0-Hecke algebra action on the Stanley-Reisner ring of the Boolean algebra

Jia Huang*

School of Mathematics, University of Minnesota, Minneapolis, MN 55455, USA

Abstract. We define an action of the 0-Hecke algebra of type A on the Stanley-Reisner ring of the Boolean algebra. By studying this action we obtain a family of multivariate noncommutative symmetric functions, which specialize to the noncommutative Hall-Littlewood symmetric functions and their (q,t)-analogues introduced by Bergeron and Zabrocki. We also obtain multivariate quasisymmetric function identities, which specialize to a result of Garsia and Gessel on the generating function of the joint distribution of five permutation statistics.

Résumé. Nous définissons une action de l'algèbre de Hecke-0 de type A sur l'anneau Stanley-Reisner de l'algèbre de Boole. En étudiant cette action, on obtient une famille de fonctions symétriques non commutatives multivariées, qui se spécialisent pour les non commutatives fonctions de Hall-Littlewood symétriques et leur (q,t)-analogues introduits par Bergeron et Zabrocki. Nous obtenons également des identités de fonction quasisymmetric multivariées, qui se spécialisent à la suite de Garsia et Gessel sur la fonction génératrice de la distribution conjointe de cinq statistiques de permutation.

Keywords: 0-Hecke algebra, Stanley-Reisner ring, Boolean algebra, noncommutative Hall-Littlewood symmetric function, multivariate quasisymmetric function.

1 Introduction

Let $\mathbb{F}$ be any field. The symmetric group $\mathfrak{S}_n$ naturally acts on the polynomial ring $\mathbb{F}[X] := \mathbb{F}[x_1, \ldots, x_n]$ by permuting the variables $x_1, \ldots, x_n$. The invariant algebra $\mathbb{F}[X]^{\mathfrak{S}_n}$, which consists of all the polynomials fixed by this $\mathfrak{S}_n$-action, is a polynomial algebra generated by the elementary symmetric functions $e_1, \ldots, e_n$. The coinvariant algebra $\mathbb{F}[X]/(\mathbb{F}[X]_+^{\mathfrak{S}_n})$, with $(\mathbb{F}[X]_+^{\mathfrak{S}_n}) = (e_1, \ldots, e_n)$, is a vector space of dimension $n!$ over $\mathbb{F}$, and when $\mathbb{F}$ has characteristic larger than n the coinvariant algebra carries the regular representation of $\mathfrak{S}_n$. A well known basis for $\mathbb{F}[X]/(\mathbb{F}[X]_+^{\mathfrak{S}_n})$ consists of the descent monomials. Garsia [7] obtained this basis by transferring a natural basis from the Stanley-Reisner ring $\mathbb{F}[\mathcal{B}_n]$ of the Boolean algebra $\mathcal{B}_n$ to the polynomial ring $\mathbb{F}[X]$. Here the *Boolean algebra* $\mathcal{B}_n$ is the set of all subsets of $[n] := \{1, 2, \ldots, n\}$ partially ordered by inclusion, and the *Stanley-Reisner ring* $\mathbb{F}[\mathcal{B}_n]$ is the quotient of the polynomial algebra $\mathbb{F}\left[y_A : A \subseteq [n]\right]$ by the ideal $\left(y_A y_B : A \text{ and } B \text{ are incomparable in } \mathcal{B}_n\right)$.

The 0-Hecke algebra $H_n(0)$ (of type A) is a deformation of the group algebra of $\mathfrak{S}_n$. It acts on $\mathbb{F}[X]$ by the Demazure operators, also known as the isobaric divided difference operators, having the same

*Email: `jiahuangustc@gmail.com`. Supported by NSF grant DMS-1001933.

invariant algebra as the $\mathfrak{S}_n$-action on $\mathbb{F}[X]$. In our earlier work [13], we showed that the coinvariant algebra $\mathbb{F}[X]/(\mathbb{F}[X]_+^{\mathfrak{S}_n})$ is also isomorphic to the regular representation of $H_n(0)$, for any field $\mathbb{F}$, by constructing another basis for $\mathbb{F}[X]/(\mathbb{F}[X]_+^{\mathfrak{S}_n})$ which consists of certain polynomials whose leading terms are the descent monomials. This and the previously mentioned connection between the Stanley-Reisner ring $\mathbb{F}[\mathcal{B}_n]$ and the polynomial ring $\mathbb{F}[X]$ motivate us to define an $H_n(0)$-action on $\mathbb{F}[\mathcal{B}_n]$.

It turns out that our $H_n(0)$-action on $\mathbb{F}[\mathcal{B}_n]$ has similar properties to the $H_n(0)$-action on $\mathbb{F}[X]$. It preserves an $\mathbb{N}^{n+1}$-multigrading of $\mathbb{F}[\mathcal{B}_n]$ and has invariant algebra equal to a polynomial algebra $\mathbb{F}[\Theta]$, where Θ is the set of *rank polynomials* θ_i (the usual analogue of e_i in $\mathbb{F}[\mathcal{B}_n]$). We show that the $H_n(0)$-action is Θ-linear and thus descends to the coinvariant algebra $\mathbb{F}[\mathcal{B}_n]/(\Theta)$. Using the descent monomials in $\mathbb{F}[\mathcal{B}_n]$ it is not hard to see that $\mathbb{F}[\mathcal{B}_n]/(\Theta)$ carries the regular representation of $H_n(0)$.

It is well known that every finite dimensional (complex) $\mathfrak{S}_n$-representation is a direct sum of simple (i.e. irreducible) $\mathfrak{S}_n$-modules, and the simple $\mathfrak{S}_n$-modules are indexed by partitions λ of n, which correspond to the Schur functions s_λ via the *Frobenius characteristic map*. Hotta-Springer [11] and Garsia-Procesi [9] discovered that the cohomology ring of the *Springer fiber* indexed by a partition μ of n is isomorphic to certain quotient ring R_μ of $\mathbb{F}[X]$, which admits a graded $\mathfrak{S}_n$-module structure corresponding to the *modified Hall-Littlewood symmetric function* $\widetilde{H}_\mu(x;t)$ via the Frobenius characteristic map. The coinvariant algebra of $\mathfrak{S}_n$ is nothing but R_{1^n}.

In our previous work [13] we established a partial analogue of the above result by showing that the $H_n(0)$-action on $\mathbb{F}[X]$ descends to R_μ if and only if $\mu = (1^k, n-k)$ is a hook, and if so then R_μ has graded quasisymmetric characteristic equal to $\widetilde{H}_\mu(x;t)$ and graded noncommutative characteristic $\widetilde{\mathbf{H}}_\mu(\mathbf{x};t)$. Here $\widetilde{\mathbf{H}}_\alpha(\mathbf{x};t)$ is the noncommutative modified Hall-Littlewood symmetric function introduced by Bergeron and Zabrocki [3] for any composition α of n. Using an analogue of the nabla operator Bergeron and Zabrocki [3] also introduced a (q,t)-analogue $\widetilde{\mathbf{H}}_\alpha(\mathbf{x};q,t)$ for any composition α. Now we provide in Theorem 1.1 below a complete representation theoretic interpretation for $\widetilde{\mathbf{H}}_\alpha(\mathbf{x};t)$ and $\widetilde{\mathbf{H}}_\alpha(\mathbf{x};q,t)$ by the $H_n(0)$-action on $\mathbb{F}[\mathcal{B}_n]$.

To state our result, we first recall the two characteristic maps for representations of $H_n(0)$ introduced by Krob and Thibon [14], which we call the *quasisymmetric characteristic* and the *noncommutative characteristic*. The simple $H_n(0)$-modules are indexed by compositions α of n and correspond to the *fundamental quasisymmetric functions* F_α via the quasisymmetric characteristic; the projective indecomposable $H_n(0)$-modules are also indexed by compositions α of n and correspond to the *noncommutative ribbon Schur functions* $\mathbf{s}_\alpha$ via the noncommutative characteristic. See §2 for details.

Theorem 1.1 *Let α be a composition of n. Then there exists a homogeneous $H_n(0)$-invariant ideal I_α of the multigraded algebra $\mathbb{F}[\mathcal{B}_n]$ such that the quotient algebra $\mathbb{F}[\mathcal{B}_n]/I_\alpha$ becomes a projective $H_n(0)$-module with multigraded noncommutative characteristic equal to*

$$\widetilde{\mathbf{H}}_\alpha(\mathbf{x};t_1,\ldots,t_{n-1}) := \sum_{\beta\preccurlyeq\alpha} \underline{t}^{D(\beta)}\mathbf{s}_\beta \quad \text{inside} \quad \mathbf{NSym}[t_1,\ldots,t_{n-1}].$$

One has $\widetilde{\mathbf{H}}_\alpha(\mathbf{x};t,t^2,\ldots,t^{n-1}) = \widetilde{\mathbf{H}}_\alpha(\mathbf{x};t)$, and obtains $\widetilde{\mathbf{H}}_\alpha(\mathbf{x};q,t)$ from $\widetilde{\mathbf{H}}_{1^n}(\mathbf{x};t_1,\ldots,t_{n-1})$ by taking $t_i = t^i$ for all $i\in D(\alpha)$, and $t_i = q^{n-i}$ for all $i \in [n-1]\setminus D(\alpha)$.

Here $D(\alpha)$ is the set of partial sums of α, the notation $\beta\preccurlyeq\alpha$ means α and β are compositions of n with $D(\beta)\subseteq D(\alpha)$, and $\underline{t}^S$ denotes the product $\prod_{i\in S} t_i$ over all elements i in a multiset S, including the repeated ones. Taking $\alpha = (1^n)$ shows that $\mathbb{F}[\mathcal{B}_n]/(\Theta)$ carries the regular representation of $H_n(0)$.

Specializations of $\widetilde{\mathbf{H}}_{1^n}(\mathbf{x}; t_1, \ldots, t_{n-1})$ include not only $\widetilde{\mathbf{H}}_\alpha(\mathbf{x}; q, t)$, but also a more general family of noncommutative symmetric functions depending on parameters associated with paths in binary trees introduced recently by Lascoux, Novelli, and Thibon [15].

Next we study the quasisymmetric characteristic of $\mathbb{F}[\mathcal{B}_n]$. We combine the usual $\mathbb{N}^{n+1}$-multigrading of $\mathbb{F}[\mathcal{B}_n]$ (recorded by $\underline{t} := t_0, \ldots, t_n$) with the length filtration of $H_n(0)$ (recorded by q) and obtain an $\mathbb{N} \times \mathbb{N}^{n+1}$-multigraded quasisymmetric characteristic for $\mathbb{F}[\mathcal{B}_n]$.

Theorem 1.2 *The $\mathbb{N} \times \mathbb{N}^{n+1}$-multigraded quasisymmetric characteristic of $\mathbb{F}[\mathcal{B}_n]$ is*

$$\begin{aligned}
\mathrm{Ch}_{q,\underline{t}}(\mathbb{F}[\mathcal{B}_n]) &= \sum_{k\geq 0} \sum_{\alpha \in \mathrm{Com}(n,k+1)} \underline{t}^{D(\alpha)} \sum_{w \in \mathfrak{S}^\alpha} q^{\mathrm{inv}(w)} F_{D(w^{-1})} \\
&= \sum_{w \in \mathfrak{S}_n} \frac{q^{\mathrm{inv}(w)} \underline{t}^{D(w)} F_{D(w^{-1})}}{\prod_{0\leq i\leq n}(1-t_i)} \\
&= \sum_{k\geq 0} \sum_{\mathbf{p} \in [k+1]^n} t_{p'_1} \cdots t_{p'_k} q^{\mathrm{inv}(\mathbf{p})} F_{D(\mathbf{p})}.
\end{aligned}$$

Here we identify F_I with F_α if $D(\alpha) = I \subseteq [n-1]$. The set $\mathrm{Com}(n,k)$ consists of all *weak compositions of n with length k*, i.e. all the sequences $\alpha = (\alpha_1, \ldots, \alpha_k)$ of k nonnegative integers with $|\alpha| := \sum_{i=1}^k \alpha_i = n$. The *descent multiset* of the weak composition α is the *multiset*

$$D(\alpha) := \{\alpha_1, \alpha_1 + \alpha_2, \ldots, \alpha_1 + \cdots + \alpha_{k-1}\}.$$

We also define $\mathfrak{S}^\alpha := \{w \in \mathfrak{S}_n : D(w) \subseteq D(\alpha)\}$. The set $[k+1]^n$ consists of all words of length n on the alphabet $[k+1]$. Given $\mathbf{p} = (p_1, \ldots, p_n) \in [k+1]^n$, we write $p'_i := \#\{j : p_j \leq i\}$, $\mathrm{inv}(\mathbf{p}) := \#\{(i,j) : 1 \leq i < j \leq n : p_i > p_j\}$, and $D(\mathbf{p}) := \{i : p_i > p_{i+1}\}$.

Let $\mathbf{ps}_{q;\ell}(F_\alpha) := F_\alpha(1, q, q^2, \ldots, q^{\ell-1}, 0, 0, \ldots)$. Applying the linear transformation $\sum_{\ell\geq 0} u_1^\ell \mathbf{ps}_{q_1;\ell+1}$ and the specialization $t_i = q_2^i u_2$ for all $i = 0, 1, \ldots, n$ to Theorem 1.2, we recover a result of Garsia and Gessel [8, Theorem 2.2] on the generating function of the joint distribution of five permutation statistics:

$$\frac{\sum_{w\in\mathfrak{S}_n} q_0^{\mathrm{inv}(w)} q_1^{\mathrm{maj}(w^{-1})} u_1^{\mathrm{des}(w^{-1})} q_2^{\mathrm{maj}(w)} u_2^{\mathrm{des}(w)}}{(u_1; q_1)_n (u_2; q_2)_n} = \sum_{\ell,k\geq 0} u_1^\ell u_2^k \sum_{(\lambda,\mu)\in B(\ell,k)} q_0^{\mathrm{inv}(\mu)} q_1^{|\lambda|} q_2^{|\mu|} \qquad (1)$$

Here $(u;q)_n := \prod_{0\leq i\leq n}(1-q^i u)$, the set $B(\ell,k)$ consists of pairs of weak compositions $\lambda = (\lambda_1, \ldots, \lambda_n)$ and $\mu = (\mu_1, \ldots, \mu_n)$ satisfying the conditions $\ell \geq \lambda_1 \geq \cdots \geq \lambda_n$, $\max\{\mu_i : 1 \leq i \leq n\} \leq k$, and $\lambda_i = \lambda_{i+1} \Rightarrow \mu_i \geq \mu_{i+1}$ (such pairs (λ, μ) are sometimes called *bipartite partitions*), and $\mathrm{inv}(\mu)$ is the number of inversion pairs in μ. Some further specializations of Theorem 1.2 imply identities of Carlitz-MacMahon [6, 17] and Adin-Brenti-Roichman [1].

The structure of this paper is as follows. Section 2 reviews the representation theory of the 0-Hecke algebra. Section 3 studies the Stanley-Reisner ring of the Boolean algebra. Section 4 defines a 0-Hecke algebra action on the Stanley-Reisner ring of the Boolean algebra. The noncommutative and quasisymmetric characteristics are discussed in Section 5 and Section 6. Finally we give some remarks and questions for future research in Section 7, including a generalization to an action of the Hecke algebra of any finite Coxeter group on the Stanley-Reisner ring of the Coxeter complex.

2 Representation theory of the 0-Hecke algebra

We review the representation theory of the 0-Hecke algebra in this section. The (type A) *Hecke algebra* $H_n(q)$ is the associative $\mathbb{F}(q)$-algebra generated by $T_1, \ldots, T_{n-1}$ with relations

$$\begin{cases} (T_i+1)(T_i-q)=0, & 1\le i\le n-1,\\ T_iT_j=T_jT_i, & 1\le i,j\le n-1,\ |i-j|>1,\\ T_iT_{i+1}T_i=T_{i+1}T_iT_{i+1}, & 1\le i\le n-2. \end{cases} \tag{2}$$

It has an $\mathbb{F}(q)$-basis $\{T_w : w\in\mathfrak{S}_n\}$ where $T_w := T_{i_1}\cdots T_{i_k}$ if $w=s_{i_1}\cdots s_{i_k}$ is a reduced expression.

Specializing $q=1$ gives the group algebra of $\mathfrak{S}_n$, with $s_i=T_i|_{q=1}$ and $w=T_w|_{q=1}$. Let $w\in\mathfrak{S}_n$. The *length* of w equals $\mathrm{inv}(w) := \#\{(i,j) : 1\le i<j\le n,\ w(i)>w(j)\}$, and the *descent set* of w is $D(w)=\{i : 1\le i\le n-1,\ w(i)>w(i+1)\}$. We write $\mathrm{des}(w) := |D(w)|$ and $\mathrm{maj}(w) := \sum_{i\in D(w)} i$.

Let α be a (weak) composition of n, and let α^c be the composition of n with $D(\alpha^c)=[n-1]\setminus D(\alpha)$. The *parabolic subgroup* $\mathfrak{S}_\alpha$ is the subgroup of $\mathfrak{S}_n$ generated by $\{s_i : i\in D(\alpha^c)\}$. The set of all minimal $\mathfrak{S}_\alpha$-coset representatives is $\mathfrak{S}^\alpha := \{w\in\mathfrak{S}_n : D(w)\subseteq D(\alpha)\}$. The *descent class* of α consists of the permutations in $\mathfrak{S}_n$ with descent set equal to $D(\alpha)$, and turns out to be an interval under the (left) weak order of $\mathfrak{S}_n$, denoted by $[w_0(\alpha), w_1(\alpha)]$. One sees that $w_0(\alpha)$ is the longest element of the parabolic subgroup $\mathfrak{S}_{\alpha^c}$, and $w_1(\alpha)$ is the longest element in $\mathfrak{S}^\alpha$ (c.f. Björner and Wachs [5, Theorem 6.2]).

Another interesting specialization of $H_n(q)$ is the *0-Hecke algebra* $H_n(0)$, with generators $\overline{\pi}_i = T_i|_{q=0}$ for $i=1,\ldots,n-1$, and an $\mathbb{F}$-basis $\{\overline{\pi}_w = T_w|_{q=0} : w\in\mathfrak{S}_n\}$. Let $\pi_i := \overline{\pi}_i+1$. Then $\pi_1,\ldots,\pi_{n-1}$ form another generating set for $H_n(0)$, with the same relations as (2) except $\pi_i^2=\pi_i$, $1\le i\le n-1$. The element $\pi_w := \pi_{i_1}\cdots\pi_{i_k}$ is well defined for any $w\in\mathfrak{S}_n$ with a reduced expression $w=s_{i_1}\cdots s_{i_k}$, and $\{\pi_w : w\in\mathfrak{S}_n\}$ is another $\mathbb{F}$-bases for $H_n(0)$. One can check that π_w equals the sum of $\overline{\pi}_u$ over all u less than or equal to w in the Bruhat order of $\mathfrak{S}_n$. In particular, $\pi_{w_0(\alpha)}$ is the sum of $\overline{\pi}_u$ for all $u\in\mathfrak{S}_{\alpha^c}$.

Norton [18] decomposed the 0-Hecke algebra $H_n(0)$ into a direct sum of projective indecomposable submodules $\mathbf{P}_\alpha := H_n(0)\cdot\overline{\pi}_{w_0(\alpha)}\pi_{w_0(\alpha^c)}$ for all $\alpha\models n$ (i.e. compositions of n). Each $\mathbf{P}_\alpha$ has an $\mathbb{F}$-basis $\left\{\overline{\pi}_w\pi_{w_0(\alpha^c)} : w\in[w_0(\alpha), w_1(\alpha)]\right\}$. Its *radical* $\mathrm{rad}\,\mathbf{P}_\alpha$ is the unique maximal $H_n(0)$-submodule spanned by $\{\overline{\pi}_w\pi_{w_0(\alpha^c)} : w\in(w_0(\alpha), w_1(\alpha)]\}$. Although $\mathbf{P}_\alpha$ itself is not necessarily simple, its *top* $\mathbf{C}_\alpha := \mathbf{P}_\alpha/\,\mathrm{rad}\,\mathbf{P}_\alpha$ is a one-dimensional simple $H_n(0)$-module with the action of $H_n(0)$ given by

$$\overline{\pi}_i = \begin{cases} -1, & \text{if } i\in D(\alpha),\\ 0, & \text{if } i\notin D(\alpha). \end{cases}$$

It follows from general representation theory of algebras (see e.g. [2, §I.5]) that $\{\mathbf{P}_\alpha : \alpha\models n\}$ and $\{\mathbf{C}_\alpha : \alpha\models n\}$ are the complete lists of pairwise non-isomorphic projective indecomposable and simple $H_n(0)$-modules, respectively.

Krob and Thibon [14] introduced a correspondence between $H_n(0)$-representations and the dual Hopf algebras QSym and $\mathbf{NSym}$, which we review next. The Hopf algebra QSym has a free $\mathbb{Z}$-basis of *fundamental quasisymmetric functions* F_α, and the dual Hopf algebra $\mathbf{NSym}$ has a dual basis of *noncommutative ribbon Schur functions* $\mathbf{s}_\alpha$, for all compositions α.

Let $M=M_0\supseteq M_1\supseteq\cdots\supseteq M_k\supseteq M_{k+1}=0$ be a composition series of $H_n(0)$-modules with simple factors $M_i/M_{i+1}\cong\mathbf{C}_{\alpha^{(i)}}$ for $i=0,1,\ldots,k$. Then the *quasisymmetric characteristic* of M is

$$\mathrm{Ch}(M) := F_{\alpha^{(0)}}+\cdots+F_{\alpha^{(k)}}.$$

The *noncommutative characteristic* of a projective $H_n(0)$-module $M \cong \mathbf{P}_{\alpha^{(1)}} \oplus \cdots \oplus \mathbf{P}_{\alpha^{(k)}}$ is

$$\mathbf{ch}(M) := \mathbf{s}_{\alpha^{(1)}} + \cdots + \mathbf{s}_{\alpha^{(k)}}.$$

It is not hard to extend these characteristic maps to $H_n(0)$-modules with gradings and filtrations.

3 Stanley-Reisner ring of the Boolean algebra

In this section we study the Stanley-Reisner ring of the Boolean algebra. The *Boolean algebra* $\mathcal{B}_n$ is the ranked poset of all subsets of $[n] := \{1, 2, \ldots, n\}$ ordered by inclusion, with minimum element $\emptyset$ and maximum element $[n]$. The rank of a subset of $[n]$ is defined as its cardinality. The *Stanley-Reisner ring* $\mathbb{F}[\mathcal{B}_n]$ of the Boolean algebra $\mathcal{B}_n$ is the quotient of the polynomial algebra $\mathbb{F}\left[y_A : A \subseteq [n]\right]$ by the ideal $\left(y_A y_B : A \text{ and } B \text{ are incomparable in } \mathcal{B}_n\right)$. It has an $\mathbb{F}$-basis $\{y_M\}$ indexed by the multichains M in $\mathcal{B}_n$, and is multigraded by the rank multisets $r(M)$ of the multichains M.

The symmetric group $\mathfrak{S}_n$ acts on the Boolean algebra $\mathcal{B}_n$ by permuting the integers $1, \ldots, n$. This induces an $\mathfrak{S}_n$-action on the Stanley-Reisner ring $\mathbb{F}[\mathcal{B}_n]$, preserving its multigrading. The *invariant algebra* $\mathbb{F}[\mathcal{B}_n]^{\mathfrak{S}_n}$ consists of all elements in $\mathbb{F}[\mathcal{B}_n]$ invariant under this $\mathfrak{S}_n$-action. One can show that $\mathbb{F}[\mathcal{B}_n]^{\mathfrak{S}_n} = \mathbb{F}[\Theta]$, where $\Theta := \{\theta_0, \ldots, \theta_n\}$. Garsia [7] showed that $\mathbb{F}[\mathcal{B}_n]$ is a free $\mathbb{F}[\Theta]$-module on the basis of descent monomials

$$Y_w := \prod_{i \in D(w)} y_{\{w(1), \ldots, w(i)\}}, \quad \forall w \in \mathfrak{S}_n. \tag{3}$$

There is an analogy between the Stanley-Reisner ring $\mathbb{F}[\mathcal{B}_n]$ and the polynomial ring $\mathbb{F}[X]$ via the *transfer map* $\tau : \mathbb{F}[\mathcal{B}_n] \to \mathbb{F}[X]$ defined by

$$\tau(y_M) := \prod_{1 \le i \le k} \prod_{j \in A_i} x_j$$

for all multichains $M = (A_1 \subseteq \cdots \subseteq A_k)$ in $\mathcal{B}_n$. It is *not* a ring homomorphism (e.g. $y_{\{1\}} y_{\{2\}} = 0$ but $x_1 x_2 \ne 0$). Nevertheless, it induces an isomorphism $\tau : \mathbb{F}[\mathcal{B}_n]/(\theta_0) \cong \mathbb{F}[X]$ of $\mathfrak{S}_n$-modules. Moreover, it sends the rank polynomials $\theta_1, \ldots, \theta_n$ to the elementary symmetric polynomials $e_1, \ldots, e_n$, and sends the descent monomials Y_w in $\mathbb{F}[\mathcal{B}_n^*]$ defined by (3) to the corresponding descent monomials in $\mathbb{F}[X]$ for all $w \in \mathfrak{S}_n$.

Example 3.1 *The Boolean algebra $\mathcal{B}_3$ consists of all subsets of $\{1,2,3\}$. Its Stanley-Reisner ring $\mathbb{F}[\mathcal{B}_3]$ is a free $\mathbb{F}[\Theta]$-module with a basis of descent monomials $Y_1 = 1$, $Y_{s_1} = y_2$, $Y_{s_2 s_1} = y_3$, $Y_{s_2} = y_{13}$, $Y_{s_1 s_2} = y_{23}$, $Y_{s_1 s_2 s_1} = y_{23} y_3$, where Θ consists of the rank polynomials $\theta_0 = y_\emptyset$, $\theta_1 := y_1 + y_2 + y_3$, $\theta_2 := y_{12} + y_{13} + y_{23}$, $\theta_3 := y_{123}$. The transfer map τ sends $\theta_1, \theta_2, \theta_3$ to e_1, e_2, e_3, and sends the six descent monomials in $\mathbb{F}[\mathcal{B}_3]$ to the six descent monomials 1, x_2, x_3, $x_1 x_3$, $x_2 x_3$, $x_2 x_3^2$ in $\mathbb{F}[x_1, x_2, x_3]$.*

The homogeneous components of $\mathbb{F}[\mathcal{B}_n]$ are indexed by multisets with elements in $\{0, \ldots, n\}$, or equivalently by weak compositions α of n. The α-homogeneous component $\mathbb{F}[\mathcal{B}_n]_\alpha$ has an $\mathbb{F}$-basis $\{y_M : r(M) = D(\alpha)\}$. Denote by $\mathrm{Com}(n,k)$ the set of all weak compositions of n with length k. If $M = (A_1 \subseteq \cdots \subseteq A_k)$ is a multichain of length k in $\mathcal{B}_n$ then we set $A_0 := \emptyset$ and $A_{k+1} := [n]$ by convention. Define $\alpha(M) := (\alpha_1, \ldots, \alpha_{k+1})$, where $\alpha_i = |A_i| - |A_{i-1}|$ for all $i \in [k+1]$. Then

$\alpha(M) \in \mathrm{Com}(n,k+1)$ and $D(\alpha(M)) = r(M)$, i.e. $\alpha(M)$ indexes the homogeneous component containing y_M. Define $\sigma(M)$ to be the minimal element in $\mathfrak{S}_n$ which sends the standard multichain $[\alpha_1] \subseteq [\alpha_1+\alpha_2] \subseteq \cdots \subseteq [\alpha_1+\cdots+\alpha_k]$ with rank multiset $D(\alpha(M))$ to M. Then $\sigma(M) \in \mathfrak{S}^{\alpha(M)}$.

The map $M \mapsto (\alpha(M), \sigma(M))$ is a bijection between multichains of length k in $\mathcal{B}_n$ and the pairs (α,σ) of $\alpha \in \mathrm{Com}(n,k+1)$ and $\sigma \in \mathfrak{S}^\alpha$. A short way to write down this encoding of M is to insert bars at the descent positions of $\sigma(M)$. For example, the length-4 multichain $\{2\} \subseteq \{2\} \subseteq \{1,2,4\} \subseteq [4]$ in $\mathcal{B}_4$ is encoded by $2||14|3|$.

There is another way to encode the multichain M. Let $p_i(M) := \min\{j \in [k+1] : i \in A_j\}, 1 \le i \le n$. So $p_i(M)$ is the first position where i appears in M. One checks that

$$\begin{cases} p_i(M) > p_{i+1}(M) \Leftrightarrow i \in D(\sigma(M)^{-1}), \\ p_i(M) = p_{i+1}(M) \Leftrightarrow i \notin D(\sigma(M)^{-1}),\ D(s_i\sigma(M)) \not\subseteq D(\alpha(M)), \\ p_i(M) < p_{i+1}(M) \Leftrightarrow i \notin D(\sigma(M)^{-1}),\ D(s_i\sigma(M)) \subseteq D(\alpha(M)). \end{cases} \tag{4}$$

The map $M \mapsto p(M) := (p_1(M),\ldots,p_n(M))$ is an bijection between the set of multichains with length k in $\mathcal{B}_n$ and the set $[k+1]^n$ of all words of length n on the alphabet $[k+1]$, for any fixed integer $k \ge 0$.

Let $p(M) = \mathbf{p} = (p_1,\ldots,p_n) \in [k+1]^n$. Then $\mathrm{inv}(p(M)) = \mathrm{inv}(\sigma(M))$. Let $\mathbf{p}' := (p'_1,\ldots,p'_k)$ where $p'_i := |\{j : p_j(M) \le i\}| = |A_i|$. Then the rank multiset of M consists of $p'_1,\ldots,p'_k$. Define $D(\mathbf{p}) := \{i \in [n-1] : p_i > p_{i+1}\}$. For example, the multichain $3|14||2|5$ corresponds to $\mathbf{p} = (2,4,1,2,5) \in [5]^5$, and one has $\mathbf{p}' = (1,3,3,4)$, $D(2,5,1,2,4) = \{2\}$.

These two encodings (with slightly different notation) were already used by Garsia and Gessel [8] in their work on generating functions of multivariate distributions of permutation statistics.

4 0-Hecke algebra action

We saw an analogy between $\mathbb{F}[\mathcal{B}_n]$ and $\mathbb{F}[X]$ in the last section. The usual $H_n(0)$-action on the polynomial ring $\mathbb{F}[X]$ is via the *Demazure operators*

$$\overline{\pi}_i(f) := \frac{x_{i+1}f - x_{i+1}s_if}{x_i - x_{i+1}}, \quad \forall f \in \mathbb{F}[X],\ 1 \le i \le n-1. \tag{5}$$

The above definition is equivalent to

$$\overline{\pi}_i(x_i^a x_{i+1}^b m) = \begin{cases} (x_i^{a-1}x_{i+1}^{b+1} + x_i^{a-2}x_{i+1}^{b+2} \cdots + \underline{x_i^b x_{i+1}^a})m, & \text{if } a > b, \\ 0, & \text{if } a = b, \\ -(\underline{x_i^a x_{i+1}^b} + x_i^{a+1}x_{i+1}^{b-1} + \cdots + x_i^{b-1}x_{i+1}^{a+1})m, & \text{if } a < b. \end{cases} \tag{6}$$

Here m is any monomial in $\mathbb{F}[X]$ containing neither x_i nor x_{i+1}. Denote by $\overline{\pi}'_i$ the operator obtained from (6) by taking only the leading term (underlined) in the lexicographic order of the result. Then $\overline{\pi}'_1,\ldots,\overline{\pi}'_{n-1}$ realize another $H_n(0)$-action on $\mathbb{F}[X]$. We call it the *transferred* $H_n(0)$*-action* because it can be obtained by applying the transfer map τ to our $H_n(0)$-action on $\mathbb{F}[\mathcal{B}_n]$, which we now define.

Let $M = (A_1 \subseteq \cdots \subseteq A_k)$ be a multichain in $\mathcal{B}_n$. Recall that $p_i(M) := \min\{j \in [k+1] : i \in A_j\}$, $1 \le i \le n$. We define

$$\overline{\pi}_i(y_M) := \begin{cases} -y_M, & p_i(M) > p_{i+1}(M), \\ 0, & p_i(M) = p_{i+1}(M), \\ s_i(y_M), & p_i(M) < p_{i+1}(M) \end{cases} \tag{7}$$

for $i = 1, \dots, n-1$. Applying the transfer map τ one recovers $\overline{\pi}'_i$. For instance, when $n = 4$ one has

$$\begin{array}{rclrcl} \overline{\pi}_1(y_{1|34||2|}) & = & y_{2|34||1|}, & \overline{\pi}'_1(x_1^4x_2x_3^3x_4^3) & = & x_1x_2^4x_3^3x_4^3, \\ \overline{\pi}_2(y_{1|34||2|}) & = & -y_{1|34||2|}, & \overline{\pi}'_2(x_1^4x_2x_3^3x_4^3) & = & -x_1^4x_2x_3^3x_4^3, \\ \overline{\pi}_3(y_{1|34||2|}) & = & 0, & \overline{\pi}'_3(x_1^4x_2x_3^3x_4^3) & = & 0. \end{array}$$

One can check that $\overline{\pi}_1, \dots, \overline{\pi}_{n-1}$ realize an $H_n(0)$-action on $\mathbb{F}[\mathcal{B}_n]$ preserving the multigrading of $\mathbb{F}[\mathcal{B}_n]$. If one set $t_i = t^i$ for $i = 1, \dots, n$, then there is an isomorphism $\mathbb{F}[\mathcal{B}_n]/(\emptyset) \cong \mathbb{F}[X]$ of graded $H_n(0)$-modules (which can be given explicitly, but *not* via the transfer map τ).

It is not hard to show that $\mathbb{F}[\mathcal{B}_n]^{H_n(0)} = \mathbb{F}[\Theta]$, where $\mathbb{F}[\mathcal{B}_n]^{H_n(0)}$ is the *invariant algebra* of the $H_n(0)$-action on $\mathbb{F}[\mathcal{B}_n]$, defined as

$$\mathbb{F}[\mathcal{B}_n]^{H_n(0)} := \{f \in \mathbb{F}[\mathcal{B}_n] : \pi_i f = f,\ i = 1, \dots, n-1\}.$$

Proposition 4.1 *The $H_n(0)$-action on $\mathbb{F}[\mathcal{B}_n]$ is Θ-linear.*

Therefore the *coinvariant algebra* $\mathbb{F}[\mathcal{B}_n]/(\Theta)$ is a multigraded $H_n(0)$-module, which is isomorphic to the regular representation of $H_n(0)$ by Theorem 1.1. This cannot be obtained simply by applying the transfer map τ, since τ is *not* a map of $H_n(0)$-modules.

5 Noncommutative characteristic

In this section we use the $H_n(0)$-action on the Stanley-Reisner ring $\mathbb{F}[\mathcal{B}_n]$ of the Boolean algebra $\mathcal{B}_n$ to provide a noncommutative analogue of the following remarkable result.

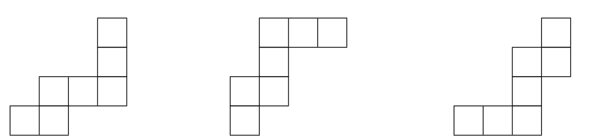
Theorem 5.1 (Hotta-Springer [11], Garsia-Procesi [9]) *For any partition $\mu = (0 < \mu_1 \le \dots \le \mu_k)$ of n, there exists an $\mathfrak{S}_n$-invariant ideal J_μ of $\mathbb{C}[X]$ such that $\mathbb{C}[X]/J_\mu$ is isomorphic to the cohomology ring of the Springer fiber indexed by μ and has graded Frobenius characteristic equal to the modified Hall-Littlewood symmetric function*

$$\widetilde{H}_\mu(X;t) = \sum_\lambda t^{n(\mu)} K_{\lambda\mu}(t^{-1}) s_\lambda \quad \text{inside} \quad \mathrm{Sym}[t]$$

where $n(\mu) := \mu_{k-1} + 2\mu_{k-2} + \dots + (k-1)\mu_1$ and $K_{\lambda\mu}(t)$ is the Kostka-Foulkes polynomial.

Example 5.2 *Tanisaki [19] gives a construction for the ideal J_μ. If $\mu = (1^k, n-k)$ is a hook then $J_{1^k,n-k}$ is generated by $e_1, \dots, e_k$ and all monomials $x_{i_1} \cdots x_{i_{k+1}}$ with $1 \le i_1 < \dots < i_{k+1} \le n$.*

Now consider a composition $\alpha = (\alpha_1, \dots, \alpha_\ell)$. The major index of α is $\mathrm{maj}(\alpha) := \sum_{i \in D(\alpha)} i$. Viewing a partition $\mu = (0 < \mu_1 \le \mu_2 \le \cdots)$ as a composition one has $\mathrm{maj}(\mu) = n(\mu)$. Recall that $\overleftarrow{\alpha} := (\alpha_\ell, \dots, \alpha_1)$ and α^c is the composition of n with $D(\alpha^c) = [n-1] \setminus D(\alpha)$. We define $\alpha' := \overleftarrow{\alpha^c} = (\overleftarrow{\alpha})^c$. One can identify α with a *ribbon diagram*, i.e. a connected skew Young diagram without 2 by 2 boxes, which has row lengths $\alpha_1, \dots, \alpha_\ell$, ordered from bottom to top. Note that a ribbon diagram is a Young diagram if and only if it is a hook. One can check that α' is the transpose of α; see the example below.



$\alpha = (2,3,1,1)$ $\qquad$ $\alpha^c = (1,2,1,3)$ $\qquad$ $\alpha' = (3,1,2,1)$

Bergeron and Zabrocki [3] introduced a noncommutative modified Hall-Littlewood symmetric function

$$\widetilde{\mathbf{H}}_\alpha(\mathbf{x};t) := \sum_{\beta\preccurlyeq\alpha} t^{\mathrm{maj}(\beta)}\mathbf{s}_\beta \quad \text{inside} \quad \mathbf{NSym}[t] \tag{8}$$

and a (q,t)-analogue

$$\widetilde{\mathbf{H}}_\alpha(\mathbf{x};q,t) := \sum_{\beta\models n} t^{c(\alpha,\beta)}q^{c(\alpha',\overleftarrow{\beta})}\mathbf{s}_\beta \quad \text{inside} \quad \mathbf{NSym}[q,t] \tag{9}$$

for every composition α, where $\mathbf{s}_\beta$ is the noncommutative ribbon Schur function indexed by β, and $c(\alpha,\beta) := \sum_{i\in D(\alpha)\cap D(\beta)} i$. In our earlier work [13] we provided a partial representation theoretic interpretation for $\widetilde{\mathbf{H}}_\alpha(\mathbf{x};t)$ when $\alpha = (1^k, n-k)$ is a hook, using the $H_n(0)$-action on the polynomial ring $\mathbb{F}[X]$ by the Demazure operators.

Theorem 5.3 ([13]) *The ideal J_μ of $\mathbb{F}[X]$ is $H_n(0)$-invariant if and only if $\mu = (1^{n-k}, k)$ is a hook, and if that holds then $\mathbb{F}[X]/J_\mu$ becomes a graded projective $H_n(0)$-module with*

$$\mathbf{ch}_t(\mathbb{F}[X]/J_\mu) = \widetilde{\mathbf{H}}_\mu(\mathbf{x};t),$$

$$\mathrm{Ch}_t(\mathbb{F}[X]/J_\mu) = \widetilde{H}_\mu(\mathbf{x};t).$$

Now we switch to the Stanley-Reisner ring $\mathbb{F}[\mathcal{B}_n]$ and define I_α to be its ideal generated by

$$\Theta_\alpha := \{\theta_i : i\in D(\alpha)\cup\{n\}\} \quad \text{and} \quad \{y_A : A\subseteq [n],\ |A|\notin D(\alpha)\cup\{n\}\}$$

for any composition α of n. The following result is a restatement of Theorem 1.1.

Theorem 5.4 *Let α be a composition of n. Then $\mathbb{F}[\mathcal{B}_n]/I_\alpha$ is a projective $H_n(0)$-module with multigraded noncommutative characteristic equal to*

$$\widetilde{\mathbf{H}}_\alpha(\mathbf{x};t_1,\ldots,t_{n-1}) := \sum_{\beta\preccurlyeq\alpha} \underline{t}^{D(\beta)}\mathbf{s}_\beta \quad \text{inside} \quad \mathbf{NSym}[t_1,\ldots,t_{n-1}].$$

One has $\widetilde{\mathbf{H}}_\alpha(\mathbf{x};t,t^2,\ldots,t^{n-1}) = \widetilde{\mathbf{H}}_\alpha(\mathbf{x};t)$, and one obtains $\widetilde{\mathbf{H}}_\alpha(\mathbf{x};q,t)$ from $\widetilde{\mathbf{H}}_{1^n}(\mathbf{x};t_1,\ldots,t_{n-1})$ by taking $t_i = t^i$ for all $i\in D(\alpha)$, and $t_i = q^{n-i}$ for all $i\in[n-1]\setminus D(\alpha)$.

Proof: There is an $\mathbb{F}$-basis for $\mathbb{F}[\mathcal{B}_\alpha]/(\Theta_\alpha)$ given by the descent monomials Y_w defined in (3) for all $w\in\mathfrak{S}^\alpha$. The result follows from the $H_n(0)$-action on this basis and (4). □

The proof of this theorem is actually simpler than the proof of our partial interpretation for $\widetilde{\mathbf{H}}_\alpha(\mathbf{x};t)$ in [13]. This is because $\overline{\pi}_i$ sends a descent monomial in $\mathbb{F}[\mathcal{B}_n]$ to either 0 or ±1 times a descent monomial, but sends a descent monomial in $\mathbb{F}[X]$ to a polynomial in general (whose leading term is still a descent monomial). We view the Stanley-Reisner ring $\mathbb{F}[\mathcal{B}_n]$ (or $\mathbb{F}[\mathcal{B}_n]/(\emptyset)$) as a $q=0$ analogue of the polynomial ring $\mathbb{F}[X]$. For an odd (i.e. $q=-1$) analogue, see Lauda and Russell [16].

Remark 5.5 *If $\alpha = (1^k, n-k)$ is a hook, one can check that the ideal $I_{1^k,n-k}$ of $\mathbb{F}[\mathcal{B}_n]$ has generators $\theta_1,\ldots,\theta_k$ and all y_A with $A\subseteq[n]$ and $|A|\notin[k]$. By Example 5.2, the images of these generators under the transfer map τ are the Tanisaki generators for the ideal $J_{1^k,n-k}$ of $\mathbb{F}[X]$, but $\tau(I_{1^k,n-k}) \neq J_{1^k,n-k}$.*

For any composition $\alpha \models n$, one can view $\widetilde{\mathbf{H}}_\alpha(\mathbf{x}; t_1, \ldots, t_{n-1})$ as a modified version of

$$\mathbf{H}_\alpha = \mathbf{H}_\alpha(\mathbf{x}; t_1, \ldots, t_{n-1}) := \sum_{\beta \preccurlyeq \alpha} \underline{t}^{D(\alpha) \setminus D(\beta)} \mathbf{s}_\beta.$$

Below are some properties satisfied by $\mathbf{H}_\alpha$, generalizing the properties of $\mathbf{H}_\alpha(\mathbf{x}; t)$ given in [3].

Proposition 5.6 *Let α and β be two compositions.*
(i) $\mathbf{H}_\alpha(0, \ldots, 0) = \mathbf{s}_\alpha$, $\mathbf{H}_\alpha(1, \ldots, 1) = \mathbf{h}_\alpha$.
(ii) $\bigcup_{n \geq 0} \{\mathbf{H}_\alpha : \alpha \models n\}$ *is a basis for* $\mathbf{NSym}[t_1, t_2, \ldots]$.
(iii) $\langle \mathbf{H}_\alpha, \mathbf{H}_\beta \rangle = (-1)^{|\alpha| + \ell(\alpha)} \delta_{\alpha, \beta^c}$ *for any pair of compositions α and β.*
(iv)

$$\mathbf{H}_\alpha \cdot \mathbf{H}_\beta = \sum_{\gamma \preccurlyeq \beta} \left(\prod_{i \in D(\beta) \setminus D(\gamma)} (t_i - t_{|\alpha| + i}) \right) \left(\mathbf{H}_{\alpha\gamma} + (1 - t_{|\alpha|}) \mathbf{H}_{\alpha \rhd \gamma} \right).$$

(v) If $n = |\alpha|$ and $t|n := (t_1, \ldots, t_{n-1}, 1, t_1, \ldots, t_{n-1}, 1, \ldots)$ then

$$\mathbf{H}_\alpha(\mathbf{x}; t_1, \ldots, t_{n-1}) \mathbf{H}_\beta(\mathbf{x}; t|n) = \mathbf{H}_{\alpha\beta}(t|n).$$

6 Quasisymmetric characteristic

Now we study the quasisymmetric characteristic of $\mathbb{F}[\mathcal{B}_n]$. The following lemma follows easily from (4).

Lemma 6.1 *Let α be a weak composition of n. Then the α-homogeneous component $\mathbb{F}[\mathcal{B}_n]_\alpha$ of the Stanley-Reisner ring $\mathbb{F}[\mathcal{B}_n]$ is an $H_n(0)$-submodule of $\mathbb{F}[\mathcal{B}_n]$ with homogeneous multigrading $\underline{t}^{D(\alpha)}$ and isomorphic to the cyclic module $H_n(0)\pi_{w_0(\alpha^c)}$.*

Since $\mathbb{F}[\mathcal{B}_n]_\alpha$ is a cyclic multigraded $H_n(0)$-module, we get an $\mathbb{N} \times \mathbb{N}^{n+1}$-multigraded quasisymmetric characteristic

$$\mathrm{Ch}_{q, \underline{t}}(\mathbb{F}[\mathcal{B}_n]_\alpha) = \sum_{w \in \mathfrak{S}^\alpha} q^{\mathrm{inv}(w)} \underline{t}^{D(\alpha)} F_{D(w^{-1})} \tag{10}$$

where q keeps track of the length filtration and $\underline{t}$ keeps track of the multigrading of $\mathbb{F}[\mathcal{B}_n]_\alpha$. This defines an $\mathbb{N} \times \mathbb{N}^{n+1}$-multigraded quasisymmetric characteristic for the Stanley-Reisner ring $\mathbb{F}[\mathcal{B}_n]$, which is explicitly given in Theorem 1.2 and restated below.

Theorem 6.2 *The $\mathbb{N} \times \mathbb{N}^{n+1}$-multigraded quasisymmetric characteristic of $\mathbb{F}[\mathcal{B}_n]$ is*

$$\begin{aligned} \mathrm{Ch}_{q, \underline{t}}(\mathbb{F}[\mathcal{B}_n]) &= \sum_{k \geq 0} \sum_{\alpha \in \mathrm{Com}(n, k+1)} \underline{t}^{D(\alpha)} \sum_{w \in \mathfrak{S}^\alpha} q^{\mathrm{inv}(w)} F_{D(w^{-1})} \\ &= \sum_{w \in \mathfrak{S}_n} \frac{q^{\mathrm{inv}(w)} \underline{t}^{D(w)} F_{D(w^{-1})}}{\prod_{0 \leq i \leq n} (1 - t_i)} \\ &= \sum_{k \geq 0} \sum_{\mathbf{p} \in [k+1]^n} t_{p'_1} \cdots t_{p'_k} q^{\mathrm{inv}(\mathbf{p})} F_{D(\mathbf{p})}. \end{aligned}$$

Proof: Use the two encodings of the multichains in $\mathcal{B}_n$ as well as the free $\mathbb{F}[\Theta]$-basis $\{Y_w : w \in \mathfrak{S}_n\}$ of descent monomials for $\mathbb{F}[\mathcal{B}_n]$ discussed in Section 3. □

Next we explain here how this theorem specializes to (1), a result of Garsia and Gessel [8, Theorem 2.2] on the multivariate generating function of the permutation statistics $\mathrm{inv}(w)$, $\mathrm{maj}(w)$, $\mathrm{des}(w)$, $\mathrm{maj}(w^{-1})$, and $\mathrm{des}(w^{-1})$ for all $w \in \mathfrak{S}_n$. First recall that

$$F_\alpha = \sum_{\substack{i_1 \geq \cdots \geq i_n \geq 1 \\ i \in D(\alpha) \Rightarrow i_j > i_{j+1}}} x_{i_1} \cdots x_{i_n}, \quad \forall \alpha \models n.$$

Let $\mathbf{ps}_{q;\ell}(F_\alpha) := F_\alpha(1, q, q^2, \ldots, q^{\ell-1}, 0, 0, \ldots)$, and $(u;q)_n := (1-u)(1-qu)(1-q^2u)\cdots(1-q^nu)$. It is not hard to check (see Gessel and Reutenauer [10, Lemma 5.2]) that

$$\sum_{\ell \geq 0} u^\ell \mathbf{ps}_{q;\ell+1}(F_\alpha) = \frac{q^{\mathrm{maj}(\alpha)} u^{\mathrm{des}(\alpha)}}{(u;q)_n}.$$

Then applying the linear transformation $\sum_{\ell \geq 0} u_1^\ell \mathbf{ps}_{q_1;\ell+1}$ and the specialization $t_i = q_2^i u_2$ for all $i = 0, 1, \ldots, n$ to Theorem 6.2 we recover (1).

A further specialization of Theorem 6.2 gives a well known result which is often attributed to Carlitz [6] but actually dates back to MacMahon [17, Volume 2, Chapter 4].

Corollary 6.3 (Carlitz-MacMahon) *Let* $[k+1]_q := 1 + q + q^2 + \cdots + q^k$. *Then*

$$\frac{\sum_{w \in \mathfrak{S}_n} q^{\mathrm{maj}(w)} u^{\mathrm{des}(w)}}{(u;q)_n} = \sum_{k \geq 0} ([k+1]_q)^n u^k.$$

Theorem 6.2 also implies the following result, which was obtained by Adin, Brenti, and Roichman [1] from the Hilbert series of the coinvariant algebra $\mathbb{F}[X]/(\mathbb{F}[X]_+^{\mathfrak{S}_n})$.

Corollary 6.4 (Adin-Brenti-Roichman) *Let* $\mathrm{Par}(n)$ *be the set of weak partitions* $\lambda = (\lambda_1, \ldots, \lambda_n)$ *with* $\lambda_1 \geq \cdots \geq \lambda_n \geq 0$, *and let* $m(\lambda) = (m_0(\lambda), m_1(\lambda), \ldots)$, *where* $m_j(\lambda) := \#\{1 \leq i \leq n : \lambda_i = j\}$. *Then*

$$\sum_{\lambda \in \mathrm{Par}(n)} \binom{n}{m(\lambda)} \prod_{i=1}^n q_i^{\lambda_i} = \frac{\sum_{w \in \mathfrak{S}_n} \prod_{i \in D(w)} q_1 \cdots q_i}{(1-q_1)(1-q_1q_2)\cdots(1-q_1\cdots q_n)}.$$

7 Remarks and questions for future research

7.1 Hecke algebra action

It is well known that the symmetric group $\mathfrak{S}_n$ is the Coxeter group of type A_{n-1}. The Stanley-Reisner ring of $\mathcal{B}_n$ is essentially the Stanley-Reisner ring of the Coxeter complex of $\mathfrak{S}_n$. The Hecke algebra $H_W(q)$ can be defined for any finite Coxeter group W. We can generalize our action $H_n(0)$-action on $\mathbb{F}[\mathcal{B}_n]$ to an $H_W(q)$-action on the Stanley-Reisner ring $\mathbb{F}(q)[\Delta(W)]$ of the Coxeter complex $\Delta(W)$ of any finite Coxeter group W. We show similar results for this $H_W(q)$-action.

7.2 Gluing the group algebra and the 0-Hecke algebra

The group algebra $\mathbb{F}W$ of a finite Coxeter group W naturally admits both actions of W and $H_W(0)$. Hivert and Thiéry [12] defined the *Hecke group algebra* of W by gluing these two actions. In type A, one can also glue the usual actions of $\mathfrak{S}_n$ and $H_n(0)$ on the polynomial ring $\mathbb{F}[X]$, but the resulting algebra is different from the Hecke group algebra of $\mathfrak{S}_n$.

Now one has a W-action and an $H_W(0)$-action on the Stanley-Reisner ring $\mathbb{F}[\Delta(W)]$. What can we say about the algebra generated by the operators s_i and $\overline{\pi}_i$ on $\mathbb{F}[\Delta(W)]$? Is it the same as the Hecke group algebra of W? If not, what properties (dimension, bases, presentation, simple and projective indecomposable modules, etc.) does it have?

7.3 Tits Building

Let $\Delta(G)$ be the Tits building of the general linear group $G = GL(n, \mathbb{F}_q)$ and its usual BN-pair over a finite field $\mathbb{F}_q$; see e.g. Björner [4]. The Stanley-Reisner ring $\mathbb{F}[\Delta(G)]$ is a q-analogue of $\mathbb{F}[\mathcal{B}_n]$. The nonzero monomials in $\mathbb{F}[\Delta(G)]$ are indexed by multiflags of subspaces of $\mathbb{F}_q^n$, and there are $q^{\mathrm{inv}(w)}$ many multiflags corresponding to a given multichain M in $\mathcal{B}_n$, where $w = \sigma(M)$. Can one obtain the multivariate quasisymmetric function identities in Theorem 1.2 by defining a nice $H_n(0)$-action on $\mathbb{F}[\Delta(G)]$?

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