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Evaluations of Hecke algebra traces at Kazhdan-Lusztig basis elements

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Abstract. For irreducible characters $\{\chi_q^\lambda \,|\, \lambda \vdash n\}$ and induced sign characters $\{\epsilon_q^\lambda \,|\, \lambda \vdash n\}$ of the Hecke algebra $H_n(q)$, and Kazhdan-Lusztig basis elements $C'_w(q)$ with w avoiding the pattern 312, we combinatorially interpret the polynomials $\chi_q^\lambda(q^{\frac{\ell(w)}{2}} C'_w(q))$ and $\epsilon_q^\lambda(q^{\frac{\ell(w)}{2}} C'_w(q))$. This gives a new algebraic interpretation of q-chromatic symmetric functions of Shareshian and Wachs. We conjecture similar interpretations and generating functions corresponding to other $H_n(q)$-traces.

Résumé. Pour les caractères irreductibles $\{\chi_q^\lambda \,|\, \lambda \vdash n\}$ et les caractères induits du signe $\{\epsilon_q^\lambda \,|\, \lambda \vdash n\}$ du algèbre de Hecke, et les éléments $C'_w(q)$ du base Kazhdan-Lusztig avec w qui évite le motif 312, nous interprétons les polinômes $\chi_q^\lambda(q^{\frac{\ell(w)}{2}} C'_w(q))$ et $\epsilon_q^\lambda(q^{\frac{\ell(w)}{2}} C'_w(q))$ de manière combinatorielle. Cette donne une nouvelle interprétation aux fonctionnes symétriques q-chromatiques de Shareshian et Wachs. Nous conjecturons des interprétations semblables et des foncionnes generatrices qui correspondent aux autres applications centrales de $H_n(q)$.

Keywords: Hecke algebra, trace, Kazhdan-Lusztig basis, tableau

1 Introduction

The *symmetric group algebra* $\mathbb{C}\mathfrak{S}_n$ and the *(Iwahori-) Hecke algebra* $H_n(q)$ have similar presentations as algebras over $\mathbb{C}$ and $\mathbb{C}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}]$ respectively, with multiplicative identity elements e and T_e, generators $s_1, \ldots, s_{n-1}$ and $T_{s_1}, \ldots, T_{s_{n-1}}$, and relations

$$\begin{array}{lll} s_i^2 = e & T_{s_i}^2 = (q-1)T_{s_i} + qT_e & \text{for } i = 1, \ldots, n-1, \\ s_is_js_i = s_js_is_j & T_{s_i}T_{s_j}T_{s_i} = T_{s_j}T_{s_i}T_{s_j} & \text{for } |i-j| = 1, \\ s_is_j = s_js_i & T_{s_i}T_{s_j} = T_{s_j}T_{s_i} & \text{for } |i-j| \geq 2. \end{array}$$

Analogous to the natural basis $\{w \,|\, w \in \mathfrak{S}_n\}$ of $\mathbb{C}\mathfrak{S}_n$ is the natural basis $\{T_w \,|\, w \in \mathfrak{S}_n\}$ of $H_n(q)$, where we define $T_w = T_{s_{i_1}} \cdots T_{s_{i_\ell}}$ whenever $s_{i_1} \cdots s_{i_\ell}$ is a reduced expression for w in $\mathfrak{S}_n$. We call ℓ the *length* of w and write $\ell = \ell(w)$. (See [Hum90].) The specialization of $H_n(q)$ at $q^{\frac{1}{2}} = 1$ is isomorphic to $\mathbb{C}\mathfrak{S}_n$. In addition to the natural bases of $\mathbb{C}\mathfrak{S}_n$ and $H_n(q)$, we have the (signless) *Kazhdan-Lusztig bases* [KL79] $\{C'_w(1) \,|\, w \in \mathfrak{S}_n\}$, $\{C'_w(q) \,|\, w \in \mathfrak{S}_n\}$, defined in terms of certain *Kazhdan-Lusztig*

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polynomials $\{P_{u,v}(q) \,|\, u, v \in \mathfrak{S}_n\}$ in $\mathbb{N}[q]$ by

$$C'_w(1) = \sum_{v \leq w} P_{v,w}(1)v, \qquad C'_w(q) = q_{e,w}^{-1} \sum_{v \leq w} P_{v,w}(q)T_v, \tag{1}$$

where $\leq$ denotes the Bruhat order and we define $q_{v,w} = q^{\frac{\ell(w)-\ell(v)}{2}}$. (See, e.g., [BB96].)

Representations of $\mathbb{C}\mathfrak{S}_n$ and $H_n(q)$ are often studied in terms of *characters*. The $\mathbb{C}$-span of the $\mathfrak{S}_n$-characters is called the space of $\mathfrak{S}_n$*-class functions*, and has dimension is equal to the number of integer partitions of n. (See [Sag01].) Three well-studied bases are the irreducible characters $\{\chi^\lambda \,|\, \lambda \vdash n\}$, induced sign characters $\{\epsilon^\lambda \,|\, \lambda \vdash n\}$, and induced trivial characters $\{\eta^\lambda \,|\, \lambda \vdash n\}$, where $\lambda \vdash n$ denotes that λ is a partition of n. The $\mathbb{C}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}]$-span of the $H_n(q)$-characters, called the space of $H_n(q)$*-traces*, has the same dimension and analogous character bases $\{\chi_q^\lambda \,|\, \lambda \vdash n\}$, $\{\epsilon_q^\lambda \,|\, \lambda \vdash n\}$, $\{\eta_q^\lambda \,|\, \lambda \vdash n\}$, specializing at $q^{\frac{1}{2}} = 1$ to the $\mathfrak{S}_n$-character bases. Each of the two spaces has a fourth basis consisting of *monomial* class functions $\{\phi^\lambda \,|\, \lambda \vdash n\}$ or traces $\{\phi_q^\lambda \,|\, \lambda \vdash n\}$, and a fifth basis consisting of *power sum* class functions $\{\psi^\lambda \,|\, \lambda \vdash n\}$ or traces $\{\psi_q^\lambda \,|\, \lambda \vdash n\}$. These are defined via the inverse Kostka numbers $\{K_{\lambda,\mu}^{-1} \,|\, \lambda, \mu \vdash n\}$ and the numbers $\{L_{\lambda,\mu}^{-1} \,|\, \lambda, \mu \vdash n\}$ of row-constant Young tableaux of shape λ and content μ by

$$\phi^\lambda \underset{\text{def}}{=} \sum_\mu K_{\lambda,\mu}^{-1}\chi^\mu, \quad \phi_q^\lambda \underset{\text{def}}{=} \sum_\mu K_{\lambda,\mu}^{-1}\chi_q^\mu, \qquad \psi^\lambda \underset{\text{def}}{=} \sum_\mu L_{\lambda,\mu}\phi^\mu, \quad \psi_q^\lambda \underset{\text{def}}{=} \sum_\mu L_{\lambda,\mu}\phi_q^\mu. \tag{2}$$

These functions are not characters. (See [BRW96], [Hai93], [Ste92].) In each space, the five bases are related to one another by the same transition matrices which relate the Schur, elementary, complete homogeneous, monomial, and power sum bases of the homogeneous degree n symmetric functions. (See, e.g., [Sta99].)

It is known that irreducible $\mathfrak{S}_n$-characters $\{\chi^\lambda \,|\, \lambda \vdash n\}$ satisfy $\chi^\lambda(w) \in \mathbb{Z}$ for all $w \in \mathfrak{S}_n$. Thus for any integer linear combination θ of these and any element $z \in \mathbb{Z}\mathfrak{S}_n$, we have $\theta(z) \in \mathbb{Z}$ as well. In some cases, we may associate sets R, S to the pair (θ, z) to combinatorially interpret the integer $\theta(z)$ as $(-1)^{|S|}|R|$. We summarize known results and open problems in the following table.

θ	$\theta(w) \in \mathbb{N}$?	interpretation of $\theta(w)$ as $(-1)^{\|S\|}\|R\|$?	$\theta(C'_w(1)) \in \mathbb{N}$?	interpretation of $\theta(C'_w(1))$ as $\|R\|$ for w avoiding 312?
η^λ	yes	yes	yes	yes
ϵ^λ	no	yes	yes	yes
χ^λ	no	open	yes	yes
ψ^λ	yes	yes	yes	yes
ϕ^λ	no	yes	conj. by Stembridge, Haiman	open

For known combinatorial interpretations of $\theta(w)$, see [BRW96]. The number $\chi^\lambda(w)$ may be computed by the well-known algorithm of Murnaghan and Nakayama. (See, e.g., [Sta99].) Otherwise, $\chi^\lambda(w)$ has no conjectured expression of the type stated above. Interpretations of $\theta(C'_w(1))$ are not known for general $w \in \mathfrak{S}_n$, but nonnegativity follows from work of Haiman [Hai93] and Stembridge [Ste91]. Interpretations of $\eta^\lambda(C'_w(1))$, $\epsilon^\lambda(C'_w(1))$, $\chi^\lambda(C'_w(1))$ for w avoiding the pattern 312 follow via straightforward arguments from results of various authors, notably Gasharov [Gas96], Karlin-MacGregor [KM59], Lindström [Lin73], Littlewood [Lit40], Merris-Watkins [MW85], Stanley-Stembridge [SS93], [Ste91]. These will be discussed in Section 3. There is no conjectured combinatorial interpretation of $\phi^\lambda(C'_w(1))$, even for w avoiding the pattern 312 although interpretations have been given for particular partitions λ by Stembridge [Ste92] and several of the authors [CSS11].

It is known that irreducible $H_n(q)$-characters $\{\chi_q^\lambda \,|\, \lambda \vdash n\}$ satisfy $\chi_q^\lambda(T_w) \in \mathbb{Z}[q]$ for all $w \in \mathfrak{S}_n$. Thus for any integer linear combination θ_q of these and any element $z \in \mathrm{span}_{\mathbb{Z}[q]}\{T_w \,|\, w \in \mathfrak{S}_n\}$, we have $\theta_q(z) \in \mathbb{Z}[q]$ as well. In some cases, we may associate sequences $(S_k)_{k\geq 0}$, $(R_k)_{\geq 0}$ of sets to the pair (θ_q, z) to combinatorially interpret $\theta_q(z)$ as $\sum_k (-1)^{|S_k|}|R_k|q^k$. We summarize known results and open problems in the following table.

θ_q	$\theta_q(T_w) \in \mathbb{N}[q]$?	interpretation of $\theta_q(T_w)$ as $\sum_k(-1)^{\vert S_k\vert}\vert R_k\vert q^k$?	$\theta(q_{e,w}C'_w(q)) \in \mathbb{N}[q]$?	interpretation of $\theta_q(q_{e,w}C'_w(q))$ as $\sum_k \vert R_k\vert q^k$ for w avoiding 312?
η_q^λ	no	open	yes	conj. in Section 4
ϵ_q^λ	no	open	yes	stated in Section 4
χ_q^λ	no	open	yes	stated in Section 4
ψ_q^λ	no	open	conj. by Haiman	conj. in Section 4
ϕ_q^λ	no	open	conj. by Haiman	open

The polynomial $\chi_q^\lambda(T_w)$, and therefore all polynomials $\theta_q(T_w)$, may be computed via a q-extension of the Murnaghan-Nakayama algorithm. (See, e.g., [KV84], [KW92], [Ram91].) Otherwise, $\theta_q^\lambda(w)$ has no conjectured expression of the type stated above. Interpretations of $\theta_q(q_{e,w}C'_w(q))$ are not known for general $w \in \mathfrak{S}_n$, but results concerning containment in $\mathbb{N}[q]$ follow principally from work of Haiman [Hai93]. For w avoiding the pattern 312, a formula for $\epsilon_q^\lambda(q_{e,w}C'_w(q))$ is given by the authors in Section 4. Work of Gasharov [Gas96] and Shareshian-Wachs [SW12] then implies a formula for $\chi_q^\lambda(q_{e,w}C'_w(q))$. Conjectures for $\psi_q^\lambda(q_{e,w}C'_w(q))$ are due to the authors and Shareshian-Wachs. These results and conjectures will also be discussed in Section 4. There is no conjectured combinatorial interpretation of $\phi_q^\lambda(q_{e,w}C'_w(q))$, even for w avoiding the pattern 312.

Another way to understand the evaluations $\theta(w)$ is to define a generating function $\mathrm{Imm}_\theta(x)$ in the polynomial ring $\mathbb{C}[x_{1,1}, \dots, x_{n,n}]$ for $\{\theta(w) \,|\, w \in \mathfrak{S}_n\}$. Similarly, we may define a generating function $\mathrm{Imm}_{\theta_q}(x)$ in a certain noncommutative ring $\mathcal{A}(n;q)$ for $\{\theta(T_w) \,|\, w \in \mathfrak{S}_n\}$. In some cases these generating functions have simple forms. We summarize known results in the following tables.

θ	nice expression for $\mathrm{Imm}_\theta(x)$?
η^λ	yes
ϵ^λ	yes
χ^λ	open
ψ^λ	yes
ϕ^λ	open

θ_q	nice expression for $\mathrm{Imm}_{\theta_q}(x)$?
η_q^λ	yes
ϵ_q^λ	yes
χ_q^λ	open
ψ_q^λ	conj. in Section 2
ϕ_q^λ	open

Nice expressions for $\mathrm{Imm}_{\eta^\lambda}(x)$ and $\mathrm{Imm}_{\epsilon^\lambda}(x)$ are due to Littlewood [Lit40] and Merris-Watkins [MW85], and a nice expression for $\mathrm{Imm}_{\psi^\lambda}(x)$ follows immediately from the usual definition of ψ. An expression for $\mathrm{Imm}_{\chi^\lambda}(x)$ as a coefficient of a generating function in two sets of variables was given by Goulden-Jackson [GJ92]. There is no conjectured nice formula for $\mathrm{Imm}_{\phi^\lambda}(x)$, although a nice formula for particular partitions λ was stated by Stembridge [Ste92]. These results will be discussed in Section 2. Nice expressions for $\mathrm{Imm}_{\eta_q^\lambda}(x)$ and $\mathrm{Imm}_{\epsilon_q^\lambda}(x)$ are due to the fourth author and Konvalinka [KS11], as is an expression for $\mathrm{Imm}_{\chi_q^\lambda}(x)$ as a coefficient in a generating function in two sets of variables. A nice expression for $\mathrm{Imm}_{\psi_q^\lambda}(x)$ is conjectured by the authors. These results and conjecture will be discussed in Section 2.

In Section 2 we discuss known descriptions of the class functions in terms of generating functions in the ring $\mathbb{C}[x_{1,1}, \ldots, x_{n,n}]$ and in a certain quantum analog $\mathcal{A}(n;q)$ of $\mathbb{C}[x_{1,1}, \ldots, x_{n,n}]$ known as the *quantum matrix bialgebra*. We also give a combinatorial interpretation of the entries of the transition matrices relating certain bases of $\mathcal{A}(n;q)$. In Section 3 we give combinatorial interpretations, using results in the previous section. In Section 4 we give new descriptions of the class functions in terms of generating functions in the rings $\mathbb{C}\otimes\Lambda$ and $\mathbb{C}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}]\otimes\Lambda$ of symmetric functions having coefficients in $\mathbb{C}$ and $\mathbb{C}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}]$. Finally, in Section 5 we draw connections to posets and to the chromatic quasisymmetric functions of Shareshian and Wachs.

2 Generating functions for $\theta(w)$ and $\theta_q(T_w)$ when θ is fixed

For a fixed $\mathfrak{S}_n$-class function θ, we create a generating function for $\{\theta(w) \,|\, w \in \mathfrak{S}_n\}$ by writing $x = (x_{i,j})$, $\mathbb{C}[x] \underset{\mathrm{def}}{=} \mathbb{C}[x_{1,1}, \ldots, x_{n,n}]$, and defining

$$\mathrm{Imm}_\theta(x) \underset{\mathrm{def}}{=} \sum_{w\in\mathfrak{S}_n} \theta(w) x_{1,w_1} \cdots x_{n,w_n} \ \in\ \mathbb{C}[x].$$

We call this polynomial the θ-*immanant*. The sign character ($w \mapsto (-1)^{\ell(w)}$) immanant and trivial character ($w \mapsto 1$) immanant are the determinant and permanent. Nice formulas for the ϵ^λ-immanants and η^λ-immanants employ determinants and permanents of submatrices of x,

$$x_{I,J} \underset{\mathrm{def}}{=} (x_{i,j})_{i\in I, j\in J}, \qquad I, J \subset [n] \underset{\mathrm{def}}{=} \{1, \ldots, n\}.$$

In particular, for $\lambda = (\lambda_1, \ldots, \lambda_r) \vdash n$ we have Littlewood-Merris-Watkins identities [Lit40], [MW85]

$$\mathrm{Imm}_{\epsilon^\lambda}(x) = \sum_{(I_1,\ldots,I_r)} \det(x_{I_1,I_1}) \cdots \det(x_{I_r,I_r}), \qquad \mathrm{Imm}_{\eta^\lambda}(x) = \sum_{(I_1,\ldots,I_r)} \mathrm{per}(x_{I_1,I_1}) \cdots \mathrm{per}(x_{I_r,I_r}), \quad (3)$$

where the sums are over all sequences of pairwise disjoint subsets of $[n]$ satisfying $|I_j| = \lambda_j$. A formula for the ψ^λ-immanant relies upon a sum over all permutations of cycle type λ,

$$\mathrm{Imm}_{\psi^\lambda}(x) = z_\lambda \sum_{\substack{w \\ \mathrm{cyc}(w)=\lambda}} x_{1,w_1} \cdots x_{n,w_n},$$

where z_λ is the product $1^{\alpha_1} 2^{\alpha_2} \cdots n^{\alpha_n} \alpha_1! \cdots \alpha_n!$, and λ has α_i parts equal to i for $i = 1, \ldots, n$. No such nice formulas are known for the χ^λ-immanants or ϕ^λ-immanants in general, although we do have a formula [Ste92, Thm. 2.8] for $\mathrm{Imm}_{\phi^\lambda}(x)$ when $\lambda_1 = \cdots = \lambda_r = k$,

$$\mathrm{Imm}_{\phi^{k^r}}(x) = \sum_{(I_1,\ldots,I_k)} \det(x_{I_1,I_2}) \det(x_{I_2,I_3}) \cdots \det(x_{I_k,I_1}),$$

where the sum is over all sequences of pairwise disjoint subsets of $[n] = [kr]$ satisfying $|I_j| = r$.

For a fixed $H_n(q)$-trace θ_q, we create a generating function for $\{\theta_q(T_w) \,|\, w \in \mathfrak{S}_n\}$ as before, but interpreting polynomials in $x = (x_{i,j})$ as elements of the *quantum matrix bialgebra* $\mathcal{A}(n;q)$, the non-commutative $\mathbb{C}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}]$-algebra generated by n^2 variables $x = (x_{1,1}, \dots, x_{n,n})$, subject to the relations

$$\begin{aligned} x_{i,\ell}x_{i,k} &= q^{\frac{1}{2}} x_{i,k}x_{i,\ell}, \qquad & x_{j,k}x_{i,\ell} &= x_{i,\ell}x_{j,k} \\ x_{j,k}x_{i,k} &= q^{\frac{1}{2}} x_{i,k}x_{j,k} \qquad & x_{j,\ell}x_{i,k} &= x_{i,k}x_{j,\ell} + (q^{\frac{1}{2}} - q^{\frac{-1}{2}})x_{i,\ell}x_{j,k}, \end{aligned} \tag{4}$$

for all indices $1 \leq i < j \leq n$ and $1 \leq k < \ell \leq n$. As a $\mathbb{C}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}]$-module, $\mathcal{A}(n;q)$ has a basis of monomials $x_{\ell_1,m_1} \cdots x_{\ell_r,m_r}$ in which index pairs appear in lexicographic order. The relations (4) allow one to express other monomials in terms of this natural basis.

As a generating function for $\{\theta_q(T_w) \,|\, w \in \mathfrak{S}_n\}$, we define

$$\mathrm{Imm}_{\theta_q}(x) \underset{\mathrm{def}}{=} \sum_{w \in \mathfrak{S}_n} \theta_q(T_w) q_{e,w}^{-1} x_{1,w_1} \cdots x_{n,w_n}$$

in $\mathcal{A}(n;q)$, and call this the θ_q*-immanant*. The $H_n(q)$ sign character ($T_w \mapsto (-1)^{\ell(w)}$) immanant and trivial character ($T_w \mapsto q^{\ell(w)}$) immanant are called the *quantum determinant* and *quantum permanent*,

$$\det_q(x) = \sum_{w \in \mathfrak{S}_n} (-q^{\frac{-1}{2}})^{\ell(w)} x_{1,w_1} \cdots x_{n,w_n}, \qquad \mathrm{per}_q(x) = \sum_{w \in \mathfrak{S}_n} (q^{\frac{1}{2}})^{\ell(w)} x_{1,w_1} \cdots x_{n,w_n}.$$

Specializing $\mathcal{A}(n;q)$, $\det_q(x)$, and $\mathrm{per}_q(x)$ at $q^{\frac{1}{2}} = 1$, we obtain the commutative polynomial ring $\mathbb{C}[x]$ and the classical determinant $\det(x)$ and permanent $\mathrm{per}(x)$.

Nice formulas for the ϵ_q^λ-immanants and η_q^λ-immanants employ quantum determinants and quantum permanents of submatrices of x. In particular, the fourth author and Konvalinka [KS11, Thm. 5.4] proved quantum analogs of the Littlewood-Merris-Watkins identities in (3),

$$\mathrm{Imm}_{\epsilon_q^\lambda}(x) = \sum_{(I_1,\dots,I_r)} \det_q(x_{I_1,I_1}) \cdots \det_q(x_{I_r,I_r}), \quad \mathrm{Imm}_{\eta_q^\lambda}(x) = \sum_{(I_1,\dots,I_r)} \mathrm{per}_q(x_{I_1,I_1}) \cdots \mathrm{per}_q(x_{I_r,I_r}), \tag{5}$$

where the sums are as in (3).

To state a nice form for the ψ_q^λ-immanant, we introduce the following definitions. Given a sequence $c = (i_1, \dots, i_k)$ of distinct elements of $[n]$ with $i_1 = \min\{i_1, \dots, i_k\}$, define the element $d_{(i_1,\dots,i_k)}(x)$ of $\mathcal{A}(n;q)$ to be the sum of all cyclic rearrangements of the monomial $x_{i_1,i_2}x_{i_2,i_3} \cdots x_{i_k,i_1}$, each weighted by $q^{j-(k+1)/2}$, where x_{i_1,i_2} appears in position j,

$$d_c(x) = q^{\frac{-(k-1)}{2}} x_{i_1,i_2}x_{i_2,i_3} \cdots x_{i_k,i_1} + q^{\frac{-(k-3)}{2}} x_{i_k,i_1}x_{i_1,i_2} \cdots x_{i_{k-1},i_k} + \cdots + q^{\frac{(k-1)}{2}} x_{i_2,i_3} \cdots x_{i_k,i_1}x_{i_1,i_2}.$$

For $w \in \mathfrak{S}_n$ having cycle type $\lambda = (\lambda_1, \dots, \lambda_r)$, define the polynomial $g_w(x)$ to be the sum, over all cycle decompositions $(c_1, \dots, c_r)$ of w with $|c_j| = \lambda_j$, of $d_{c_1}(x) \cdots d_{c_r}(x)$. For example, the permutation $w = (1,4,3)(2,7)(5,6) = (1,4,3)(5,6)(2,7)$ with its (exactly) two admissible cycle decompositions leads to the element $g_w(x) =$

$$\begin{aligned} &(q^{-1}x_{1,4}x_{4,3}x_{3,1} + x_{3,1}x_{1,4}x_{4,3} + qx_{4,3}x_{3,1}x_{1,4})(q^{\frac{-1}{2}}x_{2,7}x_{7,2} + q^{\frac{1}{2}}x_{7,2}x_{2,7})(q^{\frac{-1}{2}}x_{5,6}x_{6,5} + q^{\frac{1}{2}}x_{6,5}x_{5,6}) + \\ &(q^{-1}x_{1,4}x_{4,3}x_{3,1} + x_{3,1}x_{1,4}x_{4,3} + qx_{4,3}x_{3,1}x_{1,4})(q^{\frac{-1}{2}}x_{5,6}x_{6,5} + q^{\frac{1}{2}}x_{6,5}x_{5,6})(q^{\frac{-1}{2}}x_{2,7}x_{7,2} + q^{\frac{1}{2}}x_{7,2}x_{2,7}). \end{aligned}$$

Conjecture 2.1 *Fix $\lambda = (\lambda_1, \ldots, \lambda_r) \vdash n$. Then in $\mathcal{A}(n;q)$ we have* $\mathrm{Imm}_{\psi_q^\lambda}(x) = \sum_{\substack{w \\ \mathrm{cyc}(w)=\lambda}} g_w(x)$.

For example, when $n = 5$ and $\lambda = (3, 2)$, we represent each permutation having cycle type $(3, 2)$ as a product of a 3-cycle with least letter written first and a 2-cycle with least letter written first, $(1, 2, 3)(4, 5)$, $(1, 4, 2)(3, 5), \ldots, (3, 5, 4)(1, 2)$, and we have

$$\begin{aligned}\mathrm{Imm}_{\psi_q^{32}}(x) = {} & (q^{-1}x_{1,2}x_{2,3}x_{3,1} + x_{3,1}x_{1,2}x_{2,3} + qx_{2,3}x_{3,1}x_{1,2})(q^{\frac{-1}{2}}x_{4,5}x_{5,4} + q^{\frac{1}{2}}x_{5,4}x_{4,5}) \\ & + (q^{-1}x_{1,4}x_{4,2}x_{2,1} + x_{2,1}x_{1,4}x_{4,2} + qx_{4,2}x_{2,1}x_{1,4})(q^{\frac{-1}{2}}x_{3,5}x_{5,3} + q^{\frac{1}{2}}x_{5,3}x_{3,5}) \\ & + \cdots + (q^{-1}x_{3,5}x_{5,4}x_{4,3} + x_{4,3}x_{3,5}x_{5,4} + qx_{5,4}x_{4,3}x_{3,5})(q^{\frac{-1}{2}}x_{1,2}x_{2,1} + q^{\frac{1}{2}}x_{2,1}x_{1,2}).\end{aligned}$$

No such nice formulas are known for the χ_q^λ- or ϕ_q^λ- immanants.

To obtain values of $\epsilon_q^\lambda(T_w)$ and $\eta_q^\lambda(T_w)$ from (5), one must use the relations (4) to expand in the natural basis of (the zero-weight space $\mathrm{span}\{x_{1,w_1} \cdots x_{n,w_n} \,|\, w \in \mathfrak{S}_n\}$ of) $\mathcal{A}(n;q)$. For this purpose, it is helpful to combinatorially interpret the coefficients arising as entries in the transition matrix relating the bases $\mathcal{B}_u = \{x_{u_1,v_1} \cdots x_{u_n,v_n} \,|\, v \in \mathfrak{S}_n\}$ and the natural basis $\{x_{1,w_1} \cdots x_{n,w_n} \,|\, w \in \mathfrak{S}_n\}$. These were obtained by Lambright and the fourth author in [LS10]. To combinatorially interpret the evaluations $\{\epsilon_q^\lambda(q_{e,w}C'_w(q)) \,|\, \lambda \vdash n\}$ when w avoids the pattern 312, we prove a stronger result.

Theorem 2.2 *Fix $u, w \in \mathfrak{S}_n$ with $u \leq w$, and let $s_{i_1} \cdots s_{i_\ell}$ be the right-to-left lexicographically greatest reduced expression for u. Choose an index $k \leq \ell + 1$ and define $u' = s_{i_{k-1}} \cdots s_{i_1} u$, $w' = s_{i_{k-1}} \cdots s_{i_1} w$. Then we have*

$$x_{u_1,w_1} \cdots x_{u_n,w_n} = \sum_{v \in \mathfrak{S}_n} t_{u,w',v}(q^{\frac{1}{2}} - q^{\frac{-1}{2}}) x_{u'_1,v_1} \cdots x_{u'_n,v_n},$$

where $\{t_{u,w',v}(q) \,|\, v \in \mathfrak{S}_n\}$ are polynomials in $\mathbb{N}[q]$. Moreover, the coefficient of q^b in $t_{u,w',v}(q)$ is equal to the number of sequences $(\pi^{(0)}, \ldots, \pi^{(k-1)})$ of permutations satisfying

1. *$\pi^0 = w$, $\pi^{(k-1)} = v$,*
2. *$\pi^{(j)} \in \{s_{i_j}\pi^{(j-1)}, \pi^{(j-1)}\}$ for $j = 1, \ldots, k-1$,*
3. *$\pi^{(j)} = s_{i_j}\pi^{(j-1)}$ if $s_{i_j}\pi^{(j-1)} > \pi^{(j-1)}$,*
4. *$\pi^{(j)} = \pi^{(j-1)}$ for exactly b values of j.*

Proof: Omitted. □

We may think of each sequence $(\pi^{(0)}, \ldots, \pi^{(k-1)})$ in the above proof as a $(k-1)$-step walk from w to v in the weak order on $\mathfrak{S}_n$. After visiting $\pi^{(j)} \in \mathfrak{S}_n$, we may either revisit this permutation or move to $s_{i_j}\pi^{(j)}$, with the latter option being mandatory if s_{i_j} is a left ascent for $\pi^{(j)}$.

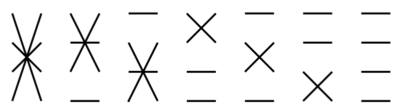
3 Descending star networks and interpretations of class functions

Call a directed planar graph G a *planar network of order n* if it is acyclic and may be embedded in a disc so that $2n$ boundary vertices labeled clockwise as *source* $1, \ldots,$ *source* n (with indegrees of 0) and *sink* $n, \ldots,$ *sink* 1 (with outdegrees of 0). In figures, we will draw sources on the left and sinks on the right, implicitly labeled $1, \ldots, n$ from bottom to top. Given a planar network G, define the *path matrix* $B = B(G) = (b_{i,j})$ of G by

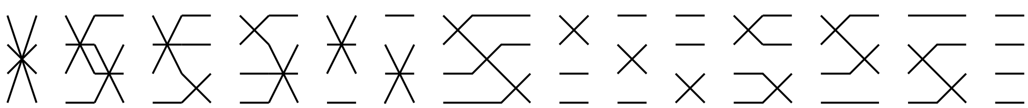
$$b_{i,j} = \text{ number of paths in } G \text{ from source } i \text{ to sink } j. \tag{6}$$

It is known that the path matrix of any planar network is totally nonnegative (TNN), i.e., that every minor of this matrix is nonnegative. This fact is known as *Lindström's Lemma*.

Call a sequence $\pi = (\pi_1, \ldots, \pi_n)$ of source-to-sink paths in a planar network a *bijective path family* if for some $w \in \mathfrak{S}_n$ with one-line notation $w_1 \cdots w_n$, each component path π_i begins at source i and terminates at sink w_i. We will say also that π has *type* w. Call a planar network a *bijective skeleton* if it is a union of n source-to-sink paths. Clearly a bijective path family can cover an entire planar network G only if G is a bijective skeleton. For $[a, b]$ a subinterval of $[n]$, let $G_{[a,b]}$ be the bijective skeleton consisting of $a-1$ horizontal edges, a "star" of $b-a+1$ edges from sources $a, \ldots, b$ to an intermediate vertex, and $b-a+1$ more edges from this vertex to sinks $a, \ldots, b$, and $n-b$ more horizontal edges. For $n = 4$, there are seven such networks: $G_{[1,4]}$, $G_{[2,4]}$, $G_{[1,3]}$, $G_{[3,4]}$, $G_{[2,3]}$, $G_{[1,2]}$, $G_{[1,1]} = \cdots = G_{[4,4]}$, respectively,

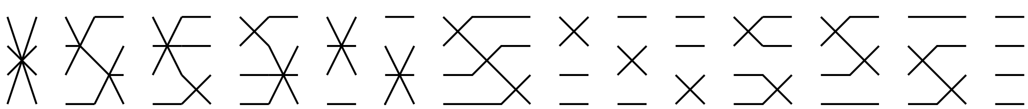


Define $G_I \circ G_J$ to be the concatenation of planar networks G_I and G_J, and consider a sequence $([c_1, d_1], \ldots, [c_r, d_r])$ of subintervals of $[n]$ satisfying $c_1 > \cdots > c_r$ and $d_1 > \cdots > d_r$, and the concatenation $G_{[c_1,d_1]} \circ \cdots \circ G_{[c_r,d_r]}$ of corresponding star networks. For $n = 4$, these are



(7)

For each such planar network G, we define a related planar network F by modifying G as follows. For $i = 1, \ldots, r-1$, if the intersection $[c_{i+1}, d_{i+1}] \cap [c_i, d_i]$ has cardinality $k > 1$, then collapse the k paths from the central vertex of $G_{[c_{i+1},d_{i+1}]}$ to the central vertex of $G_{[c_i,d_i]}$, creating a single path between these vertices. Call F a *descending star network*. For $n = 4$, the decending star networks are



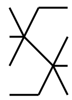
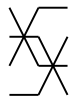
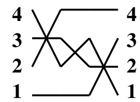
(8)

Proposition 3.1 *There are* $\frac{1}{n+1}\binom{2n}{n}$ *descending star networks of order* n*.*

Proof: (Idea.) Let F be the descending star network which corresponds as before (7) to the concatenation $G = G_{[c_1,d_1]} \circ \cdots \circ G_{[c_r,d_r]}$. Modify G to create the related network

$$G' \underset{\text{def}}{=} G_{[c_1,d_1]} \circ G_{[c_1,d_1]\cap[c_2,d_2]} \circ G_{[c_2,d_2]} \circ \cdots \circ G_{[c_{r-1},d_{r-1}]} \circ G_{[c_{r-1},d_{r-1}]\cap[c_r,d_r]} \circ G_{[c_r,d_r]}$$

by inserting $G_{[c_i,d_i]\cap[c_{i+1},d_{i+1}]}$ between $G_{[c_i,d_i]}$ and $G_{[c_{i+1},d_{i+1}]}$ for $i = 1, \ldots, r-1$. Now visually follow paths from sources to sinks, passing "straight" through each intersection, to complete a bijection to 312-avoiding permutations in $\mathfrak{S}_n$. For example, when $n = 4$ and F corresponds to $G = G_{[2,4]} \circ G_{[1,3]}$, we construct $G' = G_{[2,4]} \circ G_{[2,3]} \circ G_{[1,3]}$ and obtain the 312-avoiding permutation $w = w(F) = 3421$:

$F =$ , $G =$ , $G' =$ 

, $w = \begin{pmatrix} 1234 \\ 3421 \end{pmatrix}$.

□

For each 312-avoiding permutation $w \in \mathfrak{S}_n$, let F_w denote the descending star network corresponding to w by the bijection in the proof of Proposition 3.1. Every descending star network F_w is a bijective skeleton, and for every $v \leq w$ in the Bruhat order, there is exactly one bijective path family π of type v which covers F_w.

In a planar network G of order n, the source-to-sink paths have a natural partial order $Q = Q(G)$. If π_i is a path originating at source i, and ρ_j is a path originating at source j, then we define $\pi_i <_Q \rho_j$ if $i < j$ and π_i and ρ_j never intersect. Observe that these conditions imply the index of the sink of π_i to be less than the index of the sink of ρ_j. Let $P(G)$ be the subposet of $Q(G)$ induced by paths whose source and sink indices are equal. For each descending star network F_w, the poset $P(F_w)$ has exactly n elements: there is exactly one path from source i to sink i, for $i = 1, \ldots, n$.

To combinatorially interpret evaluations of $\mathfrak{S}_n$-class functions and $H_n(q)$-traces, we will fill (French) Young diagrams with path families $(\pi_1, \ldots, \pi_n)$ covering a descending star network F_w, and will call the resulting structures *F-tableaux*. If an F_w-tableau U contains a path family π of type v, then we also say that U has *type* v. We say that an F_w-tableau U has *shape* λ for some partition $\lambda = (\lambda_1, \ldots, \lambda_r)$ if it has λ_i cells in row i for all i. If U has λ_i cells in *column* i for all i, we say that U has shape $\lambda^\top$. In this case we define $\lambda^\top$ to be the partition whose ith part is equal to the number of cells in row i of U. Let $L(U)$ and $R(U)$ be the Young tableaux of integers obtained from U by replacing paths $\pi_1, \ldots, \pi_n$ with their corresponding source and sink indices, respectively.

We define several properties of an F-tableau in terms of the poset Q and the tableaux $L(U)$ and $R(U)$.

1. Call U *column-strict* if whenever paths $\pi_{i_1}, \ldots, \pi_{i_r}$ appear from bottom to top in a column, then we have $\pi_{i_1} <_Q \cdots <_Q \pi_{i_r}$.

2. Call U *row-semistrict* if whenever paths π_{i_1}, π_{i_2} appear consecutively (from left to right) in a row, we have $\pi_{i_1} <_Q \pi_{i_2}$ or π_{i_1} is incomparable to π_{i_2} in Q.

3. Call U *cyclically row-semistrict* if it is row-semistrict and the condition above applies also to paths π_{i_1}, π_{i_2} appearing last and first (respectively) in the same row.

4. Call U *standard* if it is column-strict and row-semistrict.

5. Call U *cylindrical* if for each row of $L(U)$ containing indices $i_1, \ldots, i_k$ from left to right, the corresponding row of $R(U)$ contains $i_2, \ldots, i_k, i_1$ from left to right.

6. Call U *row-closed* if $L(U)$ is row-strict (entries increase to the right) and if each row of $R(U)$ is a permutation of the corresponding row of $L(U)$.

For some $\mathfrak{S}_n$-class functions θ, and all 312-avoiding permutations w, we may combinatorially interpret $\theta(C'_w(1))$ in terms of a star network F_w as follows.

Proposition 3.2 *Let w avoid the pattern 312, and let F_w be the corresponding descending star network.*

1. *$\eta^\lambda(C'_w(1))$ equals the number of row-semistrict F_w-tableaux of type e and shape λ. It also equals the number of row-closed F_w-tableaux of shape λ.*

2. *$\epsilon^\lambda(C'_w(1))$ equals the number of column-strict F_w-tableaux of type e and shape $\lambda^\top$.*

3. *$\chi^\lambda(C'_w(1))$ equals the number of semistandard F_w-tableaux of type e and shape λ.*

4. *$\psi^\lambda(C'_w(1))$ equals the number of cyclically row-semistrict F_w-tableaux of type e and shape λ. It also equals the number of cylindrical F_w-tableaux of shape λ.*

5. *For $\lambda_1 \leq 2$, $\phi^\lambda(C'_w(1))$ equals zero if there exists a column-strict F_w-tableaux of type e and shape $\mu \prec \lambda$; otherwise it equals the number of column-strict F_w-tableaux of type e and shape λ.*

6. *For $\lambda = k^r$, $\phi^\lambda(C'_w(1))$ equals the number of column-strict cylindrical F_w-tableaux of shape r^k.*

Proof: (Idea.) For w avoiding 312, the path matrix $B = (b_{i,j})$ of F_w satisfies $\theta(C'_w(1)) = \mathrm{Imm}_\theta(B)$. ☐

Haiman [Hai93] and Stembridge [Ste91] have shown that we have $\chi^\lambda(C'_w(1)) \geq 0$ for all $\lambda \vdash n$ and all $w \in \mathfrak{S}_n$. However, there is no conjectured combinatorial interpretation for $\chi^\lambda(C'_w(1))$ unless w avoids 312. Haiman [Hai93] and Stembridge [Ste92] have also conjectured that we have $\phi^\lambda(C'_w(1)) \geq 0$ for all $\lambda \vdash n$ and all $w \in \mathfrak{S}_n$. There is no general conjectured combinatorial interpretation for $\phi^\lambda(C'_w(1))$, even in the case that w avoids 312, unless λ has the special form stated in Proposition 3.2.

4 Statistics on F-tableaux and interpretations of $H_n(q)$-traces

For θ an $\mathfrak{S}_n$-class function and w avoiding 312, Proposition 3.2 interprets $\theta(C'_w(1))$ as the cardinality of a set of certain F_w-tableaux. For each of these sets of F_w-tableaux, we define a statistic mapping tableaux to nonnegative integers, and show (or conjecture) that $\theta_q(q_{e,w}C'_w(q))$ is a generating function for tableaux on which the statistic takes the values $k = 0, 1, \ldots$. In each case, our statistic is based upon the number of inversions of a permutation in $\mathfrak{S}_n$. Specifically, let F be a descending star network, and let U be an F-tableau containing path family $\pi = (\pi_1, \ldots, \pi_n)$ of type w. (Thus π_i begins at source vertex i and terminates at sink vertex w_i for $i = 1, \ldots, n$.) Let (π_i, π_j) be a pair of intersecting paths in F such that π_i appears in a column of U to the left of the column containing π_j. Call (π_i, π_j) a *left inversion* in U if we have $i > j$ and a *right inversion* in U if we have $w_i > w_j$. Let $\mathrm{INV}(U)$ denote the number of left inversions in U, and let $\mathrm{RINV}(U)$ denote the number of right inversions in U.

Proofs of the validity of the tableaux interpretations in Proposition 3.2 depend upon a relationship between immanants and path matrices. To state a q-analog of this relationship, we define a map for each $n \times n$ complex matrix B by

$$\begin{aligned} \sigma_B : \mathcal{A}(n; q) &\to \mathbb{C}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}] \\ x_{1,w_1} \cdots x_{n,w_n} &\mapsto q_{e,w} b_{1,w_1} \cdots b_{n,w_n}. \end{aligned}$$

Proposition 4.1 *Let θ_q be an $H_n(q)$-trace and let $w \in \mathfrak{S}_n$ avoid the pattern 312. Then the path matrix B of F_w satisfies $\theta_q(q_{e,w}C'_w(q)) = \sigma_B(\mathrm{Imm}_{\theta_q}(x))$.*

Proof: Omitted. ☐

Theorem 4.2 *Let $w \in \mathfrak{S}_n$ avoid the pattern* 312*. For $\lambda \vdash n$ we have*

$$\epsilon_q^\lambda(q_{e,w}C'_w(q)) = \sum q^{\mathrm{INV}(U)}, \tag{9}$$

where the sum is over all column-strict F_w-tableaux U of type e and shape $\lambda^\top$. We also have

$$\chi_q^\lambda(q_{e,w}C'_w(q)) = \sum q^{\mathrm{INV}(U)}, \tag{10}$$

where the sum is over all standard F_w-tableaux U of type e and shape λ.

Proof: Omitted. The proof of (10) depends upon a result of Shareshian and Wachs [SW12]. □

Let U be an F-tableau of shape $\lambda = (\lambda_1, \dots, \lambda_r)$ containing a path family π, and let U_i be the ith row of U. Let $U_1 \circ \cdots \circ U_r$ and $U_r \circ \cdots \circ U_1$ be the F-tableaux of shape n consisting of the rows of U concatenated in increasing and decreasing order, respectively.

Conjecture 4.3 *Let $w \in \mathfrak{S}_n$ avoid the pattern* 312*. For $\lambda \vdash n$ we have*

$$\eta_q^\lambda(q_{e,w}C'_w(q)) = \sum q^{\mathrm{RINV}(U_1 \circ \cdots \circ U_r)}, \tag{11}$$

where the sum is over all row-closed F_w-tableaux U of shape λ. We also have

$$\psi_q^\lambda(q_{e,w}C'_w(q)) = \sum q^{\mathrm{INV}(U_r \circ \cdots \circ U_1)}, \tag{12}$$

where the sum is over all cylindrical F_w-tableaux U of shape λ.

Haiman [Hai93] has shown that we have $\chi_q^\lambda(q_{e,w}C'_w(q)) \in \mathbb{N}[q]$ for all $\lambda \vdash n$ and all $w \in \mathfrak{S}_n$. He has also conjectured that we have $\phi_q^\lambda(q_{e,w}C'_w(q)) \in \mathbb{N}[q]$ for all $\lambda \vdash n$ and all $w \in \mathfrak{S}_n$. There is no general conjectured combinatorial interpretation for $\phi_q^\lambda(q_{e,w}C'_w(q))$, even in the case that w avoids 312.

5 Generating functions for $\theta(C'_w(1))$, $\theta_q(q_{e,w}C'_w(q))$ when w is fixed

For each $w \in \mathfrak{S}_n$, we define a symmetric generating function for values of $\theta(C'_w(1))$ by

$$X_w = \sum_{\lambda \vdash n} \epsilon^\lambda(C'_w(1)) m_\lambda \;\in\; \Lambda_n \underset{\mathrm{def}}{=} \mathrm{span}_{\mathbb{Z}}\{m_\lambda \,|\, \lambda \vdash n\}. \tag{13}$$

Expanding X_w in various bases of the space of homogeneous degree-n symmetric functions, including the forgotten basis $\{f_\lambda \,|\, \lambda \vdash n\}$, we have

$$X_w = \sum_{\lambda \vdash n} \eta^\lambda(C'_w(1)) f_\lambda = \sum_{\lambda \vdash n} \chi^{\lambda^\top}(C'_w(1)) s_\lambda = \sum_{\lambda \vdash n} (-1)^{n-\ell(\lambda)} \psi^\lambda(C'_w(1)) \frac{p_\lambda}{z_\lambda} = \sum_{\lambda \vdash n} \phi^\lambda(C'_w(1)) e_\lambda,$$

where $\ell(\lambda)$ is the number of (nonzero) parts of λ.

The function X_w is related to the *chromatic symmetric functions* $\{X_P \,|\, P \text{ a poset }\}$ of Stanley and Stembridge [Sta95], [SS93]: if w avoids the pattern 312, then X_w is equal to the Stanley-Stembridge chromatic symmetric function $X_{P(F_w)}$. On the other hand, not all chromatic symmetric functions X_P can be expressed as X_w for appropriate $w \in \mathfrak{S}_n$, nor can all generating functions X_w be expressed as X_P for an appropriate poset P. Stanley and Stembridge [Sta95], [SS93] have conjectured that X_P is elementary nonnegative when P has no induced subposet isomorphic to the disjoint union $(\mathbf{3}+\mathbf{1})$ of a three element chain and a single element. Call such a poset $(\mathbf{3}+\mathbf{1})$-*free*. A special case of this conjecture is that X_w is elementary nonnegative for w avoiding 312. Haiman [Hai93] conjectured that X_w is elementary nonnegative for all $w \in \mathfrak{S}_n$.

For each $w \in \mathfrak{S}_n$, we define a $\mathbb{Z}[q]$-symmetric generating function for values of $\theta_q(q_{e,w}C'_w(q))$ by

$$X_{T_w} = \sum_{\lambda \vdash n} \epsilon_q^\lambda(q_{e,w}C'_w(q)) m_\lambda \ \in\ \mathbb{Z}[q] \otimes \Lambda_n = \mathrm{span}_{\mathbb{Z}[q]}\{m_\lambda \,|\, \lambda \vdash n\}. \tag{14}$$

Expanding X_{T_w} in various bases of the homogeneous degree-n graded component of $\mathbb{Z}[q] \otimes \Lambda_n$, we have

$$X_{T_w} = \sum_{\lambda \vdash n} \eta_q^\lambda(q_{e,w}C'_w(q)) f_\lambda = \sum_{\lambda \vdash n} \chi_q^{\lambda^\top}(q_{e,w}C'_w(q)) s_\lambda = \sum_{\lambda \vdash n} \frac{\psi_q^\lambda(q_{e,w}C'_w(q))}{(-1)^{n-\ell(\lambda)}} \frac{p_\lambda}{z_\lambda} = \sum_{\lambda \vdash n} \phi_q^\lambda(q_{e,w}C'_w(q)) e_\lambda.$$

The function X_{T_w} specializes at $q = 1$ to X_w, and is related to the *chromatic quasisymmetric functions* $\{X_{P,q} \,|\, P$ a labeled poset $\}$ of Shareshian and Wachs [SW12], which specialize at $q = 1$ to X_P. The function $X_{P,q}$ is itself symmetric (i.e., it belongs to $\mathbb{Z}[q] \otimes \Lambda_n$) when P is $(\mathbf{3}+\mathbf{1})$-free, $(\mathbf{2}+\mathbf{2})$-free, and labeled strategically. If w avoids the pattern 312, then by Theorem 4.2, X_{T_w} is equal to the Shareshian-Wachs chromatic symmetric function $X_{P(F_w),q}$, with each element of $P(F_w)$ labeled according to the source and sink of the path in F_w it represents. Again, not all chromatic symmetric functions $X_{P,q}$ can be expressed as X_{T_w} for appropriate $w \in \mathfrak{S}_n$, nor can all generating functions X_{T_w} be expressed as $X_{P,q}$ for an appropriate labeled poset P. Shareshian and Wachs [SW12] conjectured that $X_{P,q}$ belongs to $\mathrm{span}_{\mathbb{N}[q]}\{e_\lambda \,|\, \lambda \vdash n\}$ when P is $(\mathbf{3}+\mathbf{1})$-free, $(\mathbf{2}+\mathbf{2})$-free, and labeled appropriately. By Theorem 4.2, this is equivalent to the conjecture that X_{T_w} belongs to $\mathrm{span}_{\mathbb{N}[q]}\{e_\lambda \,|\, \lambda \vdash n\}$ for w avoiding 312. Haiman [Hai93] conjectured that X_{T_w} belongs to $\mathrm{span}_{\mathbb{N}[q]}\{e_\lambda \,|\, \lambda \vdash n\}$ for all $w \in \mathfrak{S}_n$.

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