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Moving robots efficiently using the combinatorics of CAT(0) cubical complexes

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Abstract. Given a reconfigurable system X, such as a robot moving on a grid or a set of particles traversing a graph without colliding, the possible positions of X naturally form a cubical complex $\mathcal{S}(X)$. When $\mathcal{S}(X)$ is a CAT(0) space, we can explicitly construct the shortest path between any two points, for any of the four most natural metrics: distance, time, number of moves, and number of steps of simultaneous moves.

CAT(0) cubical complexes are in correspondence with posets with inconsistent pairs (PIPs), so we can prove that a state complex $\mathcal{S}(X)$ is CAT(0) by identifying the corresponding PIP. We illustrate this very general strategy with one known and one new example: Abrams and Ghrist's "positive robotic arm" on a square grid, and the robotic arm in a strip. We then use the PIP as a combinatorial "remote control" to move these robots efficiently from one position to another.

Résumé. Etant donné un système X, qui est reconfigurable, par example un robot se déplaçant sur une grille ou bien un ensemble de particules qui traverse un graphe sans collision, toutes les positions possibles de X forment de façon naturel un c complexe cubique $\mathcal{S}(X)$. Dans le cas ou $\mathcal{S}(X)$ est un espace CAT(0), nous pouvons explicitement construire le chemin le plus court entre deux points quelconques, pour une des quatre mesures les plus naturels: la distance euclidienne, le temps, le nombre de coups, et le nombre d'étapes de mouvements simultanés.

CAT (0) complexes cubiques sont en correspondance avec les ensembles partiellement ordonnés posets des paires incompatibles (PPI), et donc nous pouvons demontrer qu'un état complexe $\mathcal{S}(X)$ est CAT (0), en identifiant le PPI correspondant. Nous illustrons cette stratégie très générale avec un example bien connu et un exemple nouveau: L'example de Abrams et Ghrist du "bras robotique positif" sur une grille carrée, et le bras robotique dans une bande. Ensuite nous utilisons le PPI come une "télécommande" combinatorique pour efficacement déplacer ces robots d'une position à une autre.

Keywords: cubical complexes, combinatorial optimization, posets, reconfigurable systems, state complexes

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1 Introduction

There are numerous contexts in mathematics, robotics, and other fields where a discrete system changes according to local, reversible moves. For example, one might consider a robotic arm moving around a

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grid, a number of particles moving around a graph, or a phylogenetic tree undergoing local mutations. Abrams, Ghrist, and Peterson [1, 8] introduced the formalism of *reconfigurable systems* to model a very wide variety of such contexts.

Perhaps the most natural and important question that arises is the *motion-planning* or *shape-planning* question: how does one efficiently get a reconfigurable system X from one position to another one? Abrams, Ghrist, and Peterson observed that the *transition graph* $G(X)$ is the 1-skeleton of the *state complex* $\mathcal{S}(X)$: a cubical complex whose vertices are the states of X, whose edges correspond to allowable moves, and whose cubes correspond to collections of moves which can be performed simultaneously. In fact, $\mathcal{S}(X)$ can be regarded as the space of all possible positions of X, including the positions in between states.

The geometry and topology of the state complex $\mathcal{S}(X)$ can help us solve the motion-planning problem for the system X. More concretely, $\mathcal{S}(X)$ is locally non-positively curved for *any* configuration system. [1, 8] Furthermore, the state complex of *some* reconfigurable systems is globally non-positively curved, or *CAT(0)*. This stronger property implies that for any two points p and q there is a unique shortest path between them. Ardila, Owen, and Sullivant [3] gave an explicit algorithm to find this path.

It is therefore extremely useful to find out when a state complex $\mathcal{S}(X)$ is CAT(0). The first groundbreaking result in this direction is due to Gromov [9], who gave a topological-combinatorial criterion for this geometric property. Roller [13] and Sageev [14], and Ardila, Owen, and Sullivant [3] then gave two completely combinatorial descriptions of CAT(0) cubical complexes. The second descripion is a bijection between rooted CAT(0) cube complexes and *posets with inconsistent pairs* (PIPs).

In this paper, we put into practice the paradigm introduced in [3] to prove that a given cubical complex X is CAT(0). The idea is simple: we identify a PIP whose corresponding (rooted) CAT(0) cubical complex is X. In principle, this method is completely general, though its implementation in a particular situation is not trivial. We illustrate this with one known and one new example of robotic arms. We close by showing how to find the shortest path between states in a CAT(0) state complex $\mathcal{S}(X)$ under four natural metrics.

2 Preliminaries

2.1 Reconfigurable systems and cubical complexes

We now sketch the basic definitions for reconfigurable systems due to Abrams, Ghrist, and Peterson and illustrate them with an example. We refer the reader to [1] and [8] for the details. Let $\mathcal{G} = (V, E)$ be a graph and $\mathcal{A}$ be a set of labels. A *state* u is a labeling of the vertices of $\mathcal{G}$ by elements of $\mathcal{A}$. Roughly speaking, a *reconfigurable system* is given by a collection of states, together with a given set of local moves called *generators* that one can perform to get from one state to another. Given a state s and a set of moves M which can be applied to s, we say that the moves in M *commute* if they can be applied simultaneously to s; that is, they are "physically independent". In this paper we will study two robotic arms moving inside a grid. Here G will represent the grid, and a labelling of G with 0s and 1s will indicate the position of the robot.

Example 2.1 (Metamorphic robots in a hexagonal lattice [7, 8]) *Consider a robot made up of identical hexagonal unit cells in the hexagonal lattice, which has the ability to pivot cells on the boundary whenever they are unobstructed. Figure 1a. shows one move, and b.-e. shows two commutative moves.*

A *cubical complex* X is a polyhedral complex obtained by gluing cubes of various dimensions, in such a way that the intersection of any two cubes is a face of both. Such a space X has a natural piecewise

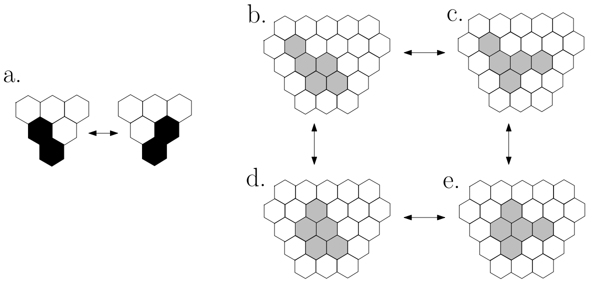

Fig. 1: a. A generator for a metamorphic robot in the hexagonal lattice. b-e. Four possible states

Euclidean metric. Any reconfigurable system gives rise to a cubical complex:

Definition 2.2 *The* state complex $\mathcal{S}(\mathcal{R})$ *of a reconfigurable system* $\mathcal{R}$ *is a cubical complex whose vertices correspond to the states of* $\mathcal{R}$*. We draw an edge between two states if they differ by an application of a single move. The* k*-cubes correspond to* k*-tuples of commutative moves.*

Figure 2 shows the state complex of a robot of 5 cells which moves following the rules of Fig. 1, and is constrained to stay inside a tunnel of width 3.

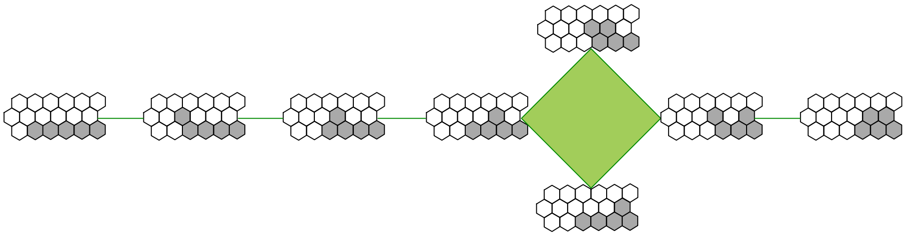

Fig. 2: The state complex of a hexagonal metamorphic robot in a tunnel.

Given a reconfigurable system $\mathcal{R}$ and a state u, there is a natural partial order on the states of $\mathcal{R}$:

Definition 2.3 *Let* $\mathcal{R}$ *be a reconfigurable system and let* u *be any "home" state. Define the* poset of states $\mathcal{R}_u$ *to be the set of states ordered by declaring that* $p \leq q$ *if there is a shortest edge-path from the home state* u *to* q *going through* p*.*

2.2 Combinatorial geometry of CAT(0) cubical complexes

We now define CAT(0) spaces, the spaces of global non-positive curvature that we are interested in. For more information, see [5, 6]. Let X be a geodesic metric space– that is, a metric space where any two points x and y are the endpoints of a curve of length $d(x, y)$. Consider a triangle T in X of side lengths a, b, c, and build a comparison triangle T' with the same lengths in the Euclidean plane. Consider a chord of length d in T which connects two points on the boundary of T; there is a corresponding comparison chord in T', say of length d'. If $d \leq d'$ for any chord in T, we say that T is a *thin triangle* in X.

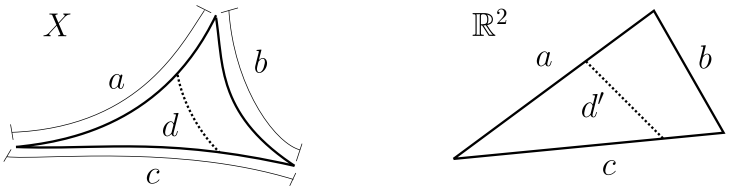

Fig. 3: A chord in a triangle in X, and the corresponding chord in the comparison triangle in the plane. The triangle in X is *thin* if $d \leq d'$ for all such chords.

Definition 2.4 *A* CAT(0) space *is a metric space having a unique geodesic between any two points, such that every triangle is thin.*

A related concept is that of a *locally CAT(0)* or *non-positively curved* metric space X. This is a space where all sufficiently small triangles are thin.

Testing whether a general metric space is CAT(0) is quite subtle. However, Gromov [9] proved that this is easier if the space is a cubical complex. He showed that a cubical complex is CAT(0) if and only if it is simply connected and the link of any vertex is a flag simplicial complex.

Ardila, Owen, and Sullivant [3] gave a purely combinatorial description of CAT(0) cube complexes, which we now describe. If X is a CAT(0) cubical complex and v is any vertex of X, we call (X, v) a *rooted CAT(0) cubical complex*. The right side of Figure 4 shows an example.

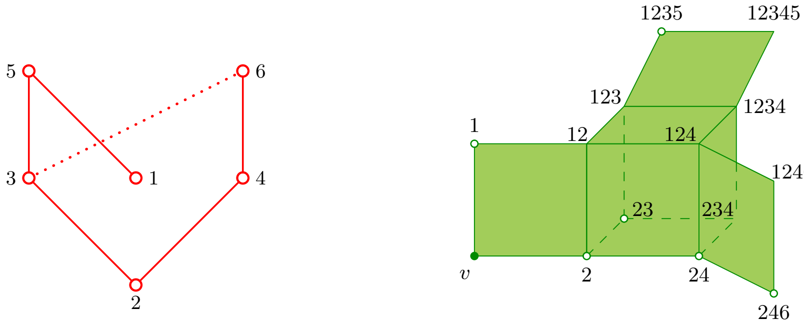

Fig. 4: A poset with inconsistent pairs and the corresponding rooted CAT(0) cubical complex.

Recall that a poset P is *locally finite* if every interval $[i, j] = \{k \in P : i \leq k \leq j\}$ is finite, and it has *finite width* if every antichain (set of pairwise incomparable elements) is finite.

Definition 2.5 *A* poset with inconsistent pairs (PIP) *is a locally finite poset P of finite width, together with a collection of* inconsistent pairs $\{p, q\}$, *such that no two comparable elements are inconsistent, and if p and q are inconsistent and $p' \geq p$ and $q' \geq q$, then p' and q' are inconsistent.*

The *Hasse diagram* of a poset with inconsistent pairs (PIP) is obtained by drawing the poset and connecting each minimal inconsistent pair with a dotted line. An inconsistent pair $\{p, q\}$ is *minimal* if there is no other inconsistent pair $\{p', q'\}$ with $p' \leq p$ and $q' \leq q$. For example, see the left side of Figure 4.

Recall that $I \subseteq P$ is an *order ideal* if $a \leq b$ and $b \in I$ imply $a \in I$. A *consistent order ideal* is one which contains no inconsistent pairs.

Definition 2.6 *If P is a poset with inconsistent pairs, we construct the* cube complex of P*, which we denote $X(P)$. The vertices of $X(P)$ are identified with the consistent order ideals of P. There will be a cube $C(I, M)$ for each pair (I, M) of a consistent order ideal I and a subset $M \subseteq I_{max}$, where I_{max} is the set of maximal elements of I. This cube has dimension $|M|$, and its vertices are obtained by removing from I the $2^{|M|}$ possible subsets of M. The cubes are naturally glued along their faces according to their labels.*

Figure 4 shows a PIP and the corresponding complex. For example, the compatible order ideal $I = \{1, 2, 3, 4\}$ and the subset $M = \{1, 4\} \subseteq I_{max}$ give rise to the square with vertices $1234, 123, 234, 23$.

Theorem 2.7 (Ardila, Owen, Sullivant) *[3] The map $P \mapsto X(P)$ is a bijection between posets with inconsistent pairs and rooted CAT(0) cube complexes.*

2.3 Reconfigurable systems and CAT(0) cubical complexes

The influential paper of Billera, Holmes, and Vogtmann [4] was one of the first to highlight the relevance of the CAT(0) property in applications. Most relevantly to this paper, the space T_n of phylogenetic trees was shown in [4] to be a CAT(0) cubical complex. This led to important consequences, such as the existence of geodesics and of "average trees" in T_n. Furthermore, after numerous partial results by many authors, Owen and Provan [11] recently gave the first polynomial time algorithm to compute geodesics in T_n. The work of Billera, Holmes, and Vogtmann was generalized in the following two directions:

Theorem 2.8 (Ardila-Owen-Sullivant) *[3] There is an algorithm to compute the geodesic between any two points in a CAT(0) cubical complex.*

Theorem 2.9 (Abrams-Ghrist, Ghrist-Peterson) *[1, 8] The state complex of a reconfigurable system is a* ***locally*** *CAT(0) cubical complex; that is, all small enough triangles are thin.*

When the state complex of a reconfigurable system is **globally** CAT(0), we can use the algorithm in Theorem 2.8 to navigate it. That will allow us to get our system from one position to another one in the optimal way. This highlights the importance of the following question:

Question 2.10 *Is the state complex of a given reconfigurable system a CAT(0) space?*

Theorem 2.7 offers a new technique to provide an affirmative answer to Question 2.10: Rooted CAT(0) cubical complexes are in bijection with PIPs; so to prove that a cubical complex is CAT(0), we "simply" have to choose a root for it, and find the corresponding PIP! In principle, this technique works for **any** reconfigurable system whose state complex X is CAT(0). In practice, it is not always easy to identify the corresponding PIP. However, we hope to convince the reader that this can be done in many interesting special cases. We will do it for one old and one new example. We introduce the two relevant robots in Section 3, and provide combinatorial proofs that their state complexes are CAT(0) in Sections 4 and 5.

3 The robotic arms

3.1 The positive robotic arm in a quadrant

The following reconfigurable system, which we call QR_n, was first introduced in [1] and shown to be CAT(0) using Gromov's topological/combinatorial criterion. Consider a robotic arm consisting of n links of unit length, attached sequentially. The robot lives inside an $n \times n$ grid, and its base is affixed to the lower left corner of the grid. Figure 5.a shows a position of the arm.

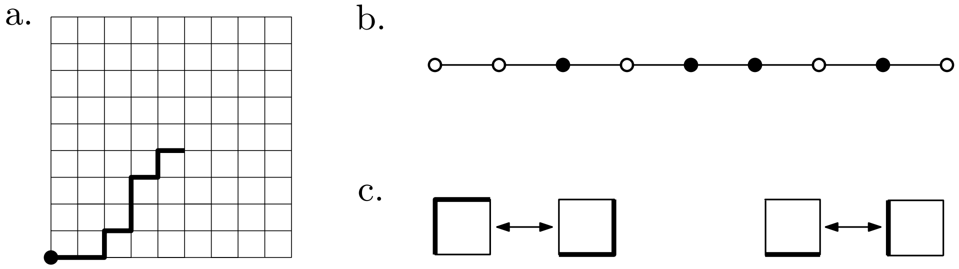

Fig. 5: a. The robotic arm in position 3568 for $n = 9$ b. the corresponding particles on a line (to be introduced later), and c. the local movements of QR_n.

The robot is free to move using the two local moves illustrated in Figure 5.c. They are: *NE-switching corners* (two consecutive links facing north and east can be switched to face east and north, and vice versa), and *NE-flipping the end* (if the last link of the robot is facing east, it can be switched to face north, and vice versa). It is clear that QR_n has 2^n possible positions, corresponding to the paths of length n which start at the southwest corner and always step east or north. We call these simply *NE-paths*.

Notation 3.1 *We will label each state of the robot using the set of its vertical steps: if a position of the robot has k links facing north at positions $a_1, \ldots, a_k$ (counting from the base), then we label it $\{a_1, \ldots, a_k\}$ or simply $a_1 \ldots a_k$.*

Notice that two states of different lengths can have the same label. We assume implicitly that the length of the robot is specified ahead of time.

3.2 The robotic arm in a strip

Now consider a robotic arm SR_n which also consists of n links of unit length, attached sequentially. The robot lives inside a $1 \times n$ grid, and its base is still affixed to the lower left corner of the grid, but the links do not necessarily have to face north and east. Figure 6 shows a position of the arm, as well as the legal moves: *switching corners* and *flipping the end*.

Again, we label a state using its vertical steps shown in Figure 6. One easily checks that the number of states of SR_n is the Fibonacci number F_{n+2}. For this reason, we call a state of SR_n an F-path.

3.2.1 The systems QR_n and SR_n as hopping particles.

Consider a board consisting of n slots on a line, and a system of indistinguishable particles hopping around the board. Any particle can hop to the slot immediately to its left or right whenever that slot is empty.

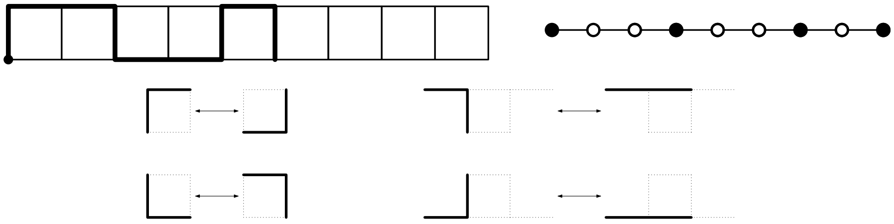

Fig. 6: The robotic arm in position 1479 for $n = 9$, the corresponding particles on a line, and the legal moves.

Particles may enter and leave the board via the rightmost slot. The following proposition is illustrated in Figure 5; for details, see [2].

Proposition 3.2 *The system QR_n is equivalent to the system of hopping particles on a board of length n.*

Now consider a similar board of n slots on a line, with indistinguishable *repellent particles* hopping around the board. The repellent particles must stay at distance at least 2 from each other.

Proposition 3.3 *The system SR_n is equivalent to the system of hopping repellent particles on a board of length n.*

4 The state complex of QR_n is CAT(0)

We now provide combinatorial proofs that the state complexes of the robots QR_n and SR_n are CAT(0). In view of Theorem 2.7, our strategy is as follows. We root the complex $\mathcal{S}(QR_n)$ at a natural vertex v. If $\mathcal{S}(QR_n)$ really is CAT(0), then Theorem 2.7 puts it in correspondence with a PIP (poset with inconsistent pairs) QP_n. We identify the candidate PIP QP_n, and prove that, under the bijection of Theorem 2.7, the PIP QP_n is mapped to the (rooted) state complex of QR_n. Therefore this complex must be CAT(0).

Definition 4.1 *Define the PIP QP_n to be the set of lattice points inside the triangle $y \geq 0, y \leq x$, and $x \leq n - 1$, with componentwise order (so $(x, y) \leq (x', y')$ if $x \leq x'$ and $y \leq y'$) and no inconsistent pairs.*

The poset QP_n has the triangular shape shown in Figure 7 for $n = 6$.

Proposition 4.2 *There is a bijection between the states of the robot QR_n and the order ideals of QP_n.*

Recall Definition 2.3. We get the following by Birkhoff's theorem:

Corollary 4.3 *If we declare the "home" state of QR_n to be the fully horizontal state, then the poset of states of QR_n is a distributive lattice.*

Let the *word of a subset* $A = \{a_1 < a_2 < \cdots < a_k\} \subseteq [n]$ be the length n word $w(A) = (a_1, a_2, \ldots a_k, (n+1), (n+1), \ldots, (n+1))$.

Proposition 4.4 *The lattice of states of QR_n is isomorphic to the poset on the subsets of $[n]$, where $A \leq B$ if $w(A) \geq w(B)$ coordinatewise.*

Having established these results about the 1-skeleton of the state complex, we now extend them to the higher-dimensional cubes.

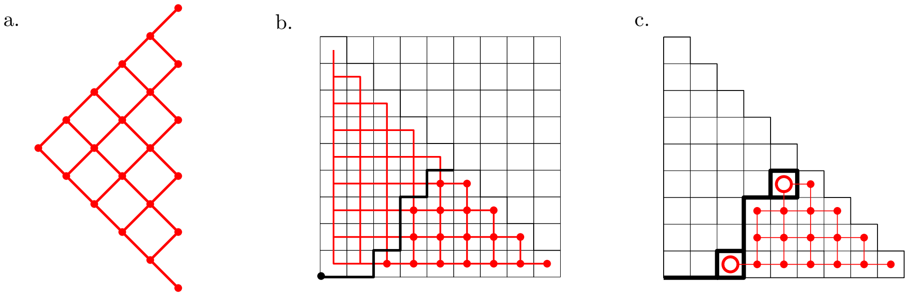

Fig. 7: a. The poset QP_6. b. A state of QR_n corresponds to an order ideal in QP_n c. The bijection between partial NE-paths and pairs (order ideal, maximal elements) of QP_n

Definition 4.5 *A* partial NE-path *is a path consisting of consecutive links which may be north edges, east edges, or unit squares, such that each unit square is attached to the rest of the path by its southwest and northeast corners. The* length *of a partial NE-path is $e + 2f$, where e is the number of edges and f is the number of squares. The partial NE-paths form a poset by containment, whose minimal elements are the NE-paths.*

To illustrate this definition, Figure 7.c shows a partial NE-path which contains the NE-path in b. Recall that $X(QP_n)$ is the rooted cube complex corresponding to the PIP QP_n under the bijection of Theorem 2.7. We use the notation of Definition 2.6.

Lemma 4.6 *The partial NE-paths of length n are in order-preserving bijection with the cubes of $X(QP_n)$.*

Lemma 4.7 *The partial NE-paths of length n are in order-preserving bijection with the cubes of the state complex $\mathcal{S}(QR_n)$.*

Proof: A k-cube C of $\mathcal{S}(QR_n)$ is given by a state u and k commutative moves $\varphi_1, \ldots, \varphi_k$ that can be applied to u. The state u is given by an NE-path, and each one of the k moves $m_1, \ldots, m_k$ corresponds to a corner of the NE-path that could be switched. The two positions of this corner before and after the move m_i form a square. Since the moves are commutative, two of these squares cannot share an edge. Adding these k squares to the NE-path u gives rise to a partial NE-path corresponding to the k-cube C.

Conversely, consider a partial NE-path with k squares. There are 2^k NE-paths contained in it, obtained by "resolving" each square into an NE or an EN corner. The resulting 2^k NE-paths form a cube of $\mathcal{S}(QR_n)$. This bijection is clearly order-preserving. □

Theorem 4.8 *The state complex of the robotic arm in an $n \times n$ grid is a CAT(0) cubical complex.*

Proof: This is an immediate consequence of Lemmas 4.6 and 4.7 and Theorem 2.7. □

As a corollary of our combinatorial description of the state complex of QR_n, we get:

Corollary 4.9 *If $q_{n,d}$ is the number of d-cubes in the state complex of the robot in a quadrant QR_n,*

$$\sum_{n,d \geq 0} q_{n,d}\, x^n y^d = \frac{1+xy}{1-2x-x^2y}.$$

5 The state complex of SR_n is CAT(0)

Now we carry out the same approach for the robotic arm in a strip SR_n.

Definition 5.1 *Define the PIP SP_n to be the set of lattice points inside the triangle $y \geq 0, y \leq 2x$, and $x \leq n-1$, with componentwise order (so $(x,y) \leq (x',y')$ if $x \leq x'$ and $y \leq y'$) and no inconsistent pairs.*

Proposition 5.2 *There is a bijection between the states of the robot SR_n and the order ideals of SP_n.*

This is proved similarly to Proposition 4.2. Details are given in [2]. If we declare the “home” state of SR_n to be the fully horizontal state h, then the poset of states of SR_n is a distributive lattice. We have,

Proposition 5.3 *The lattice of states of SR_n is isomorphic to the poset on the spread out subsets of $[n]$, where $A \leq B$ if $w(A) \geq w(B)$ coordinatewise.*

Proof: This is clear from the repellent hopping particles model for SR_n of Section 3.2.1. □

Definition 5.4 *A* partial F-path *is a partial NE-path such that the link following any vertical edge or square must be a horizontal edge.*

Recall that $X(SP_n)$ is the rooted cube complex corresponding to the PIP SP_n under the bijection of Theorem 2.7. We then have the following results. The proofs are essentially the same as those of Lemmas 4.6 and 4.7, Theorem 4.8.

Lemma 5.5 *The partial F-paths of length n are in bijection with the cubes of the state complex of $X(SP_n)$.*

Lemma 5.6 *The partial F-paths of length n are in bijection with the cubes of the state complex of SR_n.*

Theorem 5.7 *The state complex of the robotic arm in an $n \times n$ grid is a CAT(0) cubical complex.*

As a corollary of our combinatorial description of the state complex of SR_n, we get:

Corollary 5.8 *If $s_{n,d}$ is the number of d-cubes in the state complex of the robot in a strip SR_n,*

$$\sum_{n,d \geq 0} s_{n,d}\, x^n y^d = \frac{1+x+xy+x^2y}{1-x-x^2-x^3y},$$

6 Finding the optimal path between two states

Consider a robot, or some other reconfigurable system $\mathcal{R}$, whose state complex $\mathcal{S}(\mathcal{R})$ is CAT(0). As in the two examples above, there may be a natural choice of a “home state” u, such that the PIP P_u corresponding to the rooted complex $(\mathcal{S}(\mathcal{R}), u)$ has a particularly simple description. Now suppose that we want to take the robot from state a to state b in an optimal way. Equivalently, we wish to get from vertex a to vertex b of the state complex $\mathcal{S}(\mathcal{R})$.

6.1 Rerooting the complex

To find the optimal path from a to b, the first step will be to reroot the complex at a, and find the PIP P_a corresponding to the rooted CAT(0) cubical complex $(\mathcal{S}(\mathcal{R}), a)$. Fortunately, this is very easy to do.

Notation 6.1 *If p and q are an inconsistent pair in a PIP, write $p \nleftrightarrow q$.*

Proposition 6.2 *Let u and a be vertices of the CAT(0) cube complex X and let P_u and P_a be the PIPs corresponding to the rooted complexes (X, u) and (X, a) respectively. Let I be the consistent order ideal of P_u corresponding to a, and let $J = P_u - I$. The PIP P_a has an element p' corresponding to each element $p \in P_u$, and it can be described in terms of P_u as follows:*

- *If $j_1 < j_2$ in P_u, then $j_1' < j_2'$ in P_a.*
- *If $i_1 < i_2$ in P_u then $i_1' < i_2'$ in P_a.*
- *If $i < j$ in P_u then $i' \nleftrightarrow j'$ in P_a.*
- *If $j_1 \nleftrightarrow j_2$ in P_u, then $j_1' \nleftrightarrow j_2'$ in P_a.*
- *If $i \nleftrightarrow j$ in P_u then $i' < j'$ in P_a.*

Here the is and the js represent arbitrary elements of I and J, respectively.(i)

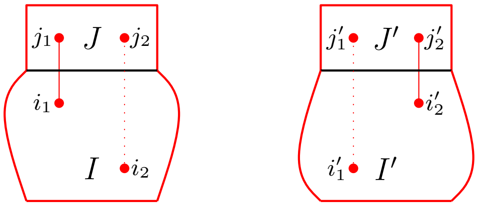

Fig. 8: The PIPs P_u and P_a before and after rerooting the CAT(0) cube complex.

Corollary 6.3 *The Hasse diagram of P_a is obtained from that of P_u by turning I upside down, and converting all solid edges from I to J into dotted edges, and vice versa.*

Note that even if P_u has no inconsistent pairs, the PIP P_a probably will have inconsistent pairs. Now that we have rerooted the complex, our goal is to get from the root a to the vertex b optimally. There are at least four notions of "optimality": we may wish to minimize Euclidean distance, number of moves, simultaneous moves, or time. We can solve these four problems.

(i) Notice that we never have $i > j$ or $i_1 \nleftrightarrow i_2$ in P_u.

6.2 Minimizing the Euclidean distance

Suppose we want to find the shortest path from a to b in the Euclidean metric of the cubical complex $\mathcal{S}(\mathcal{R})$. This can be accomplished using Ardila, Owen, and Sullivant's algorithm [3] to compute the shortest path from a to b. As explained there, a prerequisite for this is to write down the PIP P_a, which we have done in Proposition 6.2. This metric is very useful in some applications, particularly when navigating the space of phylogenetic trees [4, 11]. However, this metric does not seem natural for the robotic applications we have in mind here. It is probably more natural to consider the following three variants.

6.3 Minimizing the number of moves

Suppose we are only allowed to perform one move at a time. Geometrically, we are looking for a shortest edge-path from a to b. Let B be the consistent order ideal of P_a corresponding to vertex b in the rooted complex $(\mathcal{S}(\mathcal{R}), a)$. We can regard B as a subposet of P_a. The following description makes it clear how to construct the minimal shortest paths.

Proposition 6.4 *The shortest edge-paths from a to b are in one-to-one correspondence with the linear extensions of the poset B. Their length is $|B|$.*

6.4 Minimizing the sequence of simultaneous moves

Now suppose that we can move the robot in steps, where at each step we can perform several moves at a time with no penalty. Geometrically, we are looking for a shortest cube path from a to b, where at each step we cross a cube from the current vertex to the one across the diagonal. Again, let B be the consistent order ideal of P_a corresponding to b. Let the *depth* $d(B)$ of B be the size of the longest chain(s) in B.

Definition 6.5 *Let the* normal cube path *from a to b be the cube path given by the sequence of order ideals $\mathbf{M} : \emptyset = M_0 \subset M_1 \subset \cdots \subset M_{d(B)} = B$, where each ideal is obtained from the previous one by adding to it all the minimal elements that have not yet been added. In other words, $M_{k+1} := M_k \cup (B - M_k)_{min}$.*

The previous definition is due to Niblo and Reeves [10] in a different language; the correspondence with PIPs makes these paths more explicit. It also allows us to give a simple proof of the following result from Reeves's Ph.D. thesis [12] in [2]:

Proposition 6.6 *The shortest cube paths from a to b have size $d(B)$. In particular, the normal cube path from a to b is minimal.*

6.5 Minimizing time

Perhaps the most realistic model is to allow ourselves to move the robot continuously in time, where we can perform several moves simultaneously, as long as these moves are physically independent. We can even perform only part of a move, and perform the rest of the move later. Each move still takes one unit of time, and there is no time penalty for multitasking.

Geometrically, we are endowing each cube with the ℓ_∞ metric: For $\mathbf{x}, \mathbf{y}$ in a unit d-cube, we let $||\mathbf{x} - \mathbf{y}|| := \max(x_1 - y_1, \ldots, x_d - y_d)$. Now we are looking for a shortest path from a to b with respect to this ℓ_∞ metric. The following result, stated without proof in [1], shows that the added flexibility of performing partial moves does not actually help us move our robots more quickly.

Proposition 6.7 *The fastest paths from a to b take $d(B)$ units of time. In particular, the normal cube path from a to b is a fastest path.*

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