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HAL Id: hal-01352856
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Submitted on 16 Aug 2016

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Vertex-Coloring Edge-Weighting of Bipartite Graphs with Two Edge Weights

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received 28th Feb. 2015, revised 29th Oct. 2015, accepted 9th Dec. 2015.

Let $G$ be a graph and $S$ be a subset of $\mathbb{Z}$. A vertex-coloring $S$-edge-weighting of $G$ is an assignment of weights by the elements of $S$ to each edge of $G$ so that adjacent vertices have different sums of incident edges weights.

It was proved that every 3-connected bipartite graph admits a vertex-coloring $S$-edge-weighting for $S = \{1, 2\}$ (H. Lu, Q. Yu and C. Zhang, Vertex-coloring 2-edge-weighting of graphs, European J. Combin., 32 (2011), 22-27). In this paper, we show that every 2-connected and 3-edge-connected bipartite graph admits a vertex-coloring $S$-edge-weighting for $S \in \{\{0, 1\}, \{1, 2\}\}$. These bounds we obtain are tight, since there exists a family of infinite bipartite graphs which are 2-connected and do not admit vertex-coloring $S$-edge-weightings for $S \in \{\{0, 1\}, \{1, 2\}\}$.

Keywords: edge-weighting, vertex-coloring, 2-connected, bipartite graph

1 Introduction

In this paper, we consider only finite, undirected and simple connected graphs. For a vertex $v$ of graph $G = (V, E)$, $N_G(v)$ denotes the set of vertices which are adjacent to $v$ and $d_G(v) = |N_G(v)|$ is called the degree of vertex $v$. Let $\delta(G)$ and $\Delta(G)$ denote the minimum degree and maximum degree of graph $G$, respectively. For $v \in V(G)$ and $r \in \mathbb{Z}^+$, let $N^r_G(v) = \{u \in N(v) \mid d_G(u) = r\}$. If $v \in V(G)$ and $e \in E(G)$, we use $v \sim e$ to denote that $v$ is an end-vertex of $e$. For two disjoint subsets $S, T$ of $V(G)$, let $E_G(S, T)$ denote the subset of edges of $E(G)$ with one end in $S$ and other end in $T$ and let $e_G(S, T) = |E_G(S, T)|$. Let $G = (U, W, E)$ denote a bipartite graph with bipartition $(U, W)$ and edge set $E$.

Let $S$ be a subset of $\mathbb{Z}$. An $S$-edge-weighting of a graph $G$ is an assignment $w : E(G) \rightarrow S$. An $S$-edge-weighting $w$ of a graph $G$ induces a coloring of the vertices of $G$, where the color of vertex $v$, denoted by $c(v)$, is $\sum_{e \sim v} w(e)$. An $S$-edge-weighting of a graph $G$ is a vertex-coloring if for every edge $e = uv$, $c(u) \neq c(v)$ and we say that $G$ admits a vertex-coloring $S$-edge-weighting. If $S = \{1, 2, \ldots, k\}$, then a vertex-coloring $S$-edge-weighting of a graph $G$ is usually called a vertex-coloring $k$-edge-weighting.

For vertex-coloring edge-weighting, Karoński et al. (2004) posed the following conjecture:

*This work was supported by the National Natural Science Foundation of China No. 11471257 and the Fundamental Research Funds for the Central Universities.
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Conjecture 1.1 Every graph without isolated edges admits a vertex-coloring 3-edge-weighting.

This conjecture is still wide open. [Karoński et al., 2004] showed that Conjecture 1.1 is true for 3-colorable graphs. Recently, [Kalkowski et al., 2010] showed that every graph without isolated edges admits a vertex-coloring 5-edge-weighting. This result is an improvement on the previous bounds on \( k \) established by [Addario-Berry et al., 2007], [Addario-Berry et al., 2008], and [Wang and Yu, 2008], who obtained the bounds \( k = 30, k = 16, \) and \( k = 13 \), respectively.

Many graphs actually admit a vertex-coloring 2-edge-weighting (in fact, experiments suggest (see [Addario-Berry et al., 2008]) that almost all graphs admit a vertex-coloring 2-edge-weighting), however it is not known which ones do not. [Khatirinejad et al., 2012] explored the problem of classifying those graphs which admit a vertex-coloring 2-edge-weighting. [Chang et al., 2011] had made some progress in determining which classes of graphs admit vertex-coloring 2-edge-weighting, and proved that there exists a family of infinite bipartite graphs (e.g., the generalized \( \Theta \)-graphs) which are 2-connected and admit a vertex-coloring 3-edge-weighting but not vertex-coloring 2-edge-weightings. [Lu et al., 2011] showed that every 3-connected bipartite graph admits a vertex-coloring 2-edge-weighting.

We write

\[ G_{12} = \{ G \mid G \text{ admits a vertex-coloring } \{1, 2\}\text{-edge-weighting} \}; \]
\[ G_{01} = \{ G \mid G \text{ admits a vertex-coloring } \{0, 1\}\text{-edge-weighting} \}; \]
\[ G^*_1 = \{ G \mid G \text{ is bipartite and admits a vertex-coloring } \{1, 2\}\text{-edge-weighting} \}; \]
\[ G^*_0 = \{ G \mid G \text{ is bipartite and admits a vertex-coloring } \{0, 1\}\text{-edge-weighting} \}. \]

[Dudek and Wajc, 2011] showed that determining whether a graph belongs to \( G_{12} \) or \( G_{01} \) is NP-complete. Moreover, they showed that \( G_{12} \neq G_{01} \). The counterexamples constructed by [Dudek and Wajc, 2011] are non-bipartite.

Now we construct a bipartite graph, which admits a vertex-coloring 2-edge-weighting but not vertex-coloring \( \{0, 1\}\)-edge-weightings. Let \( C_6 \) be a cycle of length six and \( \Gamma \) be a graph obtained by connecting an isolated vertex to one of the vertices of \( C_6 \). Take two disjoint copies of \( \Gamma \). Connect two vertices of degree one of the two copies and this gives a connected bipartite graph \( G \). It is easy to prove that \( G \) admits a vertex-coloring 2-edge-weighting but not vertex-coloring \( \{0, 1\}\)-edge-weighting. Hence \( G^*_0 \neq G^*_1 \). Next we would like to propose the following problem.

Problem 1 Determining whether a graph \( G \in G^*_1 \) or \( G \in G^*_0 \) is polynomial?

In this paper, we characterize bipartite graphs which admit a vertex-coloring \( S \)-edge-weighting for \( S \in \{\{0, 1\}, \{1, 2\}\} \), and obtain the following result.

Theorem 1.2 Let \( G \) be a 3-edge-connected bipartite graph \( G = (U, W, E) \) with minimum degree \( \delta(G) \). If \( G \) contains a vertex \( u \) of degree \( \delta(G) \) such that \( G - u \) is connected, then \( G \) admits a vertex-coloring \( S \)-edge-weighting for \( S \in \{\{0, 1\}, \{1, 2\}\} \).

By Theorem 1.2, it is easy to obtain the following result, which improves and extends the result obtained by [Lu et al., 2011].

Theorem 1.3 Every 2-connected and 3-edge-connected bipartite graph \( G = (U, W, E) \) admits a vertex-coloring \( S \)-edge-weighting for \( S \in \{\{0, 1\}, \{1, 2\}\} \).
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So far, all known counterexamples of bipartite graphs, which do not have vertex-coloring $\{0, 1\}$-edge-weightings or vertex-coloring $\{1, 2\}$-edge-weightings are graphs with minimum degree 2. So we would like to propose the following problem.

**Problem 2** Does every bipartite graph with $\delta(G) \geq 3$ admit a vertex-coloring $S$-edge-weighting, where $S \in \{\{0, 1\}, \{1, 2\}\}$.

A factor of a graph $G$ is a spanning subgraph. For a graph $G$, there is a close relationship between 2-edge-weighting and graph factors. Namely, a 2-edge-weighting problem is equivalent to finding special factors with 2-edge-weighting and graph factors. Namely, a 2-edge-weighting problem is equivalent to finding special factors with pre-specified degree is an important part of edge-weighting.

Let $g, f : V(G) \to \mathbb{Z}$ be two integer-valued functions such that $g(v) \leq f(v)$ and $g(v) \equiv f(v)$ (mod 2) for all $v \in V(G)$. A factor $F$ of $G$ is called $(g, f)$-parity factor if $g(v) \leq d_F(v) \leq f(v)$ and $d_F(v) \equiv f(v)$ (mod 2) for all $v \in V(G)$. For $X \subseteq V(G)$, we write $g(X) = \sum_{x \in X} g(x)$ and $f(X)$ is defined similarly. For $(g, f)$-parity factors, Lovász obtained a sufficient and necessary condition.

**Theorem 1.4 (Lovász [1972])** A graph $G$ contains a $(g, f)$-parity factor if and only if for any two disjoint subsets $S$ and $T$ of $V(G)$, it follows that

$$\eta(S, T) = f(S) - g(T) + \sum_{x \in T} d_{G-S}(x) - \tau(S, T) \geq 0,$$

where $\tau(S, T)$ denotes the number of components $C$, called $g$-odd components of $G - S - T$ such that $g(V(C)) + c_G(V(C), T) \equiv 1 \pmod{2}$.

In the proof of the main theorems, we also need the following three lemmas.

**Theorem 1.5 (Chang et al. [2011])** Every non-trivial connected bipartite graph $G = (A, B, E)$ with $|A|$ even, admits a vertex-coloring 2-edge-weighting $w$ such that $c(u)$ is odd for $u \in A$ and $c(v)$ is even for $v \in B$.

**Theorem 1.6 (Chang et al. [2011])** Let $r \geq 3$ be an integer. Every $r$-regular bipartite graph $G$ admits a vertex-coloring 2-edge-weighting.

**Theorem 1.7 (Khatirinejad et al. [2012])** Every $r$-regular graph $G$ admits a vertex-coloring 2-edge-weighting if and only if it admits a vertex-coloring $\{0, 1\}$-edge-weighting.

## 2 Proof of Theorem 1.2

**Corollary 2.1** Every non-trivial connected bipartite graph $G = (A, B, E)$ with $|A|$ even admits a vertex-coloring $\{0, 1\}$-edge-weighting.

**Proof:** By Theorem 1.5 $G$ admits a vertex-coloring 2-edge-weighting $w$ such that $c(u)$ is odd for $u \in A$ and $c(v)$ is even for $v \in B$. Let $w'(e) = 0$ if $w(e) = 2$ and $w'(e) = 1$ if $w(e) = 1$. Then $w'$ is a vertex-coloring $\{0, 1\}$-edge-weighting of graph $G$. \qed

For completing the proof of Theorem 1.2 we need the following two technical lemmas.
Lemma 2.2 Let $G$ be a bipartite graph with bipartition $(U, W)$, where $|U| \equiv |W| \equiv 1 \pmod{2}$. Let $\delta(G) = \delta$ and $u \in U$ such that $d_G(u) = \delta$. If one of the following two conditions holds, then $G$ contains a factor $F$ such that $d_F(u) = \delta$, $d_F(x) \equiv \delta + 1 \pmod{2}$ for all $x \in U - u$, $d_F(y) \equiv \delta \pmod{2}$ for all $y \in W$ and $d_F(y) \leq \delta - 2$ for all $y \in N^\delta_G(u)$.

(i) $\delta(G) \geq 4$, $G$ is 3-edge-connected and $G - u$ is connected.

(ii) $\delta(G) = 3$, $G$ is 3-edge-connected and $|N^\delta_G(u)| \leq 2$.

Proof: Let $M$ be an integer such that $M \geq \Delta(G)$ and $M \equiv \delta \pmod{2}$. Let $m \in \{0, -1\}$ such that $m \equiv \delta \pmod{2}$.

$$g(x) = \begin{cases} \delta & \text{if } x = u, \\ m - 1 & \text{if } x \in U - u, \\ m & \text{if } x \in W, \end{cases}$$

and

$$f(x) = \begin{cases} M + 1 & \text{if } x \in U - u, \\ M & \text{if } x \in (W \cup \{u\}) - N^\delta_G(u), \\ \delta - 2 & \text{if } x \in N^\delta_G(u). \end{cases}$$

By definition, we have $g(v) \equiv f(v) \pmod{2}$ for all $v \in V(G)$. It is sufficient for us to show that $G$ contains a $(g, f)$-parity factor. Indirectly, suppose that $G$ contains no $(g, f)$-parity factors. By Theorem [1.4] there exist two disjoint subsets $S$ and $T$ such that

$$\eta(S, T) = f(S) - g(T) + \sum_{x \in T} d_{G-S}(x) - \tau(S, T) < 0,$$

where $\tau(S, T)$ denotes the number of $g$-odd components of $G - S - T$. Since $f(V(G))$ is even, by parity, we have

$$\eta(S, T) = f(S) - g(T) + \sum_{x \in T} d_{G-S}(x) - \tau(S, T) \leq -2. \quad (1)$$

Hence $S \cup T \neq \emptyset$. We choose $S$ and $T$ such that $S \cup T$ is minimal. Let $A = V(G) - S - T$.

Claim 1. $T \subseteq \{u\}$. 

Otherwise, let \( v \in T - u \) and \( T' = T - v \). We have
\[
\eta(S, T') = f(S) - g(T') + \sum_{x \in T'} d_{G - S}(x) - \tau(S, T')
\]
\[
= f(S) - (g(T) - g(v)) + \left( \sum_{x \in T} d_{G - S}(x) - d_{G - S}(v) \right) - \tau(S, T')
\]
\[
\leq f(S) - g(T) + \left( \sum_{x \in T} d_{G - S}(x) - d_{G - S}(v) \right) - \tau(S, T) + (g(v) - d_{G - S}(v) + e_G(v, A))
\]
\[
\leq f(S) - g(T) + \sum_{x \in T} d_{G - S}(x) - \tau(S, T) + (g(v) - d_{G - S}(v) + e_G(v, A))
\]
\[
= \eta(S, T) + (g(v) - d_{G - S}(v) + e_G(v, A))
\]
\[
\leq \eta(S, T) - e_G(v, A)
\]
\[
\leq \eta(S, T) \leq -2,
\]
contradicting the choice of \( S \) and \( T \).

**Claim 2.** \( S \subseteq N_G^d(u) \).

Otherwise, suppose that \( S - N_G^d(u) \neq \emptyset \) and let \( v \in S - N_G^d(u) \). Let \( S' = S - v \). We have
\[
\eta(S', T) = f(S') - g(T) + \sum_{x \in T} d_{G - S'}(x) - \tau(S', T)
\]
\[
= (f(S) - f(v)) - g(T) + \left( \sum_{x \in T} d_{G - S}(x) + e_G(v, T) \right) - \tau(S', T)
\]
\[
\leq f(S) - g(T) + \sum_{x \in T} d_{G - S}(x) - \tau(S, T) - e_G(v, A) - f(v) + e_G(v, T)
\]
\[
= f(S) - g(T) + \sum_{x \in T} d_{G - S}(x) - \tau(S, T) + (e_G(v, T) + e_G(v, A) - f(v))
\]
\[
\leq \eta(S, T) + (d_G(v) - f(v))
\]
\[
\leq \eta(S, T) \leq -2,
\]
contradicting the choice of \( S \) and \( T \) again.

We write \( \tau(S, T) = \tau \). By Claims 1 and 2, we have
\[
\eta(S, T) = f(S) - g(T) + \sum_{x \in T} d_{G - S}(x) - \tau
\]
\[
= (\delta - 2)|S| - \delta|T| + |T|(|\delta - |S||) - \tau \quad \text{(by Claims 1 and 2)}
\]
\[
= (\delta - 2 - |T||S| - \tau \quad \text{(by Claim 1)}
\]
\[
\leq -2,
\]
i.e.,

\[(\delta - 2 - |T||S| + 2 \leq \tau, \quad (2)\]

which implies \(\tau \geq 2\) since \(|T| \leq 1\).

Since \(G - u\) is connected, we may see that \(S \neq \emptyset\). Note that \(G\) is 3-edge-connected, by Claims 1 and 2, we have

\[3\tau \leq e_G(A, S \cup T)\]
\[= e_G(A, S) + e_G(A, T)\]
\[\leq (\delta - |T||S| + |T|(\delta - |S|) \quad \text{(by Claims 1 and 2)}\]
\[= (\delta - 2|T||S| + |T||S|\]
\[\text{i.e.,} \quad 3\tau \leq (\delta - 2|T||S| + |T||S|\]
\[\text{(3)}\]

Combining (2) and (3), we may see that

\[\delta|T| \geq (2\delta - |T| - 6)|S| + 6. \quad (4)\]

If \(\delta \geq 4\), then we have

\[\delta \geq \delta|T| \quad \text{(since } |T| \leq 1)\]
\[\geq (2\delta - |T| - 6)|S| + 6 \quad \text{(since } |S| \geq 1)\]
\[\geq 2\delta - |T|\]
\[\geq 2\delta - 1,\]

a contradiction. So we may assume that \(\delta = 3\). Note that \(|S| \leq |N_G^\delta(u)| \leq 2\). By (4), we have

\[3 = \delta \geq \delta|T| \geq -|T||S| + 6 \geq 4, \quad (5)\]

a contradiction again.

This completes the proof.  \(\Box\)

**Lemma 2.3** Let \(G\) be a bipartite graph with bipartition \((U, W)\), where \(|U| \equiv |W| \equiv 1 \pmod 2\). Let \(\delta(G) = \delta\) and \(u \in U\) such that \(d_G(u) = \delta\). If one of the following two conditions holds, then \(G\) contains a factor \(F\) such that \(d_F(u) = 0\), \(d_F(x) \equiv 1 \pmod 2\) for all \(x \in U - u\), \(d_F(y) \equiv 0 \pmod 2\) for all \(x \in W\) and \(d_F(y) \geq 2\) for all \(y \in N_G(u)\).

(i) \(\delta(G) \geq 4\), \(G\) is 3-edge-connected and \(G - u\) is connected.

(ii) \(\delta(G) = 3\), \(G\) is 3-edge-connected and there exists a vertex \(v \in N_G(u)\) such that \(d_G(v) > 3\).

**Proof:** Let \(M\) be an even integer such that \(M \geq \Delta(G)\). Let \(g, f : V(G) \to \mathbb{Z}\) such that

\[g(x) = \begin{cases} 
  0 & \text{if } x \in (\{u\} \cup W) - N_G(u), \\
  2 & \text{if } x \in N_G(u), \\
  -1 & \text{if } x \in U - u,
\end{cases}\]
and

\[ f(x) = \begin{cases} 
M & \text{if } x \in W \\
0 & \text{if } x = u, \\
M + 1 & \text{if } x \in U - u.
\end{cases} \]

Clearly, \( g(v) \equiv f(v) \pmod{2} \) for all \( v \in V(G) \) and \( g(V(G)) \) is even. It is also sufficient for us to show that \( G \) contains a \( (g, f) \)-parity factor.

Indirectly, suppose that \( G \) contains no \( (g, f) \)-parity factors. By Theorem 1.4 there exist two disjoint subsets \( S \) and \( T \) such that

\[ \eta(S, T) = f(S) - g(T) + \sum_{x \in T} d_{G-S}(x) - \tau(S, T) \leq -2, \tag{6} \]

where \( \tau(S, T) \) denotes the number of \( g \)-odd components of \( G - S - T \). We choose \( S \) and \( T \) such that \( S \cup T \) is minimal. Let \( A = V(G) - S - T \).

**Claim 1.** \( S \subseteq \{u\} \).

Otherwise, suppose that there exists a vertex \( v \in S - u \). Let \( S' = S - v \). Then we have

\[
\eta(S', T) = f(S') - g(T) + \sum_{y \in T} d_{G-S'}(y) - \tau(S', T) \\
= (f(S) - f(v)) - g(T) + \left( \sum_{y \in T} d_{G-S}(y) + e_G(v, T) \right) - \tau(S', T) \\
\leq f(S) - g(T) + \sum_{y \in T} d_{G-S}(y) - f(v) + e_G(v, T) - (\tau(S, T) - e_G(v, A)) \\
= \eta(S, T) - (f(v) - e_G(v, T) - e_G(v, A)) \\
\leq \eta(S, T) - (f(v) - d_G(v)) \\
\leq \eta(S, T) \leq -2,
\]

contradicting the the choice of \( S \cup T \).

**Claim 2.** \( T \subseteq N_G(u) \).
Otherwise, suppose that $T - N_G(u) \neq \emptyset$. Let $x \in T - N_G(u)$ and let $T' = T - x$. Then we have

$$\eta(S, T') = f(S) - g(T') + \sum_{y \in T'} d_{G-S}(y) - \tau(S, T')$$

$$= f(S) - (g(T) - g(x)) + \left( \sum_{y \in T} d_{G-S}(y) - d_{G-S}(x) \right) - \tau(S, T')$$

$$\leq f(S) - (g(T) - g(x)) + \left( \sum_{y \in T} d_{G-S}(y) - d_{G-S}(x) \right) - (\tau(S, T) - e_G(x, A))$$

$$= f(S) - g(T) + \sum_{y \in T} d_{G-S}(y) - \tau(S, T) - (d_{G-S}(x) - e_G(x, A)) + g(x)$$

$$= \eta(S, T) - (d_{G-S}(x) - e_G(x, A)) + g(x)$$

$$\leq \eta(S, T) \leq -2,$$

contradicting the choice of $S \cup T$.

By Claims 1 and 2, we may see that $f(S) = 0$ and $g(T) = 2|T|$. For simplicity, we write $\tau(S, T) = \tau$.

By (6), we see that

$$\tau \geq \sum_{x \in T} (d_G(x) - |S|) - 2|T| + 2,$$

which implies

$$\tau \geq \sum_{x \in T} (\delta - 1) - 2|T| + 2 \geq 2.$$  \hfill (7)

Note that $G - u$ is connected, so we have $|T| \geq 1$. Since $G$ is 3-edge-connected, we have

$$3\tau \leq \sum_{x \in T} (d_G(x) - |S|) + (\delta - |T||S|)$$

$$= \sum_{x \in T} d_G(x) + (\delta - 2|T||S|),$$

i.e.,

$$3\tau \leq \sum_{x \in T} d_G(x) + (\delta - 2|T||S|).$$  \hfill (9)

Inequalities (7) and (9) implies

$$2 \sum_{x \in T} d_G(x) + 6 \leq |S||T| + 6|T| + \delta|S| \quad \text{(since } |S| \leq 1 \text{ and } |T| \geq 1)$$

$$\leq 7|T| + \delta,$$
i.e.,

$$7|T| \geq 2 \sum_{x \in T} d_G(x) + 6 - \delta.$$  \hfill (10)

If \( \delta \geq 4 \), by (10), it follows

$$7|T| \geq 6 + \delta(2|T| - 1) \geq 8|T| + 2,$$

a contradiction. So we may assume that \( \delta = 3 \). By condition (ii), \( \sum_{x \in T} d_G(x) \geq 3|T| + 1 \). Combining (10),

$$7|T| \geq 2 \sum_{x \in T} d_G(x) + 6 - \delta \geq 2(3|T| + 1) + 3,$$

which implies \( |T| \geq 5 \), a contradiction since \( |T| \leq |N_G(u)| \leq 3 \).

This completes the proof. \( \square \)

**Proof of Theorem 1.2:** By Theorem 1.5 and Corollary 2.1 we can assume that both \(|A|\) and \(|B|\) are odd.

Firstly, we consider \( S = \{0, 1\} \). If \( G \) is 3-regular, by Theorem 1.6, then \( G \) admits a vertex-coloring 2-edge-weighting. By Theorem 1.7, \( G \) also admits a vertex-coloring \( \{0, 1\} \)-edge-weighting. So we can assume that \( \delta(G) \geq 3 \) and \( G \) is not 3-regular. If \( \delta(G) = 3 \), since \( G \) is 3-edge-connected, then \( G - x \) is connected for every vertex \( x \) of \( G \) with degree three. Hence there exists a vertex \( v \) with degree three such that \( N_G(v) \) contains a vertex with degree at least four. Let

\[
\begin{align*}
    u^* &= \begin{cases} 
    u & \text{if } \delta \geq 4, \\
    v & \text{if } \delta = 3.
    \end{cases}
\end{align*}
\]

Without loss generality, we may assume that \( u^* \in U \) and so it is a vertex satisfying the conditions of Lemma 2.3. Hence by Lemma 2.3, \( G \) contains a factor \( F \), which satisfies the following three conditions.

(i) \( d_F(u^*) = 0 \);

(ii) \( d_F(x) \equiv 1 \pmod{2} \) for all \( x \in U - u^* \);

(iii) \( d_F(y) \equiv 0 \pmod{2} \) for all \( y \in W \) and \( d_F(y) \geq 2 \) for all \( y \in N_G(u^*) \).

Clearly, \( d_F(x) \neq d_F(y) \) for all \( xy \in E(G) \). We assign weight 1 for each edge of \( E(F) \) and weight 0 for each edge of \( E(G) - E(F) \). Then we obtain a vertex-coloring \( \{0, 1\} \)-edge-weighting of graph \( G \).

Secondly, we show that \( G \) admits a vertex-coloring 2-edge-weighting. By Theorem 1.6 we may assume that \( G \) is not 3-regular. If \( \delta = 3 \), since \( G \) is 3-edge-connected, then \( G \) contains a vertex \( v' \) such that \( d_G(v') = 3 \), \( G - v' \) is connected and \( |N_G^0(v')| \leq 2 \). Let

\[
\begin{align*}
    u^* &= \begin{cases} 
    u & \text{if } \delta \geq 4, \\
    v' & \text{if } \delta = 3.
    \end{cases}
\end{align*}
\]

Then \( u^* \) is a vertex satisfying the conditions of Lemma 2.2. Hence by Lemma 2.2, \( G \) contains a factor \( F \) such that
(i) $d_F(u^*) = \delta$;
(ii) $d_F(x) \equiv \delta \pmod{2}$ for all $x \in W$ and $d_F(x) \leq \delta - 2$ for all $x \in N_G^\delta(u^*)$;
(iii) $d_F(y) \equiv \delta + 1 \pmod{2}$ for all $y \in U - u^*$.

Let $w : E(G) \to \{1, 2\}$ be a 2-edge-weighting such that $w(e) = 1$ for each $e \in E(F)$ and $w(e') = 2$ for each $e' \in E(G) - E(F)$. Clearly, $c(u^*) = \delta$. If $y \in N_G^\delta(u^*)$, since there exists an edge $e \sim y$ such that $e \notin E(F)$, then $c(y) = \sum_{e \sim y} w(e) > \delta$. If $y \in N_G(u^*) - N_G^\delta(u^*)$, then $c(y) \geq d_G(y) > \delta$. Hence $c(y) \neq c(u^*)$ for all $y \in N_G(u^*)$. For each $xy \in E(G)$, where $x \in U - u^*$ and $y \in W$, by the choice of $F$, we have $c(x) \equiv \delta + 1 \pmod{2}$ and $c(y) \equiv \delta \pmod{2}$. Hence $w$ is a vertex-coloring $\{1, 2\}$-edge-weighting of the graph $G$.

This completes the proof. \hfill $\Box$

**Corollary 2.4** Let $G$ be a 3-edge-connected bipartite graph. If $3 \leq \delta(G) \leq 5$, then $G$ admits a vertex-coloring $S$-edge-weighting for $S \in \{\{0, 1\}, \{1, 2\}\}$.

**Proof:** Since $3 \leq \delta \leq 5$ and $G$ is 3-edge-connected, then for every vertex $v$ of degree $\delta$, $G - v$ is connected. By Lemma 2.2 and Theorem 1.2 with the same proof, $G$ admits a vertex-coloring $S$-edge-weighting for $S \in \{\{0, 1\}, \{1, 2\}\}$. \hfill $\Box$

3 Conclusions

In this paper, we prove that every 2-connected and 3-edge-connected bipartite graph admits a vertex-coloring $S$-edge-weighting for $S \in \{\{0, 1\}, \{1, 2\}\}$. The generalized $\theta$-graphs is 2-connected and has a vertex-coloring 3-edge-weighting but not vertex-coloring $\{0, 1\}$-edge-weighting or vertex-coloring 2-edge-weighting. So it is an interesting problem to classify all 2-connected bipartite graphs admitting a vertex-coloring $S$-edge-weighting. Since the parity-factor problem is polynomial, then there exists a polynomial algorithm to find a vertex-coloring $S$-edge-weighting of bipartite graphs satisfying the conditions of Theorem 1.2.

**Acknowledgements**

The authors would like to thank the anonymous Reviewer for all valuable comments and suggestions to improve the quality of our paper.

**References**


