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ℓ-restricted Q-systems and quantum affine algebras

Anne-Sophie Gleitz*

Laboratoire de Mathématiques Nicolas Oresme, UCBN, Caen, France

Abstract. Kuniba, Nakanishi, and Suzuki (1994) have formulated a general conjecture expressing the positive solution of an ℓ-restricted Q-system in terms of quantum dimensions of Kirillov-Reshetikhin modules. After presenting this conjecture, we sketch a proof for the exceptional type E_6 following our preprint (2013). In types E_7 and E_8, we prove positivity for a subset of the nodes of the Dynkin diagram, and we reduce the positivity for the remaining nodes to the conjectural iterated log-concavity of certain explicit sequences of real algebraic numbers.

Résumé. Kuniba, Nakanishi et Suzuki (1994) ont formulé une conjecture générale qui exprime la solution positive d'un Q-system ℓ-restreint en fonction des dimensions quantiques de certains modules de Kirillov-Reshetikhin. Après avoir présenté cette conjecture, nous donnons une idée de la preuve pour le type exceptionnel E_6, selon notre preprint (arXiv, 2013). En types E_7 et E_8, nous démontrons la positivité pour certains sommets du diagramme de Dynkin, et nous réduisons la positivité, pour les sommets restants, à une conjecture de log-concavité itérée concernant certaines suites explicites de nombres algébriques.

Keywords: Kirillov-Reshetikhin modules, representation theory, characters, Q-systems, quantum dimension, Kuniba Nakanishi Suzuki (KNS)

1 ℓ-restricted Q-systems

Let δ be an ADE Dynkin diagram with vertex set I. Kirillov and Reshetikhin [6] have attached to δ an infinite system of algebraic equations called a Q-system, with unknowns $Q_k^{(i)}$ $(i \in I,\ k \in \mathbb{N})$ in a commutative ring. It is given by

$$\left(Q_k^{(i)}\right)^2 = Q_{k-1}^{(i)} Q_{k+1}^{(i)} + \prod_{j \sim i} Q_k^{(j)}, \qquad (i \in I,\ k \geq 1), \tag{1}$$

where, in the product, the notation $j \sim i$ means that j runs over all neighbours of i in the diagram δ. One usually imposes the initial condition:

$$Q_0^{(i)} = 1, \qquad (i \in I). \tag{2}$$

*Email: anne-sophie.gleitz@unicaen.fr

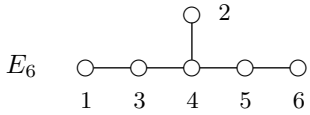

Figure 1: The Dynkin diagram for $\mathfrak{g} = E_6$ and its node numbering

Fix a positive integer ℓ. The ℓ-restricted Q-system has a finite number of unknowns

$$Q_k^{(i)} \ (i \in I,\ 0 \leq k \leq \ell)$$

satisfying the same equations (1) and (2) together with the new boundary condition:

$$Q_\ell^{(i)} = 1, \qquad (i \in I). \tag{3}$$

It is known (see [10]) that ℓ-restricted Q-systems always have a unique *positive* solution, that is, a unique solution satisfying

$$Q_k^{(i)} \in \mathbb{R}_{>0}, \qquad (i \in I,\ 1 \leq k \leq \ell - 1). \tag{4}$$

There are various motivations coming from conformal field theory and algebraic K-theory to obtain an explicit description of this positive solution [8]. In particular, if $Q_k^{(i)}$ $(i \in I,\ 0 \leq k \leq \ell)$ is this solution, then the numbers

$$x_k^{(i)} := \frac{\prod_{j \sim i} Q_k^{(j)}}{(Q_k^{(i)})^2} \in]0,1[, \qquad (i \in I,\ 1 \leq k \leq \ell - 1),$$

are the arguments of a nice dilogarithm identity, see [8, Th. 5.2, Prop. 14.1].

Example 1.1 Let us write the 2-restricted Q-system in type E_6 (see Figure 1 for the node numbering of the Dynkin diagram). Because of the boundary conditions (1) and (2), there are only 6 unknowns $Q_i := Q_1^{(i)}$ $(1 \leq i \leq 6)$, subject to

$$\begin{cases} Q_1^2 = 1 + Q_3, \\ Q_2^2 = 1 + Q_4, \\ Q_3^2 = 1 + Q_1 Q_4, \\ Q_4^2 = 1 + Q_2 Q_3 Q_5, \\ Q_5^2 = 1 + Q_4 Q_6, \\ Q_6^2 = 1 + Q_5. \end{cases}$$

Because of the unicity of the positive solution, if we further assume that $Q_i > 0$, we have

$$Q_1 = Q_6, \qquad Q_3 = Q_5,$$

due to the 2-fold symmetry of the Dynkin diagram of E_6. We are therefore reduced to the system

$$\begin{cases} Q_1^2 = 1 + Q_3, \\ Q_2^2 = 1 + Q_4, \\ Q_3^2 = 1 + Q_1 Q_4, \\ Q_4^2 = 1 + Q_2 Q_3^2, \end{cases}$$

whose unique positive solution is

$$\begin{cases} Q_1 = 1 + 2\cos\dfrac{2\pi}{7} \approx 2.246979604, \\ Q_2 = \sqrt{2}\cdot\sqrt{(4\cos^2\dfrac{2\pi}{7} + 6\cos\dfrac{2\pi}{7} + 1)\cos\dfrac{2\pi}{7}} \approx 2.801937735, \\ Q_3 = 8\cos\dfrac{2\pi}{7}\cos^2\dfrac{\pi}{7} \approx 4.048917341, \\ Q_4 = \left(2\cos\dfrac{2\pi}{7} + 1\right)\left(8\cos\dfrac{2\pi}{7}\cos^2\dfrac{\pi}{7} - 1\right) \approx 6.850855076. \end{cases}$$

If we further set

$$x_1 = \frac{Q_3}{Q_1^2}, \quad x_2 = \frac{Q_4}{Q_2^2}, \quad x_3 = \frac{Q_1 Q_4}{Q_3^2}, \quad x_4 = \frac{Q_2 Q_3^2}{Q_4^2},$$

we obtain the remarkable identity

$$2L(x_1) + L(x_2) + 2L(x_3) + L(x_4) = \frac{6\pi^2}{7},$$

where L denotes the Rogers dilogarithm function (see e.g. [14]).

2 The KNS conjecture

Kuniba, Nakanishi and Suzuki [7] have given a conjectural description of the positive solution of an ℓ-restricted Q-system in terms of quantum dimensions of Kirillov-Reshetikhin modules over the quantum affine algebra attached to δ, which we now recall.

Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$ with Dynkin diagram δ. Fix a Cartan subalgebra $\mathfrak{h}$. Let $\Phi \subset \mathfrak{h}^*$ be the root system of $\mathfrak{g}$, and Φ^+ its subset of positive roots. Let α_i, ϖ_i $(i \in I)$ denote the simple roots and the fundamental weights, respectively. We fix a symmetric bilinear form $(\cdot \mid \cdot)$ on $\mathfrak{h}^*$, normalized by

$$(\alpha_i \mid \alpha_i) = 2, \qquad (\alpha_i \mid \varpi_j) = \delta_{ij}. \tag{5}$$

We denote by

$$\rho = \sum_{i \in I} \varpi_i, \qquad \theta = \sum_i a_i \alpha_i, \tag{6}$$

the Weyl vector and the highest root, respectively. Here the a_i $(i \in I)$ are the Dynkin labels. We have

$$(\rho \mid \theta) = \sum_{i \in I} a_i = h - 1, \tag{7}$$

where h is the Coxeter number.

Let $P = \bigoplus_{i \in I} \mathbb{Z}\varpi_i$ be the weight lattice, and $P_+ = \bigoplus_{i \in I} \mathbb{N}\varpi_i$ the monoid of dominant weights. For $\lambda \in P_+$, we denote by $V(\lambda)$ the irreducible complex representation of $\mathfrak{g}$ with highest weight λ, and by $\chi(V(\lambda)) \in \mathbb{Z}[P]$ its character. The dimension of $V(\lambda)$ is given by Weyl's formula:

$$\dim V(\lambda) = \prod_{\beta \in \Phi^+} \frac{(\lambda + \rho \mid \beta)}{(\rho \mid \beta)}. \tag{8}$$

For $\zeta \in \mathbb{C}^*$, and $k \in \mathbb{Z}$ we define the ζ-integer:

$$[k]_\zeta := (\zeta^k - \zeta^{-k})(\zeta - \zeta^{-1})^{-1}.$$

The ζ-dimension of $V(\lambda)$ is the ζ-analogue of $\dim V(\lambda)$ given by

$$d_\zeta(\lambda) := \prod_{\beta \in \Phi^+} \frac{[(\lambda + \rho \mid \beta)]_\zeta}{[(\rho \mid \beta)]_\zeta}. \tag{9}$$

When $|\zeta| = 1$, this is a real number, well-defined if $[(\rho \mid \beta)]_\zeta \neq 0$ for each $\beta \in \Phi^+$.

Let $U_q(\widehat{\mathfrak{g}})$ be the Drinfeld-Jimbo quantum enveloping algebra of the affine Lie algebra $\widehat{\mathfrak{g}}$ associated with $\mathfrak{g}$ (see e.g. [4]). It has a natural subalgebra isomorphic to the quantum enveloping algebra $U_q(\mathfrak{g})$ of $\mathfrak{g}$. Here, we assume that the quantum parameter $q \in \mathbb{C}^*$ is not a root of unity. It follows that we can identify the characters of $U_q(\mathfrak{g})$ with those of $\mathfrak{g}$.

The Kirillov-Reshetikhin modules $W_{k,a}^{(i)}$ are some special irreducible finite-dimensional representations of $U_q(\widehat{\mathfrak{g}})$, depending on three parameters $i \in I$, $k \in \mathbb{N}$, $a \in \mathbb{C}^*$ (see [8, §4.2]). We will only be interested in their restrictions to $U_q(\mathfrak{g})$, which are independent of a, and will be denoted by $W_k^{(i)}$. The $U_q(\mathfrak{g})$-module $W_k^{(i)}$ is not irreducible in general. Its character can be expressed as an $\mathbb{N}$-linear combination of irreducible characters of $\mathfrak{g}$:

$$\chi\left(W_k^{(i)}\right) = \sum_{\lambda \in P_+} a_k^{(i)}(\lambda)\, \chi(V(\lambda)). \tag{10}$$

Define the ζ-dimension of a Kirillov-Reshetikhin module by

$$d_\zeta\left(W_k^{(i)}\right) := \sum_{\lambda \in P_+} a_k^{(i)}(\lambda)\, d_\zeta(\lambda). \tag{11}$$

From now on, we fix $\ell \in \mathbb{Z}_{>0}$ and we set

$$l := \ell + h, \qquad \zeta := \exp(i\pi/l). \tag{12}$$

Because of (7), the ζ-dimension (9) is well-defined for every $\lambda \in P_+$.

Conjecture 2.1 ([7, Conjecture 14.2]) *The collection of real numbers*

$$Q_k^{(i)} := d_\zeta\left(W_k^{(i)}\right), \qquad (i \in I,\ 0 \leq k \leq \ell),$$

is a solution of the ℓ-restricted Q-system. Moreover the following properties hold for any $i \in I$:

(i) $d_\zeta\left(W_k^{(i)}\right) = 0$ *for* $k \in [\![\ell + 1, l - 1]\!]$.

(ii) $Q_k^{(i)} = Q_{\ell-k}^{(i)}$ *for* $k \in [\![0, \ell]\!]$.

(iii) $Q_k^{(i)} > 0$ *for* $k \in [\![0, \ell]\!]$.

(iv) $Q_k^{(i)} < Q_{k+1}^{(i)}$ *for* $k \in [\![0, \lfloor \ell/2 \rfloor - 1]\!]$.

Example 2.2 Let us do a quick check in type A_3 for $\ell = 2$. In this simple situation the $U_q(\mathfrak{g})$-restrictions of the KR-modules $W_k^{(i)}$ are simple, and their dimensions are directly given by (9). Since $h = 4$ we have $\zeta = \exp(i\pi/6)$, and we easily compute:

$$\begin{cases} d_\zeta(W_1^{(1)}) = d_\zeta(\varpi_1) = \dfrac{\sin\frac{2\pi}{3}}{\sin\frac{\pi}{6}} = \sqrt{3}, \\ d_\zeta(W_1^{(2)}) = d_\zeta(\varpi_2) = \dfrac{1}{\sin\frac{\pi}{6}} = 2, \\ d_\zeta(W_1^{(3)}) = d_\zeta(W_1^{(1)}) = \sqrt{3}, \\ d_\zeta(W_2^{(1)}) = d_\zeta(2\varpi_1) = \dfrac{\sin\frac{\pi}{2}}{\sin\frac{\pi}{6}} \dfrac{\sin\frac{2\pi}{3}}{\sin\frac{\pi}{3}} \dfrac{\sin\frac{5\pi}{6}}{\sin\frac{\pi}{2}} = 1, \\ d_\zeta(W_2^{(2)}) = d_\zeta(2\varpi_2) = \dfrac{\sin\frac{\pi}{2}}{\sin\frac{\pi}{6}} \left(\dfrac{\sin\frac{2\pi}{3}}{\sin\frac{\pi}{3}}\right)^2 \dfrac{\sin\frac{5\pi}{6}}{\sin\frac{\pi}{2}} = 1, \\ d_\zeta(W_2^{(3)}) = d_\zeta(W_2^{(1)}) = 1. \end{cases}$$

On the other hand, the 2-restricted Q-system in type A_3 reads

$$\begin{cases} (Q_1^{(1)})^2 = Q_2^{(1)} + Q_1^{(2)} \\ (Q_1^{(2)})^2 = Q_2^{(2)} + Q_1^{(1)} Q_1^{(3)} \\ (Q_1^{(3)})^2 = Q_2^{(3)} + Q_1^{(2)} \\ Q_2^{(1)} = Q_2^{(2)} = Q_2^{(3)} = 1, \end{cases}$$

and it is easy to check that its only positive solution is given by

$$Q_1^{(1)} = Q_1^{(3)} = \sqrt{3}, \qquad Q_1^{(2)} = 2.$$

So Conjecture 2.1 holds in this case.

3 Type E

The KNS conjecture is easy to check in type A, but it remained an open problem for a long time in other types. Recently, Lee [10] has given a proof of the KNS conjecture in type D. We may and will therefore focus hereafter on type E.

Let $\mathfrak{g}$ be of exceptional type E. We number the nodes of its Dynkin diagram as in Figure 2. Recall that $h = 12$ in type E_6, $h = 18$ in type E_7, and $h = 30$ in type E_8. In [5] we have proved the following:

Theorem 3.1 *The collection of real numbers*

$$Q_k^{(i)} := d_\zeta\left(W_k^{(i)}\right), \qquad (i \in I,\ 0 \le k \le \ell),$$

is a solution of the ℓ-restricted Q-system. Moreover:

(a) In type E_6, properties (i), (ii), (iii), (iv) of Conjecture 2.1 hold for any $i \in I$. We also have

$$d_\zeta\left(W_{k+l}^{(i)}\right) = d_\zeta\left(W_k^{(i)}\right), \qquad (k \in \mathbb{N}).$$

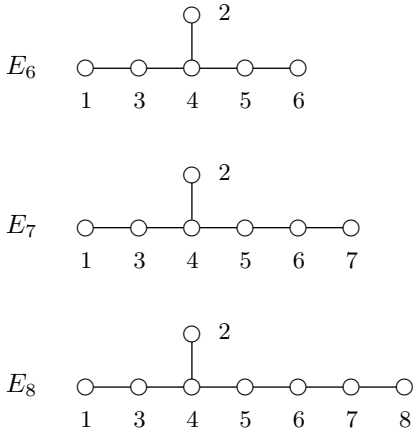

Figure 2: The Dynkin diagrams for $\mathfrak{g} = E_6, E_7, E_8$ and their node numberings

(b) In type E_7, properties (i), (ii) of Conjecture 2.1 hold for any $i \in I$, property (iii) holds for $i = 1, 2, 3, 6, 7$, and property (iv) holds for $i = 1, 2, 7$. We also have

$$\begin{aligned} d_\zeta\left(W_{k+l}^{(i)}\right) &= d_\zeta\left(W_k^{(i)}\right), \quad (k \in \mathbb{N},\ i = 1, 3, 4, 6), \\ d_\zeta\left(W_{k+l}^{(i)}\right) &= -d_\zeta\left(W_k^{(i)}\right), \quad (k \in \mathbb{N},\ i = 2, 5, 7). \end{aligned}$$

(c) In type E_8, property (i) of Conjecture 2.1 holds for any $i \in I$, property (ii) holds for $i = 1, 3, 4, 5, 6, 7, 8$, property (iii) holds for $i = 1, 3, 7, 8$, and property (iv) holds for $i = 1, 8$. We also have for any $i \in I$

$$d_\zeta\left(W_{k+l}^{(i)}\right) = d_\zeta\left(W_k^{(i)}\right), \qquad (k \in \mathbb{N}).$$

4 Outline of proof

It follows from the Kirillov-Reshetikhin conjecture, proved by Nakajima [12], that the characters $\chi(W_k^{(i)})$ $(i \in I,\ k \in \mathbb{N})$ give a solution of the unrestricted Q-system (1) (2) in the ring $\mathbb{Z}[P]$. The map $\chi(V) \mapsto d_\zeta(V)$ being additive and multiplicative, the numbers $d_\zeta(W_k^{(i)})$ $(i \in I,\ k \in \mathbb{N})$ give therefore a solution of the unrestricted Q-system in $\mathbb{R}$. So to prove Conjecture 2.1 we only need to check that

$$d_\zeta(W_\ell^{(i)}) = 1 \qquad (i \in I),$$

and that the collection $Q_k^{(i)} := d_\zeta(W_k^{(i)})$ $(i \in I,\ 0 \leq k \leq l)$ satisfies the additional properties (i), (ii), (iii), (iv).

What makes type E more difficult than type D is that there are many nodes of the Dynkin diagram with labels $a_i > 2$, and it is difficult to check positivity of the ζ-dimensions at these nodes. Moreover, there are no convenient characters formulas in the literature for the Kirillov-Reshetikhin modules, except for the extremal nodes. To get around these difficulties, we first check the KNS conjecture at the extremal nodes, and then move to the remaining nodes using the Q-system.

Let us outline the strategy of the proof in type E_6. We start from the following formulas of Chari [3, Section 3]:

$$\chi(W_k^{(i)}) = \chi(V(k\varpi_i)) \quad \text{for } i = 1, 6, \tag{13}$$

$$\chi(W_k^{(2)}) = \sum_{r=0}^{k} \chi(V(r\varpi_2)). \tag{14}$$

Using (13) and (9), we can then check rather easily the desired properties at nodes 1 and 6.

Since, by (14), $W_k^{(2)}$ is not irreducible, we need more tools at node 2. Following Lee [9, 10], we use the level l dot action of the affine Weyl group $\widehat{W}$ of $\widehat{\mathfrak{g}}$ on the weight lattice P, denoted by

$$\lambda \mapsto w \cdot \lambda, \qquad (\lambda \in P,\ w \in \widehat{W}),$$

see [5]. If λ and $w \cdot \lambda$ are dominant, it is known that

$$d_\zeta(\lambda) = (-1)^{\ell(w)} d_\zeta(w \cdot \lambda), \tag{15}$$

where $\ell(w)$ is the length of w in $\widehat{W}$. Applying (15) with $w = s_0$ and $\lambda = k\varpi_2$, we show that

$$d_\zeta(k\varpi_2) = -d_\zeta((\ell + 1 - k)\varpi_2),$$

and this formula, together with (14), allows us to check all desired properties at node 2.

Using the Q-system equations,

$$\begin{aligned}
d_\zeta(W_k^{(1)})^2 &= d_\zeta(W_{k+1}^{(1)}) d_\zeta(W_{k-1}^{(1)}) + d_\zeta(W_k^{(3)}),\\
d_\zeta(W_k^{(2)})^2 &= d_\zeta(W_{k+1}^{(2)}) d_\zeta(W_{k-1}^{(2)}) + d_\zeta(W_k^{(4)}),\\
d_\zeta(W_k^{(6)})^2 &= d_\zeta(W_{k+1}^{(6)}) d_\zeta(W_{k-1}^{(6)}) + d_\zeta(W_k^{(5)}),
\end{aligned}$$

it is then easy to deduce that

$$d_\zeta(W_\ell^{(3)}) = d_\zeta(W_\ell^{(4)}) = d_\zeta(W_\ell^{(5)}) = 1,$$

and that properties (i) and (ii) are verified at nodes 3, 4, and 5. Moreover, property (3), that is, positivity, is clearly equivalent to the fact that the sequences $d_\zeta(W_k^{(1)})$, $d_\zeta(W_k^{(2)})$ and $d_\zeta(W_k^{(6)})$ are strictly log-concave.

Recall that a finite sequence of real numbers $(a_k)_{0 \le k \le n}$ is called log-concave if

$$a_k^2 \ge a_{k-1} a_{k+1}, \qquad (1 \le k \le n-1). \tag{16}$$

It is strictly log-concave if $a_k^2 > a_{k-1} a_{k+1}$. Using well-known properties of log-concave sequences (see e.g. [13]), we show

Lemma 4.1 *For every $i \in I$, the finite sequence $d_\zeta(k\varpi_i)$ $(0 \le k \le \ell/a_i)$ is strictly log-concave and positive, where a_i is the Dynkin label at vertex i.*

We then deduce from Lemma 4.1 that $d_\zeta(W_k^{(1)})$, $d_\zeta(W_k^{(2)})$ and $d_\zeta(W_k^{(6)})$ are strictly log-concave, and from this we obtain that properties (iii) and (iv) also hold at nodes 3, 4, 5, which finishes the proof for E_6.

The proofs in type E_7 and E_8 are essentially the same as for type E_6. However log-concavity is no longer sufficient to establish positivity at some inner nodes of the Dynkin diagram, and some stronger form of log-concavity would be necessary. Let us illustrate this in type E_7.

Following [1], define $\mathfrak{L}$ to be the operator mapping the sequence of real numbers $(a_k)_{0\leq k\leq n}$ to the sequence $(b_k)_{0\leq k\leq n}$ defined by

$$b_k := a_k^2 - a_{k+1}a_{k-1}, \qquad (0 \leq k \leq n),$$

where we have set $a_{-1} = a_{n+1} = 0$. Hence (a_k) is log-concave if $\mathfrak{L}(a_k)$ is non-negative. We say that (a_k) is *i-fold log-concave* if the ith iterate $\mathfrak{L}^i(a_k)$ is non-negative, and *infinitely log-concave* if it is i-fold log-concave for all $i \in \mathbb{N}$. Interesting examples of i-fold log-concave sequences have been studied recently in combinatorics, see e.g. [1, 2, 11].

It is easy to deduce from the Q-system that the non-negativity of $Q_k^{(i)}$ for $0 \leq k \leq \ell$ and $i = 4, 5$ amounts to the fact that the sequence

$$Q_k^{(7)} = d_\zeta(k\varpi_7), \qquad (0 \leq k \leq \ell) \tag{17}$$

is 3-fold and 2-fold log-concave. We thus conjecture

Conjecture 4.2 *The sequence (17) is i-fold log-concave for $i = 2, 3$.*

In fact we believe that this sequence is even infinitely log-concave. Here are a few sample values for $Q_k^{(7)}$ when ℓ is small.

ℓ	$Q_0^{(7)}$	$Q_1^{(7)}$	$Q_2^{(7)}$	$Q_3^{(7)}$	$Q_4^{(7)}$	$Q_5^{(7)}$
2	1	2.2882456111	1			
3	1	3.7912878474	3.7912878474	1		
4	1	5.4436536231	9.0266741836	5.4436536231	1	
5	1	7.1903369448	17.142395935	17.142395935	7.1903369448	1

In type E_8, we state a similar conjecture:

Conjecture 4.3 *The sequence*

$$d_\zeta(W_k^{(8)}) = \sum_{r=0}^{k} d_\zeta(r\varpi_8)$$

is i-fold log-concave for $i = 3, 4, 5$.

This would imply that the sequences $(Q_k^{(i)})_{0\leq k\leq \ell}$ $(i = 6, 5, 4)$ are non-negative.

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