weighting for $\mathcal{S} \in \{\{0, 1\}, \{1, 2\}\}$. The generalized θ-graphs is 2-connected and has a vertex-coloring 3-edge-weighting but not vertex-coloring $\{0, 1\}$-edge-weighting or vertex-coloring 2-edge-weighting. So it is an interesting problem to classify all 2-connected bipartite graphs admitting a vertex-coloring $\mathcal{S}$-edge-weighting. Since the parity-factor problem is polynomial, then there exists a polynomial algorithm to find a vertex-coloring $\mathcal{S}$-edge-weighting of bipartite graphs satisfying the conditions of Theorem 1.2.

Acknowledgements

The authors would like to thank the anonymous Reviewer for all valuable comments and suggestions to improve the quality of our paper.

References

L. Addario-Berry, R. E. L. Aldred, K. Dalal, and B. A. Reed. Vertex coloring edge partitions. *J. Combinatorial Theory Ser. B*, 94:237–244, 2005.

L. Addario-Berry, K. Dalal, C. McDiarmid, B. A. Reed, and A. Thomason. Vertex-coloring edge-weightings. *Combintorica*, 27:1–12, 2007.

L. Addario-Berry, K. Dalal, and B. A. Reed. Degree constrained subgraphs. *Discrete Applied Math.*, 156: 1168–1174, 2008.

J. Akiyama and M. Kano. *Factors and Factorizations of Graphs — Proof Techniques in Factor Theory*. Springer-Verlag Berlin Heidelberg, 2011.

G. Chang, C. Lu, J. Wu, and Q. Yu. Vertex coloring 2-edge weighting of bipartite graphs. *Taiwanese J. Math.*, 15:1807–1813, 2011.

A. Dudek and D. Wajc. On the complexity of vertex-coloring edgeweightings. *Discrete Math. Theor. Comput. Sci.*, 13:45–50, 2011.

M. Kalkowski, M. Karoński, and F. Pfender. Vertex-coloring edge-weightings: towards the 1-2-3-conjecture. *J. Combin. Theory Ser. B*, 100:347–349, 2010.

M. Karoński, T. Łuczak, and A. Thomason. Edge weights and vertex colors. *J. Combin. Theory Ser. B*, 91:151–157, 2004.

M. Khatirinejad, R. Naserasr, M. Newman, B. Seamone, and B. Stevens. Vertex-colouring edge-weightings with two edge weights. *J. Combin. Theory Ser. B*, 14:1–20, 2012.

L. Lovász. The factorization of graphs II. *Acta Math. Acad. Sci. Hungar.*, 23:223–246, 1972.

H. Lu, Q. Yu, and C. Zhang. Vertex-coloring 2-edge-weighting of graphs. *European J. Combin.*, 32: 22–27, 2011.

T. Wang and Q. Yu. A note on vertex-coloring 13-edge-weighting. *Frontier Math. in China*, 3:1–7, 2008.