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Linear recognition of generalized Fibonacci cubes $Q_h(111)^*$

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The generalized Fibonacci cube $Q_h(f)$ is the graph obtained from the h-cube Q_h by removing all vertices that contain a given binary string f as a substring. In particular, the vertex set of the 3rd order generalized Fibonacci cube $Q_h(111)$ is the set of all binary strings $b_1b_2 \ldots b_h$ containing no three consecutive 1's. We present a new characterization of the 3rd order generalized Fibonacci cubes based on their recursive structure. The characterization is the basis for an algorithm which recognizes these graphs in linear time.

Keywords: generalized Fibonacci cube, 3rd order generalized Fibonacci cube, characterization, recognition algorithm

1 Introduction

The study of interconnection topologies is not just an important subject in the area of parallel or distributed systems, it also initiates the research work on several new interesting classes of graphs Liu and Hsu (1992). Besides hypercubes, Fibonacci cubes are the most known class of graphs which are applied as a model for interconnection network. They were studied from several points of view and they are known to possess many appealing properties Gregor (2006); Hsu (1993).

The Fibonacci cube Γ_h, $h \geq 1$, is defined as follows. The vertex set of Γ_h is the set of all binary strings $b_1b_2 \ldots b_h$ containing no two consecutive 1's. Two vertices are adjacent in Γ_h if they differ in precisely one bit. Several structural properties and applications including different metric aspects such as recursive construction, hamiltonicity, degree sequence and other enumeration have been investigated Castro and Mollard (2012); Dedó et al. (2002); Klavžar (2005); Klavžar and Mollard (2012, 2014). For an extensive survey of Fibonacci cubes see Klavžar (2013).

Suppose f is an arbitrary binary string and $h \geq 1$. The generalized Fibonacci cube, $Q_h(f)$, was introduced as the graph obtained from Q_h by removing all vertices that contain f as a substring Ilić et al. (2012). In this notation the Fibonacci cube Q_h is $Q_h(11)$. The question for which strings f, $Q_h(f)$ is

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an isometric subgraph of Q_h is raised and answered in Ilić et al. (2012) for some small lengths of f. Also, for a given large length of f, the proportion of f such that $Q_h(f)$ is an isometric subgraph of Q_h is investigated Klavžar and Shpectorov (2012).

The subclass of generalized Fibonacci cubes, the graphs $Q_h(1^k)$, have been introduced already in Hsu and Chung (1993) and further studied in Liu et al. (1994); Wasserman and Ghozati (2003); Zagaglia Salvi (1996). They are called the k-th order *generalized Fibonacci cubes (of dimension h)* in this paper (note that they were defined as "generalized Fibonacci cubes" in Hsu and Chung (1993)).

Hsu and Chung raised the following question Hsu and Chung (1993):

Question 1.1 *Given a graph, how to quickly decide whether it is the k-th order generalized Fibonacci cube of dimension h?*

The question has been answered only for Fibonacci cubes in Vesel (2015) where a linear recognition algorithm for $Q_h(11)$ is presented. This result provides a linear recognition algorithm for a closely related class of Lucas cubes Taranenko (2013).

The main contribution of this paper is a linear-time algorithm which recognizes the 3rd order generalized Fibonacci cubes. The paper is organized as follows. In the next section we give basic definitions and concepts needed in this paper. In Section 3, a new characterization of the 3rd order generalized Fibonacci cubes is given. This characterization is the basis for the algorithm presented in the last section, which recognizes these graphs in linear time.

2 Preliminaries

The *hypercube* of order h, denoted by Q_h, is the graph $G = (V, E)$ where the vertex set $V(G)$ is the set of all binary strings $u = u_1u_2 \dots u_h$, $u_i \in \{0, 1\}$, and two vertices $x, y \in V(G)$ are adjacent in Q_h if and only if x and y differ in precisely one place.

Fibonacci numbers form a sequence of non-negative integers F_n, where $F_0 = 0$, $F_1 = 1$ and for $n \geq 0$ satisfy the recurrence $F_{n+2} = F_{n+1} + F_n$.

The k-th order *generalized Fibonacci numbers* form a sequence of positive integers F^k_n, where $F^k_0 = F^k_1 = \dots = F^k_{k-2} = 0$, $F^k_{k-1} = 1$ and for $n \geq 0$ satisfy the recurrence $F^k_{n+k} = \sum_{i=0}^{k-1} F^k_{n+i}$.

We will use $[n]$ for the set $\{1, 2, \dots, n\}$ in this paper.

The *k-th order Fibonacci string* of length h is a binary string $u = u_1u_2 \dots u_h$, $u_i \in \{0, 1\}$, with $u_i \cdot u_{i+1} \dots u_{i+k-1} = 0$ for $i \in [h - k + 1]$. In other words, the k-th order Fibonacci string is a binary string without k consecutive ones.

The *3rd order generalized Fibonacci cube* $Q_h(111)$ is the subgraph of Q_h induced by the 3rd order Fibonacci strings of length h. For convenience we also set $Q_0(111) = K_1$. The 3rd order generalized Fibonacci cubes $Q_h(111)$ are shown in Fig. 1 for $h = 1, 2, 3, 4$. Note that $Q_h(111)$ is isomorphic to Q_h for $h \leq 2$, while $Q_3(111)$ is isomorphic to the vertex-deleted subgraph of Q_3 denoted by Q_3^-.

It is easy to see that the following lemma holds (see also Hsu and Chung (1993)).

Lemma 2.1 *If $h \geq 0$, then $|V(Q_h(111))| = F^3_{h+3}$.*

Let n denote the number of vertices of $Q_h(111)$.

Lemma 2.2 *Hsu and Chung (1993) If $h \geq 0$, then $|E(Q_h(111))| = \Theta(n \log n)$.*

A subgraph H of a graph G is *isometric* if $d_H(u, v) = d_G(u, v)$ for any pair of vertices u and v from H. Isometric subgraphs of hypercubes are called *partial cubes*. The *isometric dimension* of a graph G,

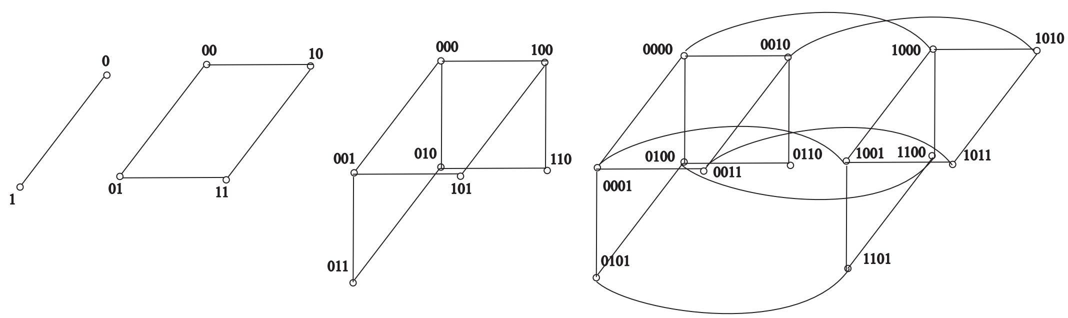

Fig. 1: the 3rd order generalized Fibonacci cubes

idim(G), is the smallest integer h such that G isometrically embeds into the h-dimensional cube. All generalized Fibonacci cubes are not partial cubes, however, $Q_h(1^s)$ isometrically embeds into Q_h as shown in Ilić et al. (2012).

Let $\alpha : V(G) \to V(Q_h)$ be an isometric embedding. We will denote the i-th coordinate of α with α^i, i.e. $\alpha = (\alpha^1, \alpha^2, \ldots, \alpha^h)$.

Let G be a connected graph and $e = xy$, $f = uv$ be two edges of G. We say that e is in relation Θ to f if $d(x,u) + d(y,v) \neq d(x,v) + d(y,u)$. Θ is reflexive and symmetric, but need not be transitive. We denote its transitive closure by Θ^*. It was proved in Winkler (1984) that G is a partial cube if and only if G is bipartite and $\Theta^* = \Theta$. The Θ-classes of a partial cube G constitute a partition of $E(G)$ and will be denoted with $1, \ldots, h$ in the sequel.

Let G be a partial cube with idim(G) = h and assume that we are given an isometric embedding of G into Q_h. Each pair $(i, \chi) \in [h] \times \{0, 1\}$ defines the semicube $W_{(i,\chi)} = \{u \in V(G) | \alpha^i(u) = \chi\}$. For any $i \in [h]$, we call $W_{(i,0)}, W_{(i,1)}$ a *complementary pair of semicubes*.

Any isometric embedding of G with idim(G) into Q_h describes the same family of semicubes and pairs of complementary semicubes, which are indexed in a different way. For a partial cube G and a complementary pair of semicubes $W_{(i,0)}, W_{(i,1)}$, the set of edges with one end vertex in $W_{(i,0)}$ and the other in $W_{(i,1)}$ forms a Θ-class i of G.

For an edge ab of G we write:
$W_{ab} = \{w \in V(G) \mid d(a,w) < d(b,w)\}$,
$W_{ba} = \{w \in V(G) \mid d(b,w) < d(a,w)\}$,
$F_{ab} = \{xy \mid xy$ edge of $E(G)$ with x in W_{ab} and y in $W_{ba}\}$,
$U_{ab} = \{w \in W_{ab} \mid w$ is an end vertex of an edge in $F_{ab}\}$,
$U_{ba} = \{w \in W_{ba} \mid w$ is an end vertex of an edge in $F_{ab}\}$.

We will need the following well known lemma, cf. (Hammack et al., 2011, Proposition 11.7).

Lemma 2.3 *Let $e = ab$ be an edge of a connected bipartite graph G and $F_{ab} = \{f \mid f \in E(G), e\Theta f\}$. Then $G - F_{ab}$ has exactly two connected components, namely $G[W_{ab}]$ and $G[W_{ba}]$.*

For an edge ab of a partial cube with an isometric embedding α into Q_h, let the vertices a and b differ in the coordinate i, i.e. $\alpha^i(a) = 0$ and $\alpha^i(b) = 1$. Then $W_{ab} = W_{(i,0)}$, $W_{ba} = W_{(i,1)}$ and $F_{ab} = \{xy \mid xy$ edge in $E(G)$ such that $\alpha^i(x) = 0$, $\alpha^i(y) = 1$ and $\alpha^j(x) = \alpha^j(y)$, for all $j \neq i\}$.

Note that the vertices of $Q_h(111)$ define an isometric embedding α into Q_h in a natural way, i.e. if $u \in V(Q_h(111))$, then $\alpha^i(u) = u_i$.

If v is a vertex of a graph G, then $N_G(v)$ denotes the set of vertices of G adjacent to v.

For $X \subseteq V(G)$ let $G[X]$ denote the subgraph of G induced by the set X.

Let 0^h stand for the vertex with zero at all coordinates and $0^{i-1}10^{h-i}$ for the vertex with one exactly at the $i-th$ coordinate for $i \in [h]$. We write xy for the concatenation of binary strings x and y. We will also denote by $e_i = 0^{i-1}10^{h-i}$ the i-th unit string in $\{0,1\}^h$.

For binary strings u and v of equal length let $u + v$ denote their sum computed bitwise modulo 2. In particular, $u + e_i$ is the string obtained from u by complementing its i-th bit.

3 Characterization

Let ab be an edge of a partial cube G for which $U_{ab} = W_{ab}$. Then $G[W_{ab}]$ is called a *peripheral subgraph* of G.

A Θ-class E of a partial cube G is called *peripheral* if at least one of $G[W_{ab}]$ and $G[W_{ba}]$ is peripheral for all $ab \in E$. It is known that every Θ-class of a Fibonacci cube is peripheral Taranenko and Vesel (2007). Here we show that the same holds for the 3rd order generalized Fibonacci cubes.

Lemma 3.1 *Every Θ-class of $Q_h(111)$ is peripheral.*

Proof: Since $Q_h(111)$ isometrically embeds into Q_h, it admits Θ-classes denoted $1, \ldots, h$. Note that an edge $e = uv$ belongs to a Θ-class i if and only if u and v differ exactly in the i-th coordinate. Suppose w.l.o.g. that $u \in W_{(i,1)}$ and $v \in W_{(i,0)}$. Let x be a vertex of $W_{(i,1)}$ and let x' be obtained from x by changing the $i-th$ position of x to 0. Since it is straightforward to see that x' is a vertex of W_{vu}, it follows that $xx' \in F_{uv}$ and $W_{uv} = W_{(i,1)} = U_{uv}$. This assertion completes the proof. □

Let f_h, $h \geq 1$, denote the number of the 3rd order Fibonacci strings of length h, i.e. $f_h = F^3_{h+3} = |V(Q_h(111))|$. For convenience, we also set $f_0 := 1$.

The following result shows the recursive structure of $Q_h(111)$ (see also Fig. 2).

Proposition 3.2 *If $W_{(i,\chi)}$, $\chi \in \{0,1\}$, is a semicube of $Q_h(111)$, then the following hold.*
(i) $Q_h(111)[W_{(1,0)}]$ is isomorphic to $Q_{h-1}(111)$.
(ii) $|W_{(1,1)}| = f_{h-2} + f_{h-3}$.
(iii) $|W_{(1,0)} \cap W_{(2,1)}| = f_{h-3} + f_{h-4}$.
(iv) $Q_h(111)[W_{(1,1)} \cap W_{(2,1)}]$ is isomorphic to $Q_{h-3}(111)$.
(v) $Q_h(111)[W_{(1,1)} \cap W_{(2,0)}]$ is isomorphic to $Q_{h-2}(111)$.
(vi) $|W_{(i,0)}| = f_{i-1}f_{h-i}$, $i \in [h]$.
(vii) If $a = 0^h$ and $b = 10^{h-1}$ (resp. $b = 0^{h-1}1$), then $W_{ab} \setminus U_{ab} = W_{(1,0)} \cap W_{(2,1)} \cap W_{(3,1)}$.

Proof: It is obvious that $u \in V(Q_{h-1}(111))$ if and only if $0u \in W_{(1,0)}$. Moreover, $uv \in E(Q_{h-1}(111))$ if and only if $0u$ and $0v$ are adjacent in $Q_h(111)$. Since $|V(Q_{h-1}(111))| = f_{h-1}$ and $|V(Q_h(111))| = f_h = f_{h-1} + f_{h-2} + f_{h-3}$, (i) and (ii) easily follow. Analogously, (iii) follows from (i) and (ii).

For (iv) note that if u is a vertex of $W_{(1,1)} \cap W_{(2,1)}$, then $u_1 = u_2 = 1$ and $u_3 = 0$. Therefore, $v \in V(Q_{h-3}(111))$ if and only if $110v \in W_{(1,1)} \cap W_{(2,1)}$ and the assertion easily follows. The proof of (v) is analogous.

For the proof of (vi), let $\mathcal{F}_h$ be the set of all the 3rd order Fibonacci strings of length h and $\mathcal{F}_h^i \subset \mathcal{F}_h$, $i \in [h]$ be the subset which includes all strings of $\mathcal{F}_h$ with 0 at the position i. In order to prove the assertion, note that $|\mathcal{F}_h| = f_h$ and $|\mathcal{F}_h^i| = |\mathcal{F}_{i-1} \times \mathcal{F}_{h-i}| = f_{i-1}f_{h-i}$.

Finally, for the proof of (vii) note that U_{ab} denotes the set of vertices of W_{ab} that are adjacent to the vertices of W_{ba}. We have $W_{ab} = W_{(1,0)}$ and $W_{ba} = W_{(1,1)}$. Moreover, $W_{ba} = W_{(1,1)} \cap (W_{(2,0)} \cup (W_{(2,1)} \cap W_{(3,0)}))$. Since every Θ-class of $Q_h(111)$ is peripheral by Lemma 3.1, we have $W_{ba} = U_{ba}$. It follows that $U_{ab} = W_{(1,0)} \cap ((W_{(2,0)} \cup (W_{(2,1)} \cap W_{(3,0)}))$. Since $W_{(1,0)} = W_{(1,0)} \cap ((W_{(2,0)} \cup (W_{(2,1)} \cap (W_{(3,0)} \cup W_{(3,1)})))$, we have $W_{ab} \setminus U_{ab} = W_{(1,0)} \setminus (W_{(1,0)} \cap ((W_{(2,0)} \cup (W_{(2,1)} \cap W_{(3,0)}))) = W_{(1,0)} \cap W_{(2,1)} \cap W_{(3,1)}$ and the assertion follows. □

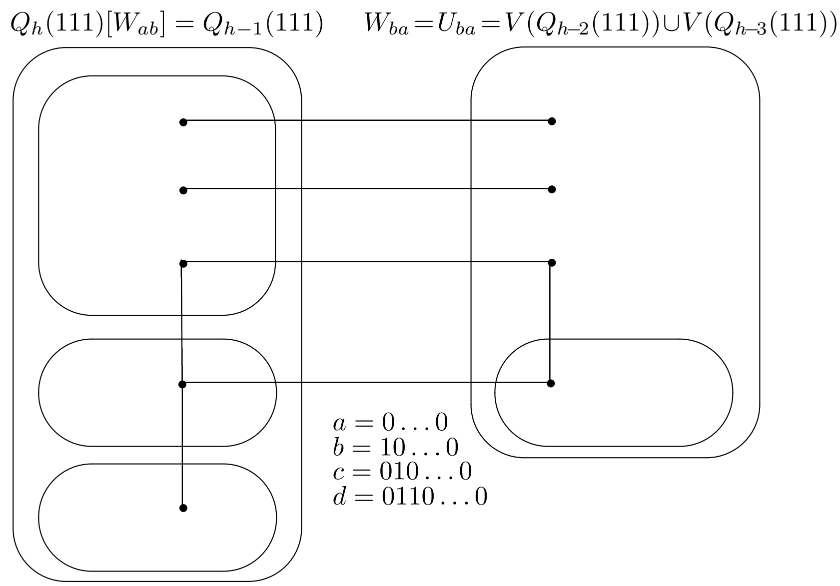

Fig. 2: The structure of $Q_h(111)$

Lemma 3.3 *Let $h \geq 9$ and $x = 10^{h-1} \in V(Q_h(111))$. If xy is an edge of $Q_h(111)[W_{(1,1)}]$ and the set $W_{(1,1)} \cap W_{yx}$ admits exactly f_{h-3} vertices, then $W_{(1,1)} \cap W_{yx} = W_{(1,1)} \cap W_{(2,1)}$.*

Proof: By Proposition 3.2(ii) we have $|W_{(1,1)}| = f_{h-2} + f_{h-3}$. Moreover, by Lemma 2.3, a vertex u of $W_{(1,1)} \cap W_{yx}$ is adjacent to a vertex v of $W_{(1,1)} \cap W_{xy}$ if and only if u and v differ in precisely the i-th coordinate, $i \geq 2$. Since $x = 10^{h-1}$, the i-th coordinate of u and v is 1 and 0, respectively. From the proof of Lemma 3.1 then it follows that $|W_{yx}| < |W_{xy}|$ and therefore we get $|W_{(1,1)} \cap W_{yx}| < |W_{(1,1)} \cap W_{xy}|$.

If $i = 2$, then $W_{(1,1)} \cap W_{yx} = W_{(1,1)} \cap W_{(2,1)}$. From Proposition 3.2(iv) (see also Fig. 2, where b and e can be replaced with x and y, respectively) we can see that $Q_h(111)[W_{(1,1)} \cap W_{yx}]$ is isomorphic to $Q_{h-3}(111)$. Thus, $|W_{(1,1)} \cap W_{xy}| = f_{h-2}$ and $|W_{(1,1)} \cap W_{yx}| = f_{h-3}$.

We will show that $|W_{(1,1)} \cap W_{(i,0)}| \neq f_{h-2}$ for every $i > 2$.

Let $i = 3$. If $u \in W_{(1,1)} \cap W_{(3,0)}$, then either $u_2 = 0$ or $u_2 = 1$. It follows from Proposition 3.2(vi) that $|W_{(1,1)} \cap W_{(3,0)}| = 2f_{h-3} > f_{h-2}$ and the case is settled.

Let $i=4$. If $u \in W_{(1,1)} \cap W_{(4,0)}$, then either $u_2=0$, or $u_2=1$ and $u_3=0$. From Proposition 3.2(vi) it follows that $|W_{(1,1)} \cap W_{(4,0)}| = 3f_{h-4} = 2f_{h-4}+f_{h-5}+f_{h-6}+f_{h-7} = f_{h-3}+f_{h-4}+f_{h-7} = f_{h-2}-f_{h-5}+f_{h-7} < f_{h-2}$.

Let $i=h$. If $u \in W_{(1,1)} \cap W_{(h,0)}$, then either $u_2=1$ and $u_3=0$ or $u_2=0$. From Proposition 3.2(vi) it follows that $|W_{(1,1)} \cap W_{(h,0)}| = f_{h-3}+f_{h-4} < f_{h-2}$.

Let $i=h-1$. If $u \in W_{(1,1)} \cap W_{(h-1,0)}$, then either $u_2=1$ and $u_3=0$ or $u_2=0$. Moreover, $u_h=0$ or $u_h=1$. From Proposition 3.2(vi) it follows that $|W_{(1,1)} \cap W_{(h-1,0)}| = 2(f_{h-4}+f_{h-5}) = f_{h-4}+f_{h-5}+f_{h-3}-f_{h-6} = f_{h-2}-f_{h-6} < f_{h-2}$.

Let $i=h-2$. Similarly as above we get $|W_{(1,1)} \cap W_{(h-2,0)}| = 4(f_{h-5}+f_{h-6}) = 2(f_{h-3}-f_{h-7}) < 2f_{h-3}-f_{h-6} = f_{h-3}+f_{h-3}-f_{h-6} = f_{h-3}+f_{h-4}+f_{h-5} = f_{h-2}$.

$h \setminus i$	2	5	6	7	8
9	81	78	77		
10	149	144	143	140	
11	274	264	264	260	259

Tab. 1: Values of $|W_{(1,1)} \cap W_{(i,0)}|$ for $i \in \{2,5,\ldots,h-3\}$, $h=\{9,10,11\}$.

For $5 \le i \le h-3$ note that for $u \in W_{(1,1)} \cap W_{(i,0)}$ we have either $u_2=0$ or $u_2=1$ and $u_3=0$. From Proposition 3.2(vi) then it follows that $|W_{(1,1)} \cap W_{(i,0)}| = f_{i-3}f_{h-i}+f_{i-4}f_{h-i}$. We will show that $(f_{i-3}+f_{i-4})f_{h-i} < f_{h-2}$, $5 \le i \le h-3$, by induction on h. For $h \in \{9,10,11\}$ the values of $|W_{(1,1)} \cap W_{(i,0)}| = (f_{i-3}+f_{i-4})f_{h-i}$ are depicted in Table 1 and we can see that $(f_{i-3}+f_{i-4})f_{h-i} < f_{h-2}$ for every i of the interest. Suppose that the assertion holds for all dimensions smaller than h, for some $h>11$. Let us consider the dimension h for $5 \le i \le h-3$. We have $(f_{i-3}+f_{i-4})f_{h-i} = (f_{i-3}+f_{i-4})f_{h-i-1}+(f_{i-3}+f_{i-4})f_{h-2-i}+(f_{i-3}+f_{i-4})f_{h-3-i}$. Note that by the induction hypothesis:

$(f_{i-3}+f_{i-4})f_{h-i-1} < f_{h-3}$, $5 \le i \le h-1$,

$(f_{i-3}+f_{i-4})f_{h-i-2} < f_{h-4}$, $5 \le i \le h-2$, and

$(f_{i-3}+f_{i-4})f_{h-i-3} < f_{h-5}$, $5 \le i \le h-3$.

It follows that $(f_{i-3}+f_{i-4})f_{h-i} < f_{h-3}+f_{h-4}+f_{h-5} = f_{h-2}$, $5 \le i \le h-3$.

Since we showed above that $(f_{i-3}+f_{i-4})f_{h-i} < f_{h-2}$ for $i \in \{h-2,h-1,h\}$, the proof is complete. □

Lemma 3.4 *Let xy be an edge of $Q_h(111)[W_{(1,1)}]$ such that $x=10^{h-1}$. If $H := Q_h(111)[W_{(1,1)} \cap W_{yx}]$, then the following hold.*

(i) If $h \in \{7,8\}$, then $|V(H)| = f_{h-3}$ if and only if $V(H) = W_{(1,1)} \cap W_{(2,1)}$ or $V(H) = W_{(1,1)} \cap W_{(h-2,1)}$. Moreover, if $V(H) = W_{(1,1)} \cap W_{(h-2,1)}$, then H is not isomorphic to $Q_{h-3}(111)$.

(ii) If $h \in \{5,6\}$, then $|V(H)| = f_{h-3}$ if and only if $V(H) = W_{(1,1)} \cap W_{(2,1)}$.

(iii) If $h \in \{3,4\}$, then $|V(H)| = f_{h-3}$ if and only if $V(H) = W_{(1,1)} \cap W_{(2,1)}$ or $V(H) = W_{(1,1)} \cap W_{(3,1)}$. Moreover, if $V(H) = W_{(1,1)} \cap W_{(3,1)}$, then H is isomorphic to $Q_{h-3}(111)$.

Proof: Note that in the graph $Q_h(111)$ we have $|W_{(1,1)}| = |W_{(1,1)} \cap W_{(i,0)}| + |W_{(1,1)} \cap W_{(i,1)}| = f_{h-2}+f_{h-3}$, $i \in \{2,\ldots,h\}$, by Proposition 3.2(ii). We can obtain analogously as in the proof of Lemma 3.3 the number of vertices of $W_{(1,1)} \cap W_{(i,0)}$ and $W_{(1,1)} \cap W_{(i,1)}$ as follows. If $i=2$, then

$|W_{(1,1)} \cap W_{(i,0)}| = f_{h-2}$ and $|W_{(1,1)} \cap W_{(i,1)}| = f_{h-3}$; if $i = 3$, then $|W_{(1,1)} \cap W_{(i,0)}| = 2f_{h-3}$; for $i \in \{4, \ldots, h\}$ we get $|W_{(1,1)} \cap W_{(i,0)}| = f_{h-3}f_{h-i} + f_{h-4}f_{h-i}$. The values of $|W_{(1,1)} \cap W_{(i,0)}|$ for every $i \in \{2, \ldots, h\}$ and for all $h = \{5, 6, 7, 8\}$ are depicted in Table 2.

$h \setminus i$	2	3	4	5	6	7	8
5	7	8	6	6			
6	13	14	12	12	11		
7	24	26	21	24	22	20	
8	44	48	39	42	44	40	37

Tab. 2: Values of $|W_{(1,1)} \cap W_{(i,0)}|$ for every $i \in \{2, \ldots, h\}$ and for all $h = \{5, 6, 7, 8\}$.

(i) Since $f_5 = 24$ and $f_6 = 44$, the first part of the assertion follows from Table 2. In order to see that $Q_7(111)[W_{(1,1)} \cap W_{(5,1)}]$ is not isomorphic to $Q_4(111)$, observe first that $V(Q_7(111)[W_{(1,1)} \cap W_{(5,1)}]) = \{$1000100, 1000101, 1000110, 1001100, 1010100, 1100100, 1001101, 1010101, 1100101, 1010110, 1100110, 1101100, 1101101$\}$.

Note that by Proposition 3.2 the graph isomorphic to $Q_4(111)$ has to admit an edge ab such that $|W_{ab}| = f_{4-1} = f_3 = 7$ and $|W_{ba}| = f_2 + f_1 = 6$.

We can see that $|W_{(1,1)} \cap W_{(5,1)} \cap W_{(2,1)}| = 5$, $|W_{(1,1)} \cap W_{(5,1)} \cap W_{(3,1)}| = 3$, $|W_{(1,1)} \cap W_{(5,1)} \cap W_{(4,1)}| = 4$, $|W_{(1,1)} \cap W_{(5,1)} \cap W_{(6,1)}| = 3$, $|W_{(1,1)} \cap W_{(5,1)} \cap W_{(7,1)}| = 5$. It follows that $Q_7(111)[W_{(1,1)} \cap W_{(5,1)}]$ cannot be isomorphic to $Q_4(111)$ and the case is settled.

The proof that $Q_8(111)[W_{(1,1)} \cap W_{(6,1)}]$ is not isomorphic to $Q_5(111)$ is analogous. We can see that $V(Q_8(111)[W_{(1,1)} \cap W_{(6,1)}]) = \{$10000100, 10000101, 10000110, 10010100, 10010101, 10010110, 10001100, 10001101, 10100100, 10100101, 10100110, 10110100, 10110101, 10110110, 10101100, 10101101, 11000100, 11000101, 11000110, 11010100, 11010101, 11010110, 11001100, 11001101$\}$.

By Proposition 3.2, the graph isomorphic to $Q_5(111)$ has to admit an edge ab such that $|W_{ab}| = f_{5-1} = f_4 = 13$ and $|W_{ba}| = f_3 + f_2 = 11$.

We can see that $|W_{(1,1)} \cap W_{(6,1)} \cap W_{(2,1)}| = 8$, $|W_{(1,1)} \cap W_{(6,1)} \cap W_{(3,1)}| = 8$, $|W_{(1,1)} \cap W_{(6,1)} \cap W_{(4,1)}| = 9$, $|W_{(1,1)} \cap W_{(6,1)} \cap W_{(5,1)}| = 6$, $|W_{(1,1)} \cap W_{(6,1)} \cap W_{(7,1)}| = 6$, $|W_{(1,1)} \cap W_{(6,1)} \cap W_{(8,1)}| = 9$. It follows that $Q_8(111)[W_{(1,1)} \cap W_{(6,1)}]$ cannot be isomorphic to $Q_5(111)$ and the case is settled.

(ii) Since we can see from Table 2 that $|W_{(1,1)} \cap W_{(i,0)}| = f_{h-2}$ only if $i = 2$, the assertion follows.

(iii) The assertion follows from Fig. 1. □

Remark 3.5 *The results of Proposition 3.2, Lemmas 3.3 and 3.4 can be analogously stated for* $b = 0^{h-1}1$, $W_{(h,\chi)}$, $W_{(h-1,\chi)}$, *and* $x = 0^{h-1}1$. *In particular, an analog to Lemma 3.3 is as follows:*

Let $h \geq 9$ *and* $x = 0^{h-1}1 \in V(Q_h(111))$. *If* xy *is an edge of* $Q_h(111)[W_{(h,1)}]$ *and the set* $W_{(h,1)} \cap W_{yx}$ *admits exactly* f_{h-3} *vertices, then* $W_{(h,1)} \cap W_{yx} = W_{(h,1)} \cap W_{(h-1,1)}$.

Lemma 3.6 *Let* $h \geq 3$ *and let* v *be a vertex of* $Q_h(111)$. *Then* $deg(v) < h$ *if*

(i) $v_i = v_{i+1} = 1$, $i \leq h-1$, *or*

(ii) $v_i = v_{i+2} = 1$, $i \leq h-2$.

Proof: (i) Since v cannot have three consecutive ones and $h \geq 3$, we have $v_{i-1} = 0$ or $v_{i+2} = 0$. Suppose w.l.o.g. that $v_{i-1} = 0$. Let $u \in N(v)$. Note that v and u differ in precisely one coordinate. Since

$v_i = v_{i+1} = 1$ and $v_{i-1} = 0$, u and v cannot differ in the $(i-1)$-st coordinate and the case is settled. The proof for (ii) is analogous. □

Proposition 3.7 *(i) If $h \geq 3$, then 0^h is the only vertex with h neighbors of degree h in $Q_h(111)$.*

(ii) Let $h \geq 4$ and $v = 0^{i-1}10^{h-i}$, $i \in [h]$. Then v has $h-2$ neighbors of degree h in $Q_h(111)$ if and only if $i = 1$ or $i = h$.

Proof:

(i) It is clear that 0^h has h neighbors of degree h in $Q_h(111)$. Assume to the contrary that $Q_h(111)$ admits a vertex v with h neighbors of degree h such that $v_i = 1$. Note that v cannot have $v_i = v_{i+1} = 1$, $i \leq h-1$, or $v_i = v_{i+2} = 1$, $i \leq h-2$, by Lemma 3.6. It follows that there exist a vertex u adjacent to v such that either $u_{i+1} = 1$ or $u_{i-1} = 1$. But then we have either $u_i = u_{i+1} = 1$ or $u_{i-1} = u_i = 1$. From Lemma 3.6 it follows that u admits less then h neighbors and we obtain a contradiction.

(ii) Note that 10^{h-1} and $0^{h-1}1$ have $h-2$ neighbors of degree h in $Q_h(111)$. Assume to the contrary that $Q_h(111)$ admits a vertex $v = 0^i10^{h-i-1}$ with exactly $h-2$ neighbors of degree h such that $2 \leq i \leq h-1$. We can see from Lemma 3.6 that $v + e_{i+1}$ and $v + e_{i-1}$ are neighbors of v with a degree less then h. Moreover, since $h \geq 4$, at least one of $v + e_{i+2}$ and $v + e_{i-2}$ exists in $Q_h(111)$. Since both have less then h neighbors in $Q_h(111)$ by Lemma 3.6, we again obtain a contradiction. □

If G is an isometric subgraph of Q_h, let $G^{i,j}$, $i, j \in [h]$, denote a graph obtained from G by exchanging the i-th and the j-th coordinate in every vertex of $V(G)$. Note that $G^{i,j}$ is an isometric subgraph of Q_h isomorphic to G. From Fig. 1 we can see

Proposition 3.8 *The vertices of $Q_3(111)[W_{(1,1)}]^{2,3}$ and $Q_4(111)[W_{(1,1)}]^{2,3}$ are the 3rd order Fibonacci strings.*

The next theorem gives a new characterization of $Q_h(111)$.

Theorem 3.9 *Let ab be an edge of a connected, bipartite graph G such that a has h neighbors of degree h and b has $h-2$ neighbors of degree h.*

If $h \geq 10$ or $h \in \{4, 5, 6, 7\}$, then G is isomorphic to $Q_h(111)$ if and only if the following conditions hold:

(i) $G[W_{ab}]$ is isomorphic to $Q_{h-1}(111)$.

(ii) F_{ab} is a matching defining an isomorphism between $G[U_{ab}]$ and $G[W_{ba}]$.

(iii) $W_{ab} \setminus U_{ab} = W_{ab} \cap W_{ca} \cap W_{dc}$ for $c \in U_{ab}$ which has $h-3$ neighbors of degree $h-1$ in $G[W_{ab}]$ and is adjacent to a vertex $d \in W_{ab} \setminus U_{ab}$.

(iv) $|W_{ab} \setminus U_{ab}| = f_{h-4}$.

Proof: Let G be isomorphic to $Q_h(111)$. From Proposition 3.7 it follows that $a = 0^h$ and $b = 10^{h-1}$ or $b = 0^{h-1}1$. Suppose first that $b = 10^{h-1}$.

(i) This is Proposition 3.2(i).

(ii) Since $a = 0^h$ and $b = 10^{h-1}$, we have $W_{ab} = W_{(1,0)}$ and $W_{ba} = W_{(1,1)}$. Thus, an edge of F_{ab} has one end-vertex with zero at the first position and one end-vertex with one at the first position. Since all Θ-classes of $Q_h(111)$ are peripheral by Lemma 3.1, it follows that $W_{ba} = U_{ba}$. Let $u, v \in W_{ba}$

and $u', v' \in W_{ab}$ such that $uu', vv' \in F_{ab}$. Since u and u' (resp. v and v') differ precisely in the first coordinate, $uv \in E(G[W_{ba}])$ if and only if $u'v' \in E(G[U_{ab}])$. It follows that $G[U_{ab}]$ and $G[W_{ba}]$ are isomorphic.

(iii) From (i) and Proposition 3.7 it follows that $c = 010^{h-2}$ or $c = 00^{h-2}1$. Since $b = 10^{h-1}$, Proposition 3.2(vii) yields $W_{ab} \setminus U_{ab} = W_{(1,0)} \cap W_{(2,1)} \cap W_{(3,1)}$, hence $c = 010^{h-2}$ and $d = 0110^{h-3}$.

(iv) Every $u \in W_{ab} \setminus U_{ab}$ has the first four coordinates fixed such that $u_4 = 0$. Hence, one can construct all the 3rd order Fibonacci strings of length $h-4$ at the coordinates $5, 6, \ldots h$ and we have $|W_{ab} \setminus U_{ab}| = f_{h-4}$.

Since the proof for $b = 0^{h-1}1$ is analogous (see also Remark 3.5), the first part of the proof is complete.

For the converse note that $G[W_{ab}]$ is isomorphic to $Q_{h-1}(111)$, thus we will first construct an isometric embedding of $G[W_{ab}]$ into $Q_{h-1}(111)$, i.e. we will assign the 3rd order Fibonacci strings of length $h-1$ to the vertices of $G[W_{ab}]$. Denote this embedding by β. Since $|N_G(a)| = h$ and G is bipartite, we have $|N_{G[W_{ab}]}(a)| = h-1$ by Lemma 2.3. Since the degree of every $u \in N_{G[W_{ab}]}(a)$ is h in G, from Lemma 2.3 it follows that the degree of u is at least $h-1$ in $G[W_{ab}]$. Moreover, since $G[W_{ab}]$ is isomorphic to $Q_{h-1}(111)$, the degree of a vertex of $G[W_{ab}]$ cannot exceed $h-1$. Hence, a has $h-1$ neighbors of degree $h-1$ in $G[W_{ab}]$. From Proposition 3.7 then it follows that we have to set $\beta(a) := (0, 0, \ldots, 0)$. Analogously, since c has $h-3$ neighbors of degree $h-1$ in $G[W_{ab}]$, we may set $\beta(c) := (1, 0, 0, \ldots, 0)$ (we could also set $\beta(c) := (0, 0, \ldots, 0, 1)$ by Proposition 3.7. We may assume that $\beta(c) := (1, 0, 0, \ldots, 0)$. It follows that for every $u \in W_{ab} \cap W_{ac}$ we have $\beta^1(u) = 0$ and for every $v \in W_{ab} \cap W_{ca}$ we have $\beta^1(v) = 1$.

Suppose first that $h \geq 10$. Note that $G[W_{ab}]$ is isomorphic to $Q_{h-1}(111)$, d belongs to $W_{ab} \cap W_{ca}$, $W_{ab} \cap W_{ca} \cap W_{dc} = W_{ab} \setminus U_{ab}$, and $|W_{ab} \setminus U_{ab}| = f_{h-4}$. Hence, by Lemma 3.3, we have to define β such that for every vertex $u \in W_{ab} \setminus U_{ab}$ we have $\beta^1(u) = \beta^2(u) = 1$ and $\beta^3(u) = 0$. For $h \in \{4, 5, 6, 7\}$ note that since β is an embeding into $Q_{h-1}(111)$, the value of h is subtracted by 1 when it is applied in Lemma 3.4 and Proposition 3.8. If $h \in \{6, 7\}$, then by Lemma 3.4(ii), we have to define β such that for every vertex $u \in W_{ab} \setminus U_{ab}$ we have $\beta^1(u) = \beta^2(u) = 1$ and $\beta^3(u) = 0$. If $h \in \{4, 5\}$, then by Lemma 3.4(iii), for every vertex $u \in W_{ab} \setminus U_{ab}$ we have either $\beta^1(u) = \beta^2(u) = 1$ and $\beta^3(u) = 0$ or $\beta^1(u) = \beta^3(u) = 1$ and $\beta^2(u) = 0$. From Proposition 3.8 it follows that above both cases are equivalent, therefore we may choose the former and conclude that for $h \geq 10$ or $h \in \{4, 5, 6, 7\}$ for every vertex $u \in W_{ab} \cap W_{ca} \cap W_{dc}$ we have $\beta^1(u) = \beta^2(u) = 1$ and $\beta^3(u) = 0$, while for every vertex $v \in W_{ab} \cap W_{ca} \cap W_{cd}$ we have $\beta^1(v) = 1$ and $\beta^2(v) = 0$. Lemma 3.3 and Proposition 3.2(iv) imply that the vertices of $W_{ab} \setminus U_{ab}$ induce $Q_{h-4}(111)$. We may then construct β for $u \in W_{ab} \cap W_{ca} \cap W_{dc}$ such that $(\beta^4(u), \beta^5(u), \ldots, \beta^{h-1}(u))$ represents the embedding of $Q_h(111)[W_{ab} \setminus U_{ab}]$ into $Q_{h-4}(111)$ and the vertices of $W_{ab} \cap W_{ca} \cap W_{cd}$ accordingly. Finally, we construct β for every vertex of $W_{ab} \cap W_{ac}$ with respect to the embedding of vertices of $W_{ab} \cap W_{ca}$ and the construction of β is complete.

We can now apply β in order to construct an embedding α of vertices of G into $Q_h(111)$ as follows. For every vertex $x \in W_{ab}$ we set $\alpha^1(x) := 0$ and $\alpha^i(x) := \beta^{i-1}(x)$, $2 \leq i \leq h$. Note that for every vertex $y \in W_{ba}$ there exist a vertex $x \in W_{ab}$ such that $xy \in F_{ab}$. Thus, we set $\alpha^1(y) := 1$ and $\alpha^i(y) := \beta^{i-1}(x)$, $2 \leq i \leq h$.

Obviously, for every vertex v of G we constructed the embedding α such that $\alpha^1(v)\alpha^2(v)\ldots\alpha^h(v)$ is a 3rd order Fibonacci string. In order to conclude the proof, we have to show that for every G we have $uv \in E(G)$ if and only if $\alpha(u)\alpha(v) \in E(Q_h(111))$. Since $G[W_{ab}]$ is isomorphic $Q_{h-1}(111)$, the claim is obvious for every $u, v \in W_{ab}$. For $u, v \in W_{ba}$ note that since the matching F_{ab} defines an isomorphism between $G[W_{ba}]$ and $G[U_{ab}]$, for every $u, v \in W_{ba}$ we have $u', v' \in U_{ab}$ such that $uu', vv' \in F_{ab}$ and

$uv \in E(G[W_{ba}])$ if and only if $u'v' \in E(G[U_{ab}])$ if and only if $\alpha(u')\alpha(v') \in E(Q_h(111))$. Finally, for $u \in W_{ab}$ and $v \in W_{ba}$ note that $uv \in E(G)$ if and only of $uv \in F_{ab}$. Since $\alpha(u)$ and $\alpha(v)$ differ in precisely one coordinate, the proof is complete. □

Since Theorem 3.9 does not include the graphs $Q_h(111)$ for $h \in \{8, 9\}$, we need the following

Proposition 3.10 *Let ab be an edge of a connected, bipartite graph G such that a has h neighbors of degree h, b has $h-2$ neighbors of degree h.*
If $h \in \{8, 9\}$, then G is isomorphic to $Q_h(111)$ if and only if the following conditions hold:

(i) $G[W_{ab}]$ *is isomorphic to* $Q_{h-1}(111)$.

(ii) F_{ab} *is a matching defining an isomorphism between* $G[U_{ab}]$ *and* $G[W_{ba}]$.

(iii) $W_{ab} \setminus U_{ab} = W_{ab} \cap W_{ca} \cap W_{dc}$ *for* $c \in U_{ab}$ *which has* $h-3$ *neighbors of degree* $h-1$ *in* $G[W_{ab}]$ *and is adjacent to a vertex* $d \in W_{ab} \setminus U_{ab}$.

(iv) $|W_{ab} \setminus U_{ab}| = f_{h-4}$.

(v) $G[W_{ab} \setminus U_{ab}]$ *is isomorphic to* $Q_{h-4}(111)$.

Proof: For the if part of the proof just note that by Lemma 3.4(i), for $h \in \{8, 9\}$ the conditions (i)-(iv) do not guarantee that $G[W_{ab} \setminus U_{ab}]$ is isomorphic to $Q_{h-4}(111)$. The rest of the proof is analogous to the proof of Theorem 3.9. □

4 Algorithm

Theorem 3.9 leads to the following recognition algorithm.

Procedure GENERALIZED3 FIBONACCI(G, h);
1. **If** $h \leq 3$ **and** G is isomorphic to one of $\{Q_1, Q_2, Q_3^-\}$ **then** ACCEPT.
2. Find $ab \in E(G)$ such that a has h neighbors of degree h and b has $h-2$ neighbors of degree h.
3. **If** no such edge exists **then** REJECT.
4. Find the sets $W_{ab}, W_{ba}, U_{ab}, U_{ba}$ and F_{ab}.
5. Construct the graph $G[W_{ab}]$.
6. Find $ac \in E(G[W_{ab}])$ such that c has $h-3$ neighbors of degree $h-1$ in $G[W_{ab}]$ and c is adjacent to a vertex $d \in W_{ab} \setminus U_{ab}$.
7. **If** no such edge exists **then** REJECT **else** find sets W_{ac}, W_{ca}, W_{dc}.
8. Verify that
   8.1. GENERALIZED3 FIBONACCI($G[W_{ab}]$, $h-1$) returns ACCEPT.
   8.2. F_{ab} is a matching which defines an isomorphism between $G[U_{ab}]$ and $G[W_{ba}]$.
   8.3. $W_{ab} \setminus U_{ab} = W_{ab} \cap W_{ca} \cap W_{dc}$.
   8.4. $|W_{ab} \setminus U_{ab}| = f_{h-4}$.
   8.5. $h \notin \{8, 9\}$ **or** $G[W_{ab} \setminus U_{ab}]$ is isomorphic to $Q_{h-4}(111)$.
9. **If** all of the foregoing conditions are fulfilled **then** ACCEPT **else** REJECT.

Before GENERALIZED3 FIBONACCI is started, some preprocessing has to be done. A given graph G is examined only if $|V(G)| = F^3_{h+3}$, for some $h \geq 1$, otherwise the graph is rejected. Moreover, we have to establish whether G is bipartite. Note that this can be done in $O(m)$ time.

The given bipartite graph G is then declared the 3rd order generalized Fibonacci cube if and only if GENERALIZED3 FIBONACCI(G,h) terminates without ever encountering a REJECT statement. In other words, GENERALIZED3 FIBONACCI for a graph G terminates with success if and only if either $h \geq 4$ and the conditions of Theorem 3.9 and Proposition 3.10 are fulfilled for G or $h \leq 3$ and G is isomorphic to one of $\{Q_1, Q_2, Q_3^-\}$. This gives us the following

Theorem 4.1 *GENERALIZED3 FIBONACCI correctly recognizes $Q_h(111)$.*

In order to find the time complexity of the algorithm, we first show two lemmas.

Lemma 4.2 *If $h \geq 3$, then*

$$\sum_{i=1}^{h} i f_i = \frac{f_1 + f_2 + (h-1)(f_{h+2} + f_h) - f_{h-1} - f_{h-2}}{2}\,.$$

Proof: We first compute $2\sum_{i=1}^{h} i f_i = \sum_{i=1}^{h} 2 i f_i$ such that we write every $2if_i$ as $if_i + i(f_{i+3} - f_{i+2} - f_{i+1})$.

$$2\sum_{i=1}^{h} i f_i = h f_h + h(f_{h+3} - f_{h+2} - f_{h+1}) + (h-1)f_{h-1} + (h-1)(f_{h+2} - f_{h+1} - f_h) +$$

$$+(h-2)f_{h-2} + (h-2)(f_{h+1} - f_h - f_{h-1}) + (h-3)f_{h-3} + (h-3)(f_h - f_{h-1} - f_{h-2}) + \dots$$

$$\dots + 3f_3 + 3(f_6 - f_5 - f_4) + 2f_2 + 2(f_5 - f_4 - f_3) + f_1 + (f_4 - f_3 - f_2)$$

$$= h f_{h+3} - f_{h+2} - (h+1)f_{h+1} + \dots + i f_i - (i-1)f_i - (i-2)f_i + (i-3)f_i + \dots + f_2 + f_1$$

We obtain above telescoping series in which all terms $f_i - (i-1)f_i - (i-2)f_i + (i-3)f_i$, $i = 3, \dots, h$, are cancelled. Thus we get

$$2\sum_{i=1}^{h} i f_i = f_1 + f_2 + h f_{h+3} - f_{h+2} - (h+1)f_{h+1}$$

$$= f_1 + f_2 + h f_{h+2} + h f_{h+1} + h f_h - f_{h+2} - (h+1)f_{h+1}$$

$$= f_1 + f_2 + h(f_{h+2} + f_h) - f_{h+2} - f_{h+1}$$

$$= f_1 + f_2 + (h-1)(f_{h+2} + f_h) - f_{h-1} - f_{h-2}.$$

The assertion now clearly follows. □

Theorem 4.3 *GENERALIZED3 FIBONACCI runs in linear time.*

Proof: Let m denote the number of edges of G.

Concerning the time complexity, we will show that the time complexity of every step of the algorithm with the exception of Step 8.1 is bounded by $O(m)$. Since for $h \leq 3$ the graphs $Q_h = Q_h(111)$ are very simple, this is obviously true for Step 1. For Steps 2 and 3 it is convenient to arrange the vector deg, such that deg_v equals the number of vertices adjacent to v for every $v \in V(G)$. We then first determine vertex a with h neighbors of degree h, by inspecting the adjacency list for every vertex of a graph. The total number of all examined entries in the adjacency list is clearly bounded by $O(m)$. If a is found, then we check its neighbors in order to find the vertex b. Analogously, this can be done again in linear time and the proof for Steps 2 and 3 is settled. Concerning Steps 4, 5, and 7 it has been shown in Jha and Slutzki (1992) that they can be performed in time linear in the number of edges of the input graph. Since the proof for Step 6 is analogous to the proof for Steps 2 and 3, we may proceed with Step 8. It has also been shown in Jha and Slutzki (1992) that Step 8.2 can be performed in time linear in the number of edges of the input graph. If we mark every vertex of $U_{ab} \cup U_{ba}$, W_{ab}, W_{ca}, and W_{dc}, then Steps 8.3 and 8.4 can be implemented to run within the same time bound. Thus, neglecting the recursive call, the total time needed to check G is bounded by $O(m)$, i.e. the cost per edge processed by a single call of the algorithm is $O(1)$.

By Lemma 2.2 we have $m = \Theta(n \log n)$. If follows that we can find a constant C such the the total time needed to check G, neglecting the recursive call, is bounded by $C\, hf_h$.

Let m_h denote the total number of edges checked by the algorithm.

From the above discussion we have

$$m_h \leq \sum_{i=1}^{h} C\, if_i = C \sum_{i=1}^{h} if_i,$$

while Lemma 4.2 yields

$$m_h \leq C\frac{f_1 + f_2 + (h-1)(f_{h+2} + f_h) - f_{h-1} - f_{h-2}}{2}$$

$$= C\frac{f_1 + f_2 + (h-1)(3f_h + 2f_{h-1} + f_{h-2}) - f_{h-1} - f_{h-2}}{2}$$

$$= C\frac{f_1 + f_2 + 3hf_h - 3f_h + 2hf_{h-1} - 3f_{h-1} + hf_{h-2} - 2f_{h-2}}{2}$$

$$= C\frac{f_1 + f_2 + (3h-3)f_h + (2h-3)f_{h-1} + (h-2)f_{h-2}}{2}$$

$$\leq C\frac{5h-6}{2}f_h \leq \frac{5}{2}Chf_h.$$

We showed that the total number of edges processed by GENERALIZED3 FIBONACCI(G, h) cannot exceed $\frac{5}{2}|E(G)|$. Moreover, since we showed that the cost per edge processed by a single call of the the algorithm is $O(1)$, the proof is complete. □

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