Semi-skyline augmented fillings and non-symmetric Cauchy kernels for stair-type shapes

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Abstract. Using an analogue of the Robinson-Schensted-Knuth (RSK) algorithm for semi-skyline augmented fillings, due to Sarah Mason, we exhibit expansions of non-symmetric Cauchy kernels $\prod_{(i,j)\in\eta}(1 - x_i y_j)^{-1}$, where the product is over all cell-coordinates (i, j) of the stair-type partition shape η, consisting of the cells in a NW-SE diagonal of a rectangle diagram and below it, containing the biggest stair shape. In the spirit of the classical Cauchy kernel expansion for rectangle shapes, this RSK variation provides an interpretation of the kernel for stair-type shapes as a family of pairs of semi-skyline augmented fillings whose key tableaux, determined by their shapes, lead to expansions as a sum of products of two families of key polynomials, the basis of Demazure characters of type A, and the Demazure atoms. A previous expansion of the Cauchy kernel in type A, for the stair shape was given by Alain Lascoux, based on the structure of double crystal graphs, and by Amy M. Fu and Alain Lascoux, relying on Demazure operators, which was also used to recover expansions for Ferrers shapes.

Résumé. En utilisant an analogue de l'algorithme de Robinson-Schensted-Knuth (RSK) pour remplissages des lignes d'horizon augmentées, proposé par Sarah Mason, nous donnons des développements d'un noyau de Cauchy non symétrique, $\prod_{(i,j)\in\eta}(1 - x_i y_j)^{-1}$, dans le cas où les paires (i, j) sont les coordonnées des cellules d'une partition η du type escalier dans un rectangle, contenant la plus grande partition escalier de ce rectangle. Dans l'esprit du développement classique sur les diagrammes rectangulaires, cette variation de RSK fournit une somme des produits de deux familles de polynômes clefs, engendrée par paires de remplissages des lignes d'horizon augmentées dont les formats définissent tableaux clefs, à savoir, la base des caractères de Demazure du type A et les Demazure atomes. Un développement du noyau de Cauchy non symétrique pour le type A, dans le cas de la partition escalier, a été donné par Alain Lascoux en employant la structure des graphes cristallins doublés, et par Amy M. Fu et Alain Lascoux, en se basant aux opérateurs de Demazure, qui a été aussi utilisé pour obtenir des expansions sur diagrammes de Ferrers.

Keywords: Non-symmetric Cauchy kernels, Demazure character, key polynomial, Demazure operator, semi-skyline augmented filling, RSK analogue.

[†]This work was partially supported by the Centro de Matemática da Universidade de Coimbra (CMUC), funded by the European Regional Development Fund through the program COMPETE and by the Portuguese Government through the FCT - Fundação para a Ciência e a Tecnologia under the project PEst-C/MAT/UI0324/2011.

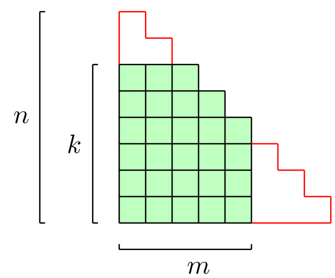
[‡]This work was partially supported by the Centro de Matemática da Universidade de Coimbra (CMUC), funded by the European Regional Development Fund through the program COMPETE and by the Portuguese Government through the FCT - Fundação para a Ciência e a Tecnologia under the project PEst-C/MAT/UI0324/2011, and Fundação para a Ciência e a Tecnologia, Grant SFRH / BD / 33700 / 2009.

1 Introduction

Given the general Lie algebra $\mathfrak{gl}_n(\mathbb{C})$, and its quantum group $U_q(\mathfrak{gl}_n)$, finite-dimensional representations of $U_q(\mathfrak{gl}_n)$ are also classified by the highest weight. Let λ be a dominant integral weight (that is, a partition) and $V(\lambda)$ the integrable representation with highest weight λ and u_λ the highest weight vector. For a given permutation w in the symmetric group $\mathfrak{S}_n$, minimum for the Bruhat order in the class modulo the stabilizer of λ, the Demazure module is defined to be $V_w(\lambda) := U_q(\mathfrak{g})^{>0}.u_{w\lambda}$, and the Demazure character is the character of $V_w(\lambda)$. Kashiwara (1991) has associated with λ a crystal graph $\mathfrak{B}_\lambda$, which can be realised as a coloured directed graph whose vertices are all semi-standard Young tableaux (SSYTs) of shape λ in the alphabet $[n]$, and the edges are coloured with a colour i, for each pair of crystal operators $f_i,\ e_i$, such that there exists a coloured i-arrow from the vertex P to P' if and only if $f_i(P) = P'$, equivalently, $e_i(P') = P$, for $1 \le i \le n-1$. Littelmann (1995) conjectured and Kashiwara (1993) proved that the intersection of a crystal basis of V_λ with $V_w(\lambda)$ is a crystal basis for $V_w(\lambda)$. The resulting subset $\mathfrak{B}_{w\lambda} \subseteq \mathfrak{B}_\lambda$ is called Demazure crystal, and the Demazure character corresponding to λ and w, is the sum of the monomial weights of SSYTs in the Demazure crystal $\mathfrak{B}_{w\lambda}$.

Demazure characters (or key polynomials) are also defined through Demazure operators (or isobaric divided differences). They were introduced by Demazure (1974) for all Weyl groups and were studied combinatorially, in the case of $\mathfrak{S}_n$, by Lascoux and Schützenberger (1990) who produce a crystal structure. The simple transpositions s_i of $\mathfrak{S}_n$ act on vectors $v \in \mathbb{N}^n$ by $s_i v := (v_1, \dots, v_{i+1}, v_i \dots, v_n)$, for $1 \le i \le n-1$, and induce an action of $\mathfrak{S}_n$ on $\mathbb{Z}[x_1, \dots, x_n]$ by considering vectors v as exponents of monomials $x^v := x_1^{v_1} x_2^{v_2} \cdots x_n^{v_n}$. Two families of Demazure operators $\pi_i,\ \widehat{\pi}_i$ on $\mathbb{Z}[x_1, \dots, x_n]$ are defined by $\pi_i f = \frac{x_i f - x_{i+1} s_i(f)}{x_i - x_{i+1}}$ and $\widehat{\pi}_i f = \pi_i f - f$, for $1 \le i \le n-1$. For the partition λ and $w = s_{i_N} \cdots s_{i_2} s_{i_1}$ a reduced decomposition in $\mathfrak{S}_n$, one defines the type A key polynomials indexed by $w\lambda$, $\kappa_{w\lambda}(x) = \pi_{i_N} \cdots \pi_{i_2} \pi_{i_1} x^\lambda$ and $\widehat{\kappa}_{w\lambda}(x) = \widehat{\pi}_{i_N} \cdots \widehat{\pi}_{i_2} \widehat{\pi}_{i_1} x^\lambda$, the latter consisting of all monomials in $\kappa_{w\lambda}$ which do not appear in $\kappa_{\sigma\lambda}$ for any $\sigma < w$ in the Bruhat order. Thereby key polynomials can be decomposed into non intersecting pieces $\kappa_{w\lambda}(x) = \sum_{\nu \le w} \widehat{\kappa}_{\nu\lambda}(x)$, where the ordering on permutations is the Bruhat order in $\mathfrak{S}_n$. In Lascoux and Schützenberger (1990) they are called *standard basis* and in Mason (2009) Demazure atoms. The Demazure character corresponding to w and λ can be expressed in terms of the Demazure operator and is equivalent to the key polynomial $\kappa_{w\lambda}$. Lascoux and Schützenberger (1990) have given a combinatorial interpretation for Demazure operators in terms of crystal operators to produce a crystal graph structure. Let P be a SSYT of shape λ and define the set $f_{s_i}(P) := \{f_i^m(P) : m \ge 0\} \setminus \{0\}$. If P is the head of an i-string of the crystal graph $\mathfrak{B}_\lambda$, $\pi_i(x^P)$ is the sum of the monomial weights of all SSYTs in $f_{s_i}(P)$. In particular, when Y is the Yamanouchi tableau of shape λ, the set $f_w(Y) := \{f_{i_N}^{m_N} \dots f_{i_1}^{m_1}(Y) : m_k \ge 0\} \setminus \{0\}$ constitutes the vertices of the Demazure crystal $\mathfrak{B}_{w\lambda}$, and $\kappa_{w\lambda}$ is the sum of all monomial weights over the Demazure crystal. The top of this crystal graph $\widehat{\mathfrak{B}}_{w\lambda} := \mathfrak{B}_{w\lambda} \setminus \bigcup_{\sigma < w} \mathfrak{B}_{\sigma\lambda}$ defines the Demazure atom $\widehat{\kappa}_{w\lambda}(x)$ which is combinatorially characterised by Lascoux and Schützenberger (1990) as the sum of the monomial weights of all SSYTs whose right key is $key(w\lambda)$.

As the sum of the monomial weights over all crystal graph $\mathfrak{B}_\lambda$ gives the Schur polynomial s_λ, each SSYT of shape λ appears in precisely one such polynomial, henceforth, the Demazure atoms form a decomposition of Schur polynomials. Specialising the combinatorial formula for nonsymmetric Macdonald polynomials $E_\gamma(x;q;t)$, given in Haglund et al. (2008), by setting $q = t = 0$, implies that $E_\gamma(x;0;0)$ is the sum of the monomial weights of all semi-skyline augmented fillings (SSAF) of shape γ which are fillings of composition diagrams with positive integers, weakly decreasing upwards along columns, and the rows satisfy an inversion condition. These polynomials are also a decomposition of the Schur polynomial s_λ, with $\gamma^+ = \lambda$. Semi-skyline augmented fillings are in bijection with semi-standard Young tableaux of the same content whose right key is the unique key with content the shape of the SSAF, Mason (2006/08). Therefore, Demazure atoms $\widehat{\kappa}_{w\lambda}(x)$ and $E_{w\lambda}(x;0;0)$ are equal, Mason (2009). Semi-skyline augmented fillings also satisfy a variation of the Robinson-Schensted-Knuth algorithm which commutes with the usual RSK and retains its symmetry. We are, therefore, endowed with a machinery to exploit expansions of non-symmetric Cauchy kernels $\prod_{(i,j)\in\eta}(1-x_iy_j)^{-1}$, where the product is over all cell-coordinates (i,j) of the diagram η in the French convention. Our main Theorem 4.2 exhibits a bijection between biwords in lexicographic order, whose biletters are cell-coordinates in a NW-SE diagonal of a rectangle and below it, containing the biggest stair shape, and pairs of SSAFs whose shapes satisfy an inequality in the Bruhat order. This allows to apply this variation of RSK for SSAFs to provide expansions for the green diagram $\eta = (m^{n-m+1}, m-1, \ldots, n-k+1)$, $1 \le m, k \le n$, $n+1 \le m+k$, depicted below. The formulas are explicit in the tableaux generating them.



The paper is organised as follows. In Section 2, we recall the tableau criterion for the Bruhat order in $\mathfrak{S}_n$, and its extension to weak compositions. In Section 3, we review the necessary theory of SSAFs, the variation of Schensted insertion and RSK for SSAFs. In Section 4, we give our main result, Theorem 4.2, and, in the last section, we apply it to the expand the Cauchy kernel for stair-type shapes.

2 Key tableaux a criterion for the Bruhat order in $\mathfrak{S}_n$

Let $\mathbb{N}$ denote the set of non-negative integers. Fix a positive integer n, and define $[n]$ the set $\{1,\ldots,n\}$. A weak composition $\gamma = (\gamma_1 \ldots, \gamma_n)$ is a vector in $\mathbb{N}^n$. If $\gamma_i = \cdots = \gamma_{i+k-1}$, for some $k \ge 1$, then we also write $\gamma = (\gamma_1 \ldots, \gamma_{i-1}, \gamma_i^k, \gamma_{i+k}, \ldots, \gamma_n)$. A partition is a weak composition whose entries are in weakly decreasing order, that is, $\gamma_1 \ge \cdots \ge \gamma_n$. Every composition γ determines a unique partition γ^+ obtained by arranging the entries of γ in weakly decreasing order. A partition $\lambda = (\lambda_1, \ldots, \lambda_n)$ is identified with its Young diagram $dg(\lambda)$ in French convention, an array of left-justified cells with λ_i cells in row i from the bottom, for $1 \le i \le n$. The cells are located in the diagram $dg(\lambda)$ by their row and column indices (i,j), where $1 \le i \le n$ and $1 \le j \le \lambda_i$. A filling of shape λ is a map $T : dg(\lambda) \to [n]$. A semi-standard Young tableau (SSYT) of shape λ is a filling of $dg(\lambda)$ weakly increasing in each row from left to right and strictly increasing up in each column. The content or weight of SSYT T is the weak composition $c(T) = (\alpha_1, \ldots, \alpha_n)$ such that T has α_i cells with entry i. A key is a SSYT such that the set of entries in the $(j+1)^{th}$ column is a subset of the set of entries in the j^{th} column, for all j. There is a bijection in Reiner and Shimozono (1995) between weak compositions in $\mathbb{N}^n$ and keys in the alphabet $[n]$

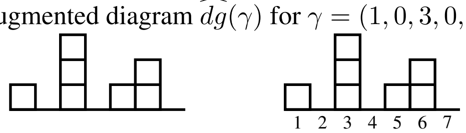
given by $\gamma \to key(\gamma)$, where $key(\gamma)$ is the key such that for all j, the first γ_j columns contain the letter j. Any key tableau is of the form $key(\gamma)$ with γ its content and γ^+ the shape.

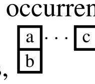
Suppose u and v are two rearrangements of a partition λ. We write $u \leq v$ in the (strong) *Bruhat order* whenever $key(u) \leq key(v)$ for the entrywise comparison. If σ and β are in $\mathfrak{S}_n$, $\sigma \leq \beta$ in the Bruhat order if and only if $\sigma(n, n-1, \dots, 1) \leq \beta(n, n-1, \dots, 1)$ as weak compositions.

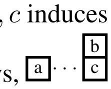
3 Semi-skyline augmented fillings

3.1 Definitions and properties

We follow most of the time the conventions and terminology in Haglund et al. (2005, 2008) and Mason (2006/08, 2009). A weak composition $\gamma = (\gamma_1, \dots, \gamma_n)$ is visualised as a diagram consisting of n columns, with γ_j boxes in column j. Formally, the *column diagram* of γ is the set $dg'(\gamma) = \{(i, j) \in \mathbb{N}^2 : 1 \leq j \leq n, 1 \leq i \leq \gamma_j\}$ where the coordinates are in French convention, the abscissa i indexing the rows, and the ordinate j indexing the columns. (The prime reminds that the components of γ are the columns.) The number of cells in a column is called the height of that column and a cell a in a column diagram is denoted $a = (i, j)$, where i is the row index and j is the column index. The *augmented diagram* of γ, $\widehat{dg}(\gamma) = dg'(\gamma) \cup \{(0, j) : 1 \leq j \leq n\}$, is the column diagram with n extra cells adjoined in row 0. This adjoined row is called the *basement* and it always contains the numbers 1 through n in strictly increasing order. The shape of $\widehat{dg}(\gamma)$ is defined to be γ. For example, the column diagram $dg'(\gamma)$ and the augmented diagram $\widehat{dg}(\gamma)$ for $\gamma = (1, 0, 3, 0, 1, 2, 0)$ are respectively,

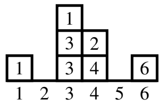


An *augmented filling* F of an augmented diagram $\widehat{dg}(\gamma)$ is a map $F : \widehat{dg}(\gamma) \to [n]$, which can be pictured as an assignment of positive integer entries to the non-basement cells of $\widehat{dg}(\gamma)$. Let $F(i)$ denote the entry in the i^{th} cell of the augmented diagram encountered when F is read across rows from left to right, beginning at the highest row and working down to the bottom row. This ordering of the cells is called the *reading order*. A cell $a = (i, j)$ precedes a cell $b = (i', j')$ in the reading order if either $i' < i$ or $i' = i$ and $j' > j$. The reading word of F is obtained by recording the non-basement entries in reading order. The content of an augmented filling F is the weak composition $c(F) = (\alpha_1, \dots, \alpha_n)$ where α_i is the number of non-basement cells in F with entry i, and n is the number of basement elements. The *standardization* of F is the unique augmented filling that one obtains by sending the i^{th} occurrence of j in the reading order to $i + \sum_{m=1}^{j-1} \alpha_m$. Let $a, b, c \in \widehat{dg}(\gamma)$ three cells situated as follows, 

where a and c are in the same row, possibly the first row, possibly with cells between them, and the height of the column containing a and b is greater than or equal to the height of the column containing c. Then the triple a, b, c is an *inversion triple of type 1* if and only if after standardization the ordering from smallest to largest of the entries in cells a, b, c induces a counterclockwise orientation. Similarly, consider three cells $a, b, c \in \widehat{dg}(\gamma)$ situated as follows, 

where a and c are in the same row (possibly the basement) and the column containing b and c has strictly greater height than the column containing a. The triple a, b, c is an *inversion triple of type 2* if and only if after standardization ordering from smallest to largest of the entries in cells $a,\ b,\ c$ induces a clockwise orientation.

Define a *semi-skyline augmented filling* (SSAF) of an augmented diagram $\widehat{dg}(\gamma)$ to be an augmented filling F such that every triple is an inversion triple and columns are weakly decreasing from bottom to top. The shape of the semi-skyline augmented filling is γ and denoted by $sh(F)$. The picture below is an example of a semi-skyline augmented filling with shape $(1, 0, 3, 2, 0, 1)$, reading word 1321346 and content $(2, 1, 2, 1, 0, 1)$.



The entry of a cell in the first row of a SSAF is equal to the basement element where it sits and, thus, in the first row the cell entries increase from left to the right. For any weak composition γ in $\mathbb{N}^n$, there is at least one SSAF with shape γ, by putting γ_i cells with entries i in the top of the basement element i.

In Mason (2006/08) a sequence of lemmas provide several conditions on triples of cells in a SSAF. We recall a property regarding an inversion triple of type 2 which will be used in the proof of our main theorem. Given a cell a in SSAF F define $F(a)$ to be the entry in a.

Remark 3.1 1. *If* $\{a, b, c\}$ *is a type* 2 *inversion triple in* F *then* $F(a) < F(b) \leq F(c)$.

3.2 An analogue of Schensted insertion and RSK for SSAF.

The fundamental operation of the Robinson-Schensted-Knuth (1970) (RSK) algorithm is Schensted insertion which is a procedure for inserting a positive integer k into a SSYT T. Mason (2006/08) defines a similar procedure for inserting a positive integer k into a SSAF F, which is used to describe an analogue of the RSK algorithm. If F is a SSAF of shape γ, we set $F := (F(j))$, where $F(j)$ is the entry in the j^{th} cell in reading order, with the cells in the basement included, and j goes from 1 to $n + \sum_{i=1}^{n} \gamma_i$. If $\hat{j}$ is the cell immediately above j and the cell is empty, set $F(\hat{j}) = 0$. The operation $k \to F$, for $k \leq n$, is defined as follows.

Procedure. The insertion $k \to F$:

1. Set $i := 1$, set $x_1 := k$, set $p_0 = \emptyset$, and set $j := 1$.

2. If $F(j) < x_i$ or $F(\hat{j}) \geq x_i$, then increase j by 1 and repeat this step. Otherwise, set $x_{i+1} := F(\hat{j})$ and set $F(\hat{j}) := x_i$. Set $p_i = (b + 1, a)$, where (b, a) is the j^{th} cell in reading order. (This means that the entry x_i "$bumps$" the entry x_{i+1} from the cell p_i.)

3. If $x_{i+1} \neq 0$ then increase i by 1, increase j by 1, and repeat step 2.

4. Set t_k equal to p_i, which is the termination cell, and terminate the algorithm.

The procedure terminates in finitely many steps and the result is a SSAF. Based on this Schensted insertion analogue, it is given a weight preserving and a shape rearranging bijection Ψ between SSYT and SSAF over the alphabet $[n]$. The bijection Ψ is defined to be the insertion, from right to left, of the column word which consists of the entries of each column, read top to bottom from columns left to rigth, of a SSYT into the empty SSAF with basement $[n]$. The bijection together with the shape of $\Psi(T)$ provides the right key of T, $K_+(T)$, a notion due to Lascoux and Schützenberger (1990).

Theorem 3.1 [Mason (2009)] *Given an arbitrary SSYT* T*, let* γ *be the shape of* $\Psi(T)$*. Then* $K_+(T) = key(\gamma)$.

It should be observed that Willis (2011) gives another way to calculate the right key of a SSYT.

Given the alphabet $[n]$, the RSK algorithm is a bijection between biwords in lexicographic order and pairs of SSYT of the same shape over $[n]$. Equipped with the Schensted insertion anlogue Mason

(2006/08) applies the same procedure to find an analogue Φ of the RSK for SSAF. This bijection has an advantage over the classical RSK because it comes along with the extra pair of right keys.

The two line array $w = \begin{pmatrix} i_1 & i_2 & \cdots & i_l \\ j_1 & j_2 & \cdots & j_l \end{pmatrix}$, $i_r < i_{r+1}$, or $i_r = i_{r+1}$ & $j_r \leq j_{r+1}$, $1 \leq i, j \leq l-1$, with $i_r,\ j_r \in [n]$, is called a biword in lexicographic order over the alphabet $[n]$. The map Φ defines a bijection between the set $\mathbb{A}$ of all biwords w in lexicographic order in the alphabet $[n]$, and pairs of SSAFs whose shapes are rearrangements of the same partition in $\mathbb{N}^n$ and the contents are respectively those of the second and first rows of w. Let $\mathbb{SSAF}$ be the set of all SSAFs with basement $[n]$.

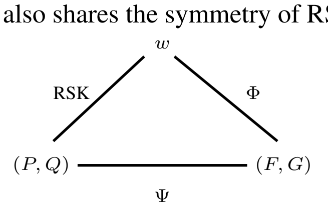
Procedure. The map $\Phi : \mathbb{A} \longrightarrow \mathbb{SSAF} \times \mathbb{SSAF}$. Let $w \in \mathbb{A}$.

1. Set $r := l$, where l is the number of biletters in w. Let $F = \emptyset = G$, where $\emptyset$ is the empty $SSAF$.
2. Set $F := (j_r \to F)$. Let h_r be the height of the column in $(j_r \to F)$ at which the insertion procedure $(j_r \to F)$ terminates.
3. Place i_r on top of the leftmost column of height $h_r - 1$ in G such that doing so preserves the decreasing property of columns from bottom to top. Set G equal to the resulting figure.
4. If $r - 1 \neq 0$, repeat step 2 for $r := r - 1$. Else terminate the algorithm.

Remark 3.2 1. *The entries in the top row of the biword are weakly increasing when read from left to right. Henceforth, if $h_r > 1$, placing i_r on top of the leftmost column of height $h_r - 1$ in G preserves the decreasing property of columns. If $h_r = 1$, the i_r^{th} column of G does not contain an entry from a previous step. It means that number i_r sits on the top of basement i_r.*
2. *Let h be the height of the column in F at which the insertion procedure $(j \to F)$ terminates. Remark 3.1, implies that there is no column of height $h + 1$ in F to the right.*

Corollary 3.2 [Mason (2006/08, 2009)] *The RSK algorithm commutes with the above analogue Φ. That is, if (P, Q) is the pair of SSYT produced by RSK algorithm applied to biword w, then $(\Psi(P), \Psi(Q)) = \Phi(w)$, and $K_+(P) = key(sh(\Psi(P)))$, $K_+(Q) = key(sh(\Psi(Q)))$.*

This result is summarised in the following scheme from which, in particular, it is clear the RSK analogue Φ also shares the symmetry of RSK.



$$c(P) = c(Q) = c(F) = c(G),$$
$$sh(F)^+ = sh(G)^+ = sh(P) = sh(Q),$$
$$K_+(P) = key(sh(F)),\ K_+(Q) = key(sh(G)).$$

4 Main Theorem

We give a bijection between biwords, in lexicographic order, whose biletters are cell-coordinates in a NW-SE diagonal of a rectangle diagram, and below it, containing the biggest stair shape, and pairs of SSAFs whose shapes satisfy an inequality in the Bruhat order.

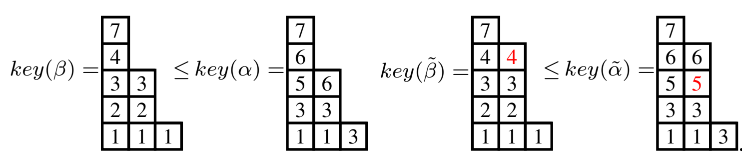
Lemma 4.1 *Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ and $\beta = (\beta_1, \beta_2, \ldots, \beta_n)$ be two weak compositions in $\mathbb{N}^n$, rearrangements of each other, with $key(\beta) \leq key(\alpha)$. Given $k \in \{1, \ldots, n\}$, let $k' \in \{1, \ldots, n\}$ be such that $\beta_{k'}$ is the left most entry of β satisfying $\alpha_k = \beta_{k'}$. Then if $\tilde{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_k + 1, \ldots, \alpha_n)$ and $\tilde{\beta} = (\beta_1, \beta_2, \ldots, \beta_{k'} + 1, \ldots, \beta_n)$, it holds $key(\tilde{\beta}) \leq key(\tilde{\alpha})$.*

Proof: Let $k,\ k' \in \{1,\ldots,n\}$ as in the lemma, and put $\alpha_k = \beta_{k'} = m \geq 1$. (The proof for $m=0$ is left to the reader. The case of interest for our problem is $m>0$ which is related with the procedure of map Φ.) This means that k appears exactly in the first m columns of $key(\alpha)$, and k' is the smallest number that does not appear in column $m+1$ of $key(\beta)$ but appears exactly in the first m columns. Let t be the row index of the cell with entry k' in column m of $key(\beta)$. Every entry less than k' in column m of $key(\beta)$ appears in column $m+1$ as well, and since in a key tableau each column is contained in the previous one, this imply that the first t rows of columns m and $m+1$ of $key(\tilde\beta)$ are equal. The only difference between $key(\tilde\beta)$ and $key(\beta)$ is in columns $m+1$, from row t to the top. Similarly if z is the row index of the cell with entry k in column $m+1$ of $key(\tilde\alpha)$, the only difference between $key(\tilde\alpha)$ and $key(\alpha)$ is in columns $m+1$ from row z to the top. To obtain column $m+1$ of $key(\tilde\beta)$, shift in the column $m+1$ of $key(\beta)$ all the cells with entries $> k'$ one row up, and add to the position left vacant (of row index t) a new cell with entry k'. The column $m+1$ of $key(\tilde\alpha)$ is obtained similarly, by shifting one row up in the column $m+1$ of $key(\alpha)$ all the cells with entries $> k$ and adding a new cell with entry k in the vacant position. Put $p = \min\{t,z\}$ and $q = \max\{t,z\}$. We divide the columns $m+1$ in each pair $key(\beta),\ key(\tilde\beta)$ and $key(\alpha),\ key(\tilde\alpha)$ into three parts: the first, from row one to row $p-1$; the second, from row p to row q; and the third, from row $q+1$ to the top row. The first parts of column $m+1$ of $key(\tilde\beta)$ and $key(\beta)$ are the same, equivalently, for $key(\tilde\alpha)$ and $key(\alpha)$. The third part of column $m+1$ of $key(\tilde\beta)$ consists of row q plus the third part of $key(\beta)$, equivalently, for $key(\tilde\alpha)$ and $key(\alpha)$. As columns $m+1$ of $key(\beta)$ and $key(\alpha)$ are entrywise comparable, the same happens to the third parts of columns $m+1$ in $key(\tilde\beta)$ and $key(\tilde\alpha)$. It remains to analyse the second parts of the pair $key(\tilde\beta),\ key(\tilde\alpha)$ which we split into two cases according to the relative magnitude of p and q.

Case 1. $p=t<q=z$. Let $k' < b_t < \cdots < b_{z-1}$ and $d_t < \cdots < d_{z-1} < k$ be respectively the cell entries of the second parts of columns $m+1$ in the pair $key(\tilde\beta),\ key(\tilde\alpha)$. By construction $k' < b_t \leq d_t < d_{t+1}$, $b_i < b_{i+1} \leq d_{i+1}$, $t<i<z-2$, and $b_{z-1} \leq d_{z-1} < k$, and, therefore, the second parts are also comparable.

Case 2. $p=z\leq q=t$. In this case, the assumption on k' implies that the first q rows of columns m and $m+1$ of $key(\tilde\beta)$ are equal. On the other hand, since column m of $key(\beta)$ is less or equal than column m of $key(\alpha)$, which is equal to the column m of $key(\tilde\alpha)$ and in turn is less or equal to column $m+1$ of $key(\tilde\alpha)$, forces by transitivity that the second part of column $m+1$ of $key(\tilde\beta)$ is less or equal than the corresponding part of $key(\tilde\alpha)$. □

We illustrate the lemma with $\beta = (3,2^2,1,0^2,1)$, $\alpha=(2,0,3,0,1,2,1)$, $\tilde\beta = (3,2^3,0^2,1)$, and $\tilde\alpha = (2,0,3,0,2^2,1)$,



Theorem 4.2 *Let w be a biword in lexicographic order in the alphabet $[n]$, and let $\Phi(w)=(F,G)$. For each biletter $\begin{pmatrix} i \\ j \end{pmatrix}$ in w one has $i+j\leq n+1$ if and only if $key(sh(G)) \leq key(\omega sh(F))$, where ω is the longest permutation of $\mathfrak{S}_n$. Moreover, if the first [respectively the second] row of w is a word in the alphabet $[m]$, with $1\leq m\leq n$, the shape of G [respectively F] has the last $n-m$ entries equal to zero.*

Proof: *"Only if part"*. We prove by induction on the number of biletters of w. If w is the empty word then F and G are the empty semi-skyline and there is nothing to prove. Let $w' = \begin{pmatrix} i_{p+1} & i_p & \cdots & i_1 \\ j_{p+1} & j_p & \cdots & j_1 \end{pmatrix}$ be a biword in lexicographic order such that $p \geq 0$ and $i_t + j_t \leq n+1$ for all $1 \leq t \leq p+1$, and $w = \begin{pmatrix} i_p \cdots i_1 \\ j_p \cdots j_1 \end{pmatrix}$ such that $\Phi(w) = (F, G)$. Let $F' := (j_{p+1} \to F)$ and h the height of the column in F' at which the insertion procedure terminates. There are two possibilities for h which the third step of the algorithm procedure of Φ requires to consider.

• $h = 1$. It means j_{p+1} is sited on the top of the basement element j_{p+1} in F and therefore i_{p+1} goes to the top of the basement element i_{p+1} in G. Let G' be the semi-skyline obtained after placing i_{p+1} in G. As $i_{p+1} \leq i_t$, for all t, i_{p+1} is the bottom entry of the first column in $key(sh(G'))$ whose remain entries constitute the first column of $key(sh(G)$. Suppose $n+1-j_{p+1}$ is added to the row z of the first column in $key(\omega sh(F))$ by shifting one row up all the entries above it. Let $i_{p+1} < a_1 < \cdots < a_z < a_{z+1} < \cdots < a_l$ and $b_1 < b_2 < \cdots < n+1-j_{p+1} < b_z < \cdots < b_l$ be respectively the cell entries of the first columns in the pair $key(sh(G'))$, $key(\omega sh(F'))$, where $a_1 < \cdots < a_z < \cdots < a_l$ and $b_1 < \cdots < b_z < \cdots < b_l$ are respectively the cell entries of the first columns in the pair $key(sh(G))$, $key(\omega sh(F))$. If $z = 1$, as $i_{p+1} \leq n+1-j_{p+1}$ and $a_i \leq b_i$ for all $1 \leq i \leq l$, then $key(sh(G')) \leq key(\omega sh(F'))$. If $z > 1$, as $i_{p+1} < a_1 \leq b_1 < b_2$, we have $i_{p+1} \leq b_1$ and $a_1 \leq b_2$. Similarly $a_i \leq b_i < b_{i+1}$, and $a_i < b_{i+1}$, for all $2 \leq i \leq z-2$. Moreover $a_{z-1} \leq b_{z-1} < n+1-j_{p+1}$, therefore, $a_{z-1} < n+1-j_{p+1}$. Also $a_i \leq b_i$ for all $z \leq i \leq l$. Hence. $key(sh(G')) \leq key(\omega sh(F'))$.

• $h > 1$. Place i_{p+1} on the top of the leftmost column of height $h-1$. This means by Lemma 4.1 $key(sh(G')) \leq key(\omega sh(F'))$.

"If part". We prove the contrapositive statement. If there exists a biletter $\begin{pmatrix} i \\ j \end{pmatrix}$ in w such that $i + j > n+1$, then at least one entry of $key(sh(G))$ is strictly bigger than the corresponding entry of $key(\omega sh(F))$. Let $w = \begin{pmatrix} i_p \cdots i_1 \\ j_p \cdots j_1 \end{pmatrix}$ be a biword in lexicographic order on the alphabet $[n]$, and $\begin{pmatrix} i_t \\ j_t \end{pmatrix}$ the first biletter in w, from right to left, with $i_t + j_t > n+1$. Set $F_0 = G_0 := \emptyset$, and for $d \geq 1$, let (F_d, G_d) be the pair of SSAFs obtained by the procedure of map Φ applied to $\begin{pmatrix} i_d \\ j_d \end{pmatrix}$ and (F_{d-1}, G_{d-1}). First apply the map Φ to the biword $\begin{pmatrix} i_{t-1} \cdots i_1 \\ j_{t-1} \cdots j_1 \end{pmatrix}$ to obtain the pair (F_{t-1}, G_{t-1}) of SSAFs whose right keys satisfy, by the "only if part" of the theorem, $key(sh(G_{t-1})) \leq key(\omega sh(F_{t-1}))$. Now insert j_t to F_{t-1}. As $i_k + j_k \leq n+1$, for $1 \leq k \leq t-1$, $i_k + j_k \leq n+1 < i_t + j_t$ and $i_t \leq i_k$, $1 \leq k \leq t-1$, then $j_t > j_k$, $1 \leq k \leq t-1$ and since w is in lexicographic order it implies $i_t < i_{t-1}$. Therefore j_t sits on the top of the basement element j_t in F_{t-1} and i_t sits on the top of the basement element i_t in G_{t-1}. It means that $n+1-j_t$ is added to the first row and first column of $key(\omega sh(F_{t-1}))$ and all entries in this column are shifted one row up. Similarly i_t is added to the first row and first column of $key(sh(G_{t-1}))$ and all the entries in this column are shifted one row up. As $i_t > n+1-j_t$ then the first columns of $key(sh(G_t))$ and $key(\omega sh(F_t))$ respectively, are not entrywise comparable, and we say that we have a "*problem*" in the key-pair $(key(sh(G_t)), key(\omega sh(F_t)))$. From now on "*problem*" means $i_t > n+1-j_t$ in some row of a pair of columns in the key-pair $(key(sh(G_d)), key(\omega sh(F_d)))$, with $d \geq t$. Let $d \geq t$ and denote by J the column with basement j_t in F_d, and by I the column with basement i_t in G_d. Let

$|J|$ and $|I|$ denote respectively the height of J and I, and let r_i and k_i denote the number of columns of height $\geq i \geq 1$, respectively, to the right of J and to the left of I.

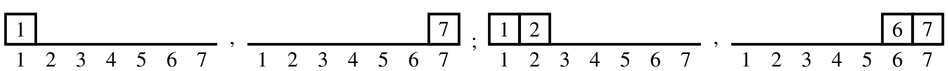
Classification of the "problem": For any $d \geq t$, either there exists $s \geq 1$ such that $|J|, |I| \geq s$, $r_s = k_s > 0$; or $1 \leq |J| \leq |I|$, and there exists $1 \leq f \leq |J|$, such that $k_i > r_i$, for $1 \leq i < f$, and $k_i = r_i = 0$, for $i \geq f$. In the first case, one has a "problem" in the $(r_s + 1)^{th}$ rows of the s^{th} columns in the key-pair $(key(sh(G_d)), key(\omega sh(F_d)))$. In the second case, one has a problem in the bottom of the $|J|^{th}$ columns.

The proof of this classification is mainly based on the Remark 3.2 which says that no insertion can terminate, to the left of J, on the top of a column of height $|J| - 1$ or $h - 1$ such that $r_h > r_{h+1}$, and, on the fact, that an insertion terminating to the right of J or on the top of J will contribute with a cell to the left or to the top of I. Therefore the original "problem" in the key-pair $(key(sh(G_t)), key(\omega sh(F_t)))$ will appear in another row or column in $(key(sh(G_d)), key(\omega sh(F_d)))$ but will never disappear with new insertions. Finally, if the second row of w is over the alphabet $[m]$, there is no cell on the top of the basement of F greater than m. Therefore, the shape of F has the last $n - m$ entries equal to zero. The other case is similar. □

Remark 4.1 *In the previous theorem if the rows of w are swapped, one obtains the biword $\tilde{w}$ such that $\Phi(\tilde{w}) = (G, F)$ with $key(sh(F) \leq key(\omega sh(G))$. Moreover, given $\nu \in \mathbb{N}^n$ and $\beta \leq \omega\nu$, there exists always a pair (F, G) of SSAFs with shapes ν and β respectively.*

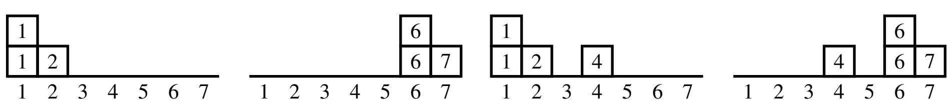
Two examples are now given to illustrate Theorem 4.2.

1. Given $w = \begin{pmatrix} 4 & 6 & 6 & 7 \\ 4 & 1 & 2 & 1 \end{pmatrix}$, $\Phi(w)$ and the key-pair $key(sh(G)) \leq key(\omega sh(F))$ are calculated.



$sh(F_1) = (1, 0^6)$ $sh(G_1) = (0^6, 1)$ $sh(F_2) = (1, 1, 0^5)$ $sh(G_2) = (0^5, 1, 1)$

$key(sh(G_1)) = 7 = key(\omega sh(F_1))$ $key(sh(G_2)) = 67 = key(\omega sh(F_2))$

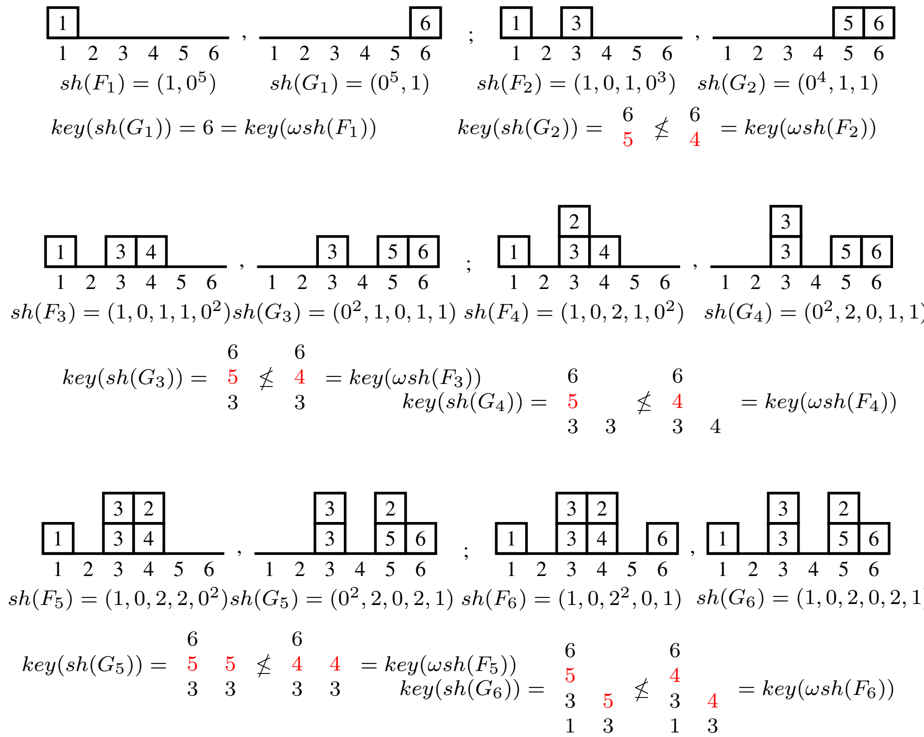


$sh(F_3) = (2, 1, 0^5)$ $sh(G_3) = (0^5, 2, 1)$ $sh(F_4) = (2, 1, 0, 1, 0^3)$ $sh(G_4) = (0^3, 1, 0, 2, 1)$

$key(sh(G_3)) = \begin{array}{cc} 7 & \\ 6 & 6 \end{array} \leq \begin{array}{cc} 7 & \\ 6 & 7 \end{array} = key(\omega sh(F_3))$

$key(sh(G_4)) = \begin{array}{cc} 7 & \\ 6 & \\ 4 & 6 \end{array} \leq \begin{array}{cc} 7 & \\ 6 & \\ 4 & 7 \end{array} = key(\omega sh(F_4))$

2. Let $w = \begin{pmatrix} 1 & 2 & 3 & 3 & 5 & 6 \\ 6 & 3 & 2 & 4 & 3 & 1 \end{pmatrix}$, with $n = 6,\ i_2 = 5 > 6 + 1 - 3$. We calculate $\Phi(w)$ whose key-pair $key(sh(G)), key(\omega sh(F))$ is not entrywise comparable.



5 Expansions of Cauchy kernels in stair-type shapes

The well-known Cauchy identity expresses the product $\prod_{i=1}^{n}\prod_{j=1}^{m}(1-x_iy_j)^{-1}$ as a sum of products of Schur functions in $x=(x_1,x_2,\dots,x_n)$ and $y=(y_1,y_2,\dots,y_m)$,

$$\prod_{(i,j)\in(m^n)}(1-x_iy_j)^{-1}=\prod_{i=1}^{n}\prod_{j=1}^{m}(1-x_iy_j)^{-1}=\sum_{\lambda}s_\lambda(x)s_\lambda(y), \tag{1}$$

over all partitions λ of length $\leq \min\{n,m\}$. Using either the RSK correspondence or the Φ correspondence, the Cauchy formula (1) can be interpreted as a bijection between monomials on the left hand side and pairs of SSYTs or SSAFs on the right. Now we replace in the Cauchy kernel the rectangle (m^n) by the stair-type shape $\lambda=(m^{n-m+1},m-1,\dots,n-k+1)$, with $1\leq m,k\leq n$, and $n+1\leq m+k$. In particular, when $m=n=k$, one has the stair-partition $\lambda=(n,n-1,\dots,1)$, that is, the cells (i,j) in the NW-SE diagonal of the square diagram (n^n) and below it. Thus $(i,j)\in\lambda$ if and only if $i+j\leq n+1$. Lascoux (2003) has given the following expansion for the non-symmetric Cauchy kernel in the stair shape, using double crystal graphs, and also Fu and Lascoux (2009), based on algebraic properties of Demazure operators,

$$\prod_{i+j\leq n+1}(1-x_iy_j)^{-1}=\sum_{\nu\in\mathbb{N}^n}\widehat{\kappa}_\nu(x)\kappa_{\omega\nu}(y), \tag{2}$$

where κ and $\hat{\kappa}$ are the two families of key polynomials, and ω is the longest permutation of $\mathfrak{S}_n$. Theorem 4.2 allows us to give an expansion of the non-symmetric Cauchy kernel for $\lambda = (m^{n-m+1}, m-1, m-2, \ldots, 1)$, for $1 \le m \le n$, and its conjugate $\overline{\lambda}$, which includes, in particular, the stair case shape (2),

$$\prod_{(i,j)\in\lambda} (1 - x_i y_j)^{-1} = \sum_{\substack{\nu\in\mathbb{N}^n \\ \nu=(\nu_1,\ldots,\nu_m,0^{n-m})}} \widehat{\kappa}_\nu(y)\kappa_{\omega\nu}(x), \tag{3}$$

$$\prod_{(i,j)\in\overline{\lambda}} (1 - x_i y_j)^{-1} = \sum_{\substack{\nu\in\mathbb{N}^n \\ \nu=(\nu_1,\ldots,\nu_m,0^{n-m})}} \widehat{\kappa}_\nu(x)\kappa_{\omega\nu}(y). \tag{4}$$

Write $\prod_{(i,j)\in\lambda}(1 - x_i y_j)^{-1} = \sum_{c\ge 0} x_{i_1} y_{j_1} \cdots x_{i_c} y_{j_c}$, where $(i_l, j_l) \in \lambda,\ i_l + j_l \le n+1,\ 1 \le i \le n,\ 1 \le j \le m,\ \ 1 \le l \le c$. Each monomial $x_{i_1} y_{j_1} \cdots x_{i_c} y_{j_c}$ is in correspondence with the biword $\binom{i_c\ \cdots\ i_1}{j_c\ \cdots\ j_1}$, whose image by Φ is the pair (F, G) of SSAFs. That is, $x_{i_1} y_{j_1} \cdots x_{i_c} y_{j_c} = y^F x^G$, where $sh(F)$ has the last $n-m$ entries equal zero, and $sh(G) \le \omega sh(F)$. Therefore,

$$\prod_{(i,j)\in\lambda} (1 - x_i y_j)^{-1} = \sum_{\substack{\nu\in\mathbb{N}^n \\ \nu=(\nu_1,\ldots,\nu_m,0^{n-m})}} \sum_{\substack{(F,G)\in SSAF \\ sh(F)=\nu \\ sh(G)\le\omega\nu}} y^F x^G = \sum_{\substack{\nu\in\mathbb{N}^n \\ \nu=(\nu_1,\ldots,\nu_m,0^{n-m})}} \sum_{\substack{F\in SSAF \\ sh(F)=\nu}} y^F \sum_{\substack{G\in SSAF \\ sh(G)\le\omega\nu}} x^G$$

$$= \sum_{\substack{\nu\in\mathbb{N}^n \\ \nu=(\nu_1,\ldots,\nu_m,0^{n-m})}} \Big(\sum_{\substack{P\in SSYT \\ sh(P)=\nu^+ \\ K_+(P)=key(\nu)}} y^P \Big)\Big(\sum_{\substack{Q\in SSYT \\ sh(Q)=\nu^+ \\ K_+(Q)=key(\beta) \\ \beta\le\omega\nu}} x^Q \Big) = \sum_{\substack{\nu\in\mathbb{N}^n \\ \nu=(\nu_1,\ldots,\nu_m,0^{n-m})}} \widehat{\kappa}_\nu(y)\kappa_{\omega\nu}(x). \tag{5}$$

The Cauchy kernel expansion (4) for the conjugate shape $\overline{\lambda} = (n, n-1, \ldots, n-m+1)$, with $1 \le m \le n$, is a consequence of (3), since $(i,j) \in \overline{\lambda}$ if and only if $(j,i) \in \lambda$, and the symmetry of Φ. When $n = m,\ \lambda = (n, n-1, \ldots, 1) = \overline{\lambda}$, and the symmetry of Φ means the two identities (2) and (3) are equivalent. Finally, as a refinement of (5), we obtain the expansion for the shape $\lambda = (m^{n-m+1}, m-1, \ldots, n-k+1)$, where $1 \le m \le k \le n$, and $n+1 \le m+k$,

$$\begin{aligned}
\prod_{(i,j)\in\lambda} (1 - x_i y_j)^{-1} &= \sum_{\substack{\nu\in\mathbb{N}^n \\ \nu=(\nu_1,\ldots,\nu_m,0^{n-m})}} \sum_{sh(F)=\nu} y^F \sum_{\substack{\beta\in\mathbb{N}^n \\ \beta=(\beta_1,\ldots,\beta_k,0^{n-k}) \\ \beta\le\omega\nu}} \sum_{sh(G)=\beta} x^G \\
&= \sum_{\substack{\nu\in\mathbb{N}^n \\ \nu=(\nu_1,\ldots,\nu_m,0^{n-m})}} \widehat{\kappa}_\nu(y)\pi^{-1}_{>k}\kappa_{\omega\nu}(x),
\end{aligned} \tag{6}$$

where $\pi^{-1}_{>k}\kappa_{\omega\nu}$ is the polynomial weight of the crystal subgraph defined by the colours $1, \ldots, k-1$, in the Demazure crystal graph $\mathfrak{B}_{\omega\nu}$. It means we are considering all the tableaux in the $\mathfrak{B}_{\omega\nu}$ with entries less or equal than k, and so all the tableaux in $\mathfrak{B}_{\omega\nu}$ with right key such that the entries are less or equal than k. It is equivalent to all SSAFs with content in $\mathbb{N}^k$, and shape rearrangement of $\omega\nu$ with zeros in the $n-k$

last entries. For $\overline{\lambda}=(m^{n-m+1},m-1,\ldots,n-k+1)$, where $1\leq k\leq m\leq n$, and $n+1\leq m+k$, one has from (6),

$$\prod_{(i,j)\in\overline{\lambda}}(1-x_iy_j)^{-1}=\prod_{(j,i)\in\lambda}(1-x_iy_j)^{-1}=\sum_{\substack{\nu\in\mathbb{N}^n\\ \nu=(\nu_1,\ldots,\nu_k,0^{n-k})}}\widehat{\kappa}_\nu(x)\pi_{>m}^{-1}\kappa_{\omega\nu}(y),$$

where $\pi_{>m}^{-1}\kappa_{\omega\nu}(y)$ is defined similarly as above, swapping k with m. All these identities are equivalent to those obtained by Lascoux (2003) regarding the shapes discussed here.

Acknowledgements

We thank to Vic Reiner for pointing us to extend our main theorem to stair-type shapes, and to Vivien Pons for letting us know about the implementation of key polynomials for Sage-Combinat. Pons (2011/12).

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