FPSAC 2014, Chicago, USA DMTCS proc. **AT**, 2014, 61–70

A generalization of the carries process

Takahiko Fujita[1*] Fumihiko Nakano[2†] Taizo Sadahiro[3‡]

[1]*Faculty of Science and Engineering, Dep. of Industrial and Systems Engineering, Chuo University, Tokyo, Japan*
[2]*Dep. of Mathematics, Gakushuin University, Tokyo, Japan*
[3]*Dep. of Information Science, Tsuda Collge, Tokyo, Japan*

We consider a carries process which is a generalization of that by Holte in the sense that (i) we take various digit sets, and (ii) we also consider negative base. Our results are : (i) eigenvalues and eigenvectors of the transition probability matrices, and their connection to combinatorics and representation theory, (ii) an application to the computation of the distribution of the sum of i.i.d. uniform r.v.'s on $[0, 1]$, (iii) a relation to riffle shuffle.

Nous considérons un processus qui est une généralisation de porte que par Holte en ce sens que (i) nous prenons divers ensembles de chiffres, et (ii) nous considérons également une base négative. Nos résultats sont les suivants: (i) valeurs et vecteurs propres des matrices de probabilité de transition, et leur lien avec la combinatoire et la théorie de la représentation, (ii) une application de calcul de la distribution de la somme des iid s 'sur r.v. uniforme $[0, 1]$, (iii) une relation à feuilleter shuffle.

Keywords: Carries process, Eulerian number, Riffle shuffle

1 Introduction

We consider to add n base-b numbers, and aim to study the behavior of carries to the next digit :

Carry	C_k	C_{k-1}	$\cdots$	C_1	$C_0 = 0$
Addends	$\cdots$	$X_{1,k}$	$\cdots$	$X_{1,2}$	$X_{1,1}$
		$\vdots$		$\vdots$	$\vdots$
	$\cdots$	$X_{n,k}$	$\cdots$	$X_{n,2}$	$X_{n,1}$
Sum		S_k	$\cdots$	S_2	S_1

where $X_{1,k}, X_{2,k}, \cdots, X_{n,k}$ is the addends in the k-th digit which lie in the **digit set** : $X_{1,k}, X_{2,k}, \cdots, X_{n,k} \in \mathcal{D} := \{0, 1, \cdots, b-1\}$, and C_k is the carry to the $(k+1)$-th digit. More precisely, in the k-th digit, $C_k = j$ if and only if

$$C_{k-1} + X_{1,k} + \cdots + X_{n,k} = jb + r, \quad r \in \mathcal{D}.$$

*Email: `rankstatistics@gmail.com`
†Email: `fumihiko@math.gakushuin.ac.jp` partially supported by JSPS grant Kiban-C 22540140
‡Email:`sadahiro@tsuda.ac.jp` partially supported by JSPS grant Kiban-C 23540155

The C_k's take their values in the carry set $\mathcal{C}(n) := \{0, 1, \cdots, n-1\}$. Taking $X_{1,k}, \cdots, X_{n,k}$ uniformly at random from $\mathcal{D}$, $\{C_k\}_{k=0}^{\infty}$ is a Markov chain which is called **the carries process**. Holte[4] found many beautiful properties of the carries process, e.g.,
(1) The transition probability $P_{ij} := \mathbf{P}\left(C_{k+1} = j \,|\, C_k = i\right)$ is given by

$$\begin{aligned} P_{ij} &:= b^{-n} \sum_{r=0}^{\lfloor z_{ij}/b \rfloor} (-1)^r \begin{pmatrix} n+1 \\ r \end{pmatrix} \begin{pmatrix} n + z_{ij} - br \\ n \end{pmatrix} \\ z_{ij} &:= (j+1)b - (i+1), \quad i, j \in \mathcal{C}(n). \end{aligned}$$

(2) The eigenvalues and the left eigenvectors of $P = \{P_{ij}\}$ are given by

$$\begin{aligned} P &= L^{-1} D L \\ D &:= \mathrm{diag}\left(1, \frac{1}{b}, \cdots, \frac{1}{b^{n-1}}\right), \quad L := \{v_{ij}(n)\}_{0 \le i,j \le n-1} \\ v_{ij}(n) &:= \sum_{r=0}^{j} (-1)^r \begin{pmatrix} n+1 \\ r \end{pmatrix} (j - r + 1)^{n-i} \end{aligned}$$

In other words, the eigenvalues are the powers of b^{-1}, and the left eigenvectors are independent of b, so that all transition probability matrices are simultaneously diagonalized :

$$P(b_1)P(b_2) = P(b_1 b_2).$$

(3) The top row eigenvector $\{v_{0,j}\}_{j=0}^{n-1}$ of L, which is proportional to the stationary distribution, is equal to the array of n-th Eulerian number $\{E(n,j)\}_{j=0}^{n-1}$; $E(n,j)$ is equal to the number of permutations of n items with j descents.
(4) All the row eigenvectors of L satisfy the following recursion relation.

$$v_{i,j}(n) \quad = \quad (j+1) v_{i,j}(n-1) + (n-j) v_{i,j-1}(n-1).$$

(5) The right eigenvectors are also explicitly computed and described by the Stirling numbers.

Diaconis-Fulman [1, 2, 3] further studied the carries process and found a connection to the riffle shuffle and the Foulkes character in representation theory : they found that
(1) The distribution of the carries process is equal to that of the process of the descent of the b-shuffle of n cards.
(2) The matrix of left eigenvectors coincides with the Foulkes character table of S_n.
(3) In the type B process, which is a variant of the carries process, the stationary distribution is proportional to the Macmahon number, which gives us the statistics of the type B-descent of the hyperoctahedral group (signed permutation group). Moreover, the type B carries process is related to the descent of the type B shuffle on the hyperoctahedral group.

The purpose of our project is to extend these properties to a generalization to the carries process, some of which discussed here are taken from [6], [7]. Because we also discuss the case of negative base, we treat them simultaneously. Let $\pm b \in \mathbf{Z} (b \ge 2)$ be the base and let

$$\mathcal{D}_d := \{d, d+1, \cdots, d+b-1\}$$

be the digit set such that

$$1 - b \le d \le 0$$

to have $0 \in \mathcal{D}_d$. Then any $x \in \mathbf{N}$ has the unique representation

$$\begin{aligned} x &= a_N(+b)^N + a_{N-1}(+b)^{N-1} + \cdots + a_0, \quad a_k \in \mathcal{D}_d, \\ x &= a'_N(-b)^N + a'_{N-1}(-b)^{N-1} + \cdots + a'_0, \quad a'_k \in \mathcal{D}_d. \end{aligned}$$

In adding n numbers under this representation, let $C^{\pm}_{k-1}$ be the carry from the $(k-1)$-th digit which belongs to a set $\mathcal{C}(\pm b, n)$ to be given in Proposition 2.1. In the k-th digit, we take $X_{1,k}, \cdots, X_{n,k}$ uniformly at random from $\mathcal{D}_d$ and then the next carry is equal to $C^{\pm}_k = j \in \mathcal{C}(\pm b, n)$ if and only if

$$C^{\pm}_{k-1} + X_{1,k} + \cdots + X_{n,k} = j(\pm b) + r, \quad r \in \mathcal{D}_d.$$

The process $\{C^{\pm}_k\}_{k=0}^{\infty}$ is Markovian with state space $\mathcal{C}(\pm b, n)$, which we call the **n-carries process over** $(\pm b, \mathcal{D}_d)$. Holte's carries process corresponds to the case where the base is positive and $d = 0$.

2 The Transition Probability

Let

$$l = l(\pm b, d) := \begin{cases} \frac{d}{b-1} & ((+b)\text{-case}) \\ -\frac{b+d}{b+1} & ((-b)\text{-case}) \end{cases}.$$

Then the carry set $\mathcal{C}(\pm b, n)$ is explicitly given by the proposition below.

Proposition 2.1 *[6]*
(1) The carry set $\mathcal{C}(\pm b, n)$ in the n-carries process over $(\pm b, \mathcal{D}_d)$ is given by

$$\begin{aligned} \mathcal{C}(\pm b, n) &= \{s, s+1, \cdots, t\} \\ s &:= \lfloor (n-1)l \rfloor = -\lceil (n-1)(-l) \rceil, \quad t := \lceil (n-1)(l+1) \rceil. \end{aligned}$$

(2) The number of elements of $\mathcal{C}(\pm b, n)$ is equal to

$$\sharp\mathcal{C}(\pm b, n) = \begin{cases} n & ((n-1)l \in \mathbf{Z}) \\ n+1 & ((n-1)l \notin \mathbf{Z}). \end{cases}$$

We remark that if $(n-1)l \notin \mathbf{Z}$, $\sharp\mathcal{C}(\pm, n)$ is larger than that of the Holte's carries process ($\sharp S$ is the number of elements of a finite set S). This fact can be understood from the following consideration (we discuss $(+b)$-case only). Let

$$F := \left\{ \frac{x_1}{b} + \frac{x_2}{b^2} + \cdots + \frac{x_k}{b_k} \,\middle|\, k \in \mathbf{N}, x_1, \cdots, x_k \in \mathcal{D}_d \right\}$$

be the set of numbers which have finite b-expansion. Then $F \subset (l, l+1)$ and F is dense in $(l, l+1)$. When we add n numbers $a^1, a^2, \cdots, a^n \in F$, the carry going beyond the decimal point is equal to c_0, if and only if $a^1 + a^2 + \cdots + a^n \in c_0 + F$. Since F is dense in $(l, l+1)$, so is $\overbrace{F + \cdots + F}^{n} = \{a^1 + \cdots + a^n \mid a^1, \cdots, a^n \in F\}$ in $(nl, n(l+1))$. Therefore, $c_0 \in \mathcal{C}(b,n)$ if and only if

$$(c_0 + (l, l+1)) \cap (nl, n(l+1)) \neq \emptyset.$$

For example, let $b = 4$, $n = 2$, and $d = -1$. Then $F+F$ is dense in $(-2/3, 4/3)$, which intersects exactly three intervals : $-1 + (-2/3, 4/3)$, $(-2/3, 4/3)$, and $1 + (-2/3, 4/3)$. Therefore $\mathcal{C}(4,2) = \{-1, 0, 1\}$.

We introduce the following notation which is an important parameter to describe our results.

$$p = p(\pm b, d, n) := \frac{1}{1 - \langle (n-1)l \rangle} = \begin{cases} 1 & ((n-1)l \in \mathbf{Z}) \\ \langle (n-1)(-l) \rangle^{-1} & ((n-1)l \notin \mathbf{Z}) \end{cases}$$

where $\langle x \rangle := x - \lfloor x \rfloor$ is the fractional part of x. $\sharp\mathcal{C}(b,n) = n$ if and only if $p = 1$, including the case of Holte's carries process. For simplicity, we shall modify the suffix of the transition probability such that it ranges from 0 to $\sharp\mathcal{C}(\pm b, n) - 1$.

$$\tilde{P}^{\pm}_{i,j} := P^{\pm}_{i+s,j+s} = \mathbf{P}\left(C^{\pm}_1 = j + s \mid C^{\pm}_0 = i + s\right), \quad i, j = 0, 1, \cdots, \sharp\mathcal{C}(b,n) - 1$$

where $s := \lfloor (n-1)l \rfloor = \min \mathcal{C}(\pm b, n)$.

Proposition 2.2 *[6]*

$$\begin{aligned} \tilde{P}^{\pm}_{i,j} &= \frac{1}{b^n} \sum_{r \geq 0} (-1)^r \binom{n+1}{r} \binom{n + A(i,j) - br}{n} 1(A(i,j) - br \geq 0) \qquad (1) \\ A(i,j) &:= \begin{cases} \left(j + \frac{1}{p}\right) b - \left(i + \frac{1}{p}\right) & ((+b) - case) \\ \left(-j + 1 - \frac{1}{p}\right) b - \left(i + \frac{1}{p}\right) + nb & ((-b) - case) \end{cases} \\ i, j &= 0, 1, \cdots, \sharp\mathcal{C}(\pm b, n) - 1. \end{aligned}$$

where $1(E)$ is the indicator function of the event E, that is, $1(E) = 1$ if E is true and $1(E) = 0$ otherwise.

When $p = 1$(resp. $p = 2$), $\tilde{P}^+$ coincides with that of Holte's carries process (resp. with that of the type B process). Proof of Proposition 2.2 implies that the carries process is determined by $l(\pm b, d)$ only. For instance, if $l(+b, d) = l(-b', d')$ (e.g., $b = 4$, $d = -1$ and $b' = 2$, $d' = -1$) then n-carries process over $(+b, \mathcal{D}_d)$ and n-carries process over $(-b', \mathcal{D}_{d'})$ have the same distribution for any $n \in \mathbf{N}$.

The matrix $\tilde{P}^{\pm} = \{\tilde{P}^{\pm}_{ij}\}$ is determined by the triple $(\pm b, n, p)$. It is clearly not true in general that for given $(\pm b, n, p)$ we can find d with $-(b-1) \leq d \leq 0$ such that $\tilde{P}^{\pm}$ is the transition probability of the n-carries process over $(\pm b, \mathcal{D}_d)$. However, there are some cases where $\tilde{P}$ is a stochastic matrix, even if it does not correspond to carries processes.

Proposition 2.3 *[7] $\tilde{P}^{\pm}$ is a stochastic matrix if $p \geq 1$ and $\frac{(\pm b) - 1}{p} \in \mathbf{Z}$.*

In these cases, $\tilde{P}^{\pm}$ defines a Markov chain $\{\kappa^{\pm}_r\}_{r=0}^{\infty}$ on $\{0, 1, \cdots, \sharp\mathcal{C}(b,n) - 1\}$ which we call **$(\pm b, n, p)$-carries process**. We shall henceforth assume that the conditions in Proposition 2.3 are always satisfied.

3 Eigenvectors of $\tilde{P}$

3.1 Left Eigenvectors

Theorem 3.1 *[6] The eigenvalues and the left eigenvectors of $\tilde{P}^{\pm}$ are given by*

$$\begin{aligned} \tilde{P}^{\pm} &= L_p^{-1} D_b^{\pm} L_p \\ \text{where} \quad D_b^{\pm} &:= \mathit{diag}\left(1, \left(\pm\frac{1}{b}\right), \cdots, \left(\pm\frac{1}{b}\right)^{\sharp\mathcal{C}(\pm b,n)-1}\right) \\ L_p &:= \{v_{ij}^{(p)}(n)\}_{0\le i,j\le \sharp\mathcal{C}(\pm b,n)-1} \\ v_{ij}^{(p)}(n) &:= \sum_{r=0}^{j} (-1)^r \binom{n+1}{r} \{p(j-r)+1\}^{n-i}. \end{aligned}$$

Remark 3.2 *When $p = 2$, $\{v_{0,k}^{(2)}(n)\}_{k=0}^{n}$ is called the (array of) MacMahon numbers : $v_{0,k}^{(2)}(n)$ is equal to the number of elements of the hyperocthedral group (signed permutation group) whose type B descent is equal to k. More generally, if $p \in \mathbf{N}$, $v_{0,k}^{(p)}(n)$ is equal to the number of elements in the colored permutation group $G_{p,n} \simeq \mathbf{Z}_p \wr S_n$ whose descent is equal to k [8].*

Remark 3.3 *Miller [5] studied the Foulkes character in the general complex reflection groups. According to his results, if $p \in \mathbf{N}$, the array of left eigenvectors $\{v_{ij}^{(p)}(n)\}_{ij}$ coincides with the Foulkes character table of $\mathbf{Z}_p \wr S_n$.*

Remark 3.4

$$\sum_{j=0}^{\sharp\mathcal{C}(b,n)-1} v_{ij}^{(p)}(n) = \begin{cases} p^n n! & (i=0) \\ 0 & (i = 1, 2, \cdots, \sharp\mathcal{C}(b,n)-1) \end{cases}$$

Remark 3.5 $\{v_{i,j}^{(p)}(n)\}$ *satisfy the following recursion relation.*

$$\begin{aligned} v_{i,j}^{(p)}(n) &= (pj+1)v_{i,j}^{(p)}(n-1) + \{p(n+1-j)-1\}v_{i,j-1}^{(p)}(n-1) \\ & \quad v_{i,-1}^{(p)}(n) = 0, \quad v_{i,n+1}^{(p)}(n) = 0. \end{aligned}$$

Remark 3.6 *Let p^* be the dual exponent of $p \neq 1$: $\frac{1}{p} + \frac{1}{p^*} = 1$. Then the row eigenvectors of L_{p^*} are the "reverse" of those of L_p in the following sense.*

$$v_{i,j}^{(p^*)}(n) = (-1)^i \left(\frac{p^*}{p}\right)^{n-i} v_{i,n-j}^{(p)}(n).$$

For instance, when $b = 4$, $d = -1$, $n \in 3\mathbf{N}+2$, we have $p = 3$, and when $b = 7$, $d = -4$, $n \in 3\mathbf{N}+2$, we have $p = \frac{3}{2}$. Hence their stationary distributions are reverse of each other.

3.2 Right Eigenvector

Theorem 3.7 *[7] Let*

$$R_p := L_p^{-1} = \{u_{ij}^{(p)}(n)\}_{i,j}$$

be the matrix composed of the right eigenvectors of $\tilde{P}^{\pm}$. Then its elements are given by

$$u_{ij}^{(p)}(n) = \sum_{k=i}^{n} \sum_{l=n-j}^{k} \frac{s(k,l)(-1)^{n-j-l}}{k!\, p^l} \begin{pmatrix} l \\ n-j \end{pmatrix} \begin{pmatrix} n-i \\ n-k \end{pmatrix}$$

where $s(n,k)$ is the Stirling number of the first kind :

$$s(n,k) := (-1)^{n-k} \sharp \{\sigma \in S_n \,|\, \sigma \text{ has } k \text{ cycles } \}.$$

The formula in Theorem 3.7 appears in [5] in a different but related context. It is then a straightforward computation to show that $R_p = L_p^{-1}$.

Remark 3.8 *The duality of L_p (Remark 3.6) implies that*

$$u_{ij}^{(p^*)}(n) = \left(-\frac{p}{p^*}\right)^{n-j} u_{n-i,j}^{(p)}(n).$$

In other words, the column vectors of R_p and R_{p^*} are reverse of each other, up to constants.

4 Correlation

As in [2], Theorem 3.7 makes it easy to compute the correlations between carries at different steps. We recall that $\{\kappa_r^{\pm}\}_{r=0}^{\infty}$ is the $(\pm b, n, p)$-carries process.

Theorem 4.1 *[7] Let $\mathbf{E}[\,\cdot\,|\kappa_0^{\pm} = i]$ be the expectation value conditioned $\kappa_0^{\pm} = i$. Then we have*

$$(1) \qquad \mathbf{E}[\kappa_r^{\pm} \,|\, \kappa_0^{\pm} = i] = \frac{1}{(\pm b)^r}\left(i + \frac{1}{p} - \frac{n+1}{2}\right) - \frac{1}{p} + \frac{n+1}{2},$$

$$(2) \qquad Var\,(\kappa_r^{\pm} \,|\, \kappa_0^{\pm} = i) = \frac{n+1}{12}\left(1 - \frac{1}{(\pm b)^{2r}}\right),$$

$$(3) \qquad Cov\,(\kappa_s^{\pm}, \kappa_{s+r}^{\pm} \,|\, \kappa_0^{\pm} = i) = \frac{1}{(\pm b)^r}\frac{n+1}{12}\left(1 - \frac{1}{(\pm b)^{2s}}\right).$$

Theorem 4.2 *[7] Let $\mathbf{E}_{\pi}[\,\cdot\,]$ be the expectation value conditioned that $\kappa_0^{\pm}$ obeys the stationary distribution. Then we have*

$$(1) \qquad \mathbf{E}_{\pi}[\kappa_0^{\pm}] = \frac{n+1}{2} - \frac{1}{p},$$

$$(2) \qquad Cov_{\pi}(\kappa_r^{\pm}, \kappa_0^{\pm}) = \frac{1}{(\pm b)^r} \cdot \frac{n+1}{12}.$$

We note that the variances and covariances do not depend on p. For $(-b)$-case, κ_r^- and κ_0^- are negatively correlated when r is odd.

5 Limit Theorem

Let

$$\left\langle \begin{array}{c} n \\ k \end{array} \right\rangle_p := \sum_{r=0}^{k} (-1)^r \left(\begin{array}{c} n+1 \\ r \end{array} \right) \{p(k-r)+1\}^n$$

be the extension of the stationary distribution $\mathbf{v}_0^{(p)} := \{v_{0,k}^{(p)}(n)\}_k$ to the case of arbitrary real number $p \in \mathbf{R}$. Then it gives the distribution of the sum of the uniformly distributed i.i.d. random variables on $[0,1]$.

Theorem 5.1 *[6] Let $Y_1, \cdots, Y_n$ be the i.i.d. uniform random variables on $[0,1]$. Then*

$$\mathbf{P}\left(Y_1 + \cdots + Y_n \in [k-1, k] + \frac{1}{p} \right) = \left\langle \begin{array}{c} n \\ k \end{array} \right\rangle_p / (p^n n!), \quad k = 0, 1, \cdots, n.$$

For $p = 1, 2$, it is given in [2].

Example

Let $P_n(a,b) := \mathbf{P}(Y_1 + \cdots + Y_n \in [a,b])$. Then if $n = 3, p = 1$,

$$P_3(0,1) = \frac{1}{6}, \quad P_3(1,2) = \frac{4}{6}, \quad P_3(2,3) = \frac{1}{6}$$

which are proportional to the Eulerian numbers. If $n = 3, p = 2$,

$$P_3\left(0, \frac{1}{2}\right) = \frac{1}{2^3 \cdot 3!}, \quad P_3\left(\frac{1}{2}, \frac{3}{2}\right) = \frac{23}{2^3 \cdot 3!}, \quad P_3\left(\frac{3}{2}, \frac{5}{2}\right) = \frac{23}{2^3 \cdot 3!}, \quad P_3\left(\frac{5}{2}, 3\right) = \frac{1}{2^3 \cdot 3!}$$

which are proportional to the MacMahon numbers.

6 Relation to the Riffle Shuffle

6.1 $(+b)$-case

We study the relationship between the carries process and the (generalized) riffle shuffles, which is a generalization to any $p \in \mathbf{N}$ of the work by Diaconis-Fulman [1, 2]. First of all, we set a ordering on the set $\Sigma := [n] \times \mathbf{Z}_p$ as follows ($[n] := \{1, 2, \cdots, n\}$).

$$\begin{aligned}
&(1,0) < (2,0) < \cdots < (n,0) < \\
&(1,p-1) < (2,p-1) < \cdots < (n,p-1) < \\
&(1,p-2) < (2,p-2) < \cdots < (n,p-2) < \\
&\cdots \\
&(1,1) < \cdots < (n,1).
\end{aligned}$$

For $q \in \mathbf{Z}_p$ let $T_q : \Sigma \to \Sigma$ be the shift given by $T_q(i,r) := (i, r+q)$, $(i,r) \in [n] \times \mathbf{Z}_p$. The group $G_{p,n} \simeq \mathbf{Z}_p \wr S_n$ of colored permutations is the set of bijections $\sigma : \Sigma \to \Sigma$ s.t. $\sigma(i,1) = T_1(\sigma(1,0))$. Writing $(\sigma(i), r_i) := \sigma(i,0)$, $i = 1, 2, \cdots, n$, $\sigma \in G_{p,n}$ is characterized by $((\sigma(1), r_1), (\sigma(2), r_2), \cdots, (\sigma(n), r_n))$.

We say that $\sigma \in G_{p,n}$ have a descent at i if and only if $(\sigma(i), r_i) > (\sigma(i+1), r_{i+1})$ (if $i = 1, 2, \cdots, n-1$) or $r_n \neq 0$. We denote by $d(\sigma)$ the number of descents of σ. Let $b = pc+1$, $c \in \mathbf{N}$. The $(+b, n, p)$**-shuffle** is defined as follows. (i) Divide the n "cards with colors" $(1, r_1), \cdots, (n, r_n)$, $r_j \in \mathbf{Z}_p$, into b-piles, and (ii) Riffle them together as the usual riffle shuffle, except that for the $(jp + r)$-th pile ($j = 0, \cdots, c$, $r = 0, \cdots p-1$, counted from the 0-th), apply T_r-shift to them. Riffle shuffle (resp. type B shuffle) is the special case where $p = 1$ (resp. where $p = 2$).

Let $\{\sigma_r\}_{r=0}^{\infty}$ be the sequence of $(+b, n, p)$-shuffles starting at $\sigma_0 = id$, which is a Markov chain on $G_{p,n}$. Then we have the following theorem, which is a straightforward generalization of the results by Diaconis-Fulman[1, 2].

Theorem 6.1 *[7] Let $b = pc+1$, $c \in \mathbf{N}$ and let $\{\kappa_r^+\}_{r=0}^{\infty}$ be the $(+b, n, p)$-carries process with $\kappa_0^+ = 0$. Then for any $N \in \mathbf{N}$ and for any $j \in \{0, 1, \cdots, n\}$*

$$\mathbf{P}\left(\kappa_N^+ = j \,|\, \kappa_0^+ = 0\right) = \mathbf{P}\left(d(\sigma_N) = j \,|\, \sigma_0 = id\right).$$

6.2 $(-b)$-case

We consider a "shuffle" corresponding to the carries process for negative base, for $p = 1, 2$. For given $\sigma = (\sigma(1), \cdots, \sigma(n)) \in S_n$, let $R_1\sigma \in S_n$

$$(R_1\sigma)(k) := n + 1 - \sigma(k), \quad k = 1, 2, \cdots, n$$

be its "reverse". Let

$$S_1^- := R_1 \circ (b\text{-shuffle})$$

be the operation of carrying out the reverse after a b-shuffle (b-shuffle means the $(+b, n, 1)$-shuffle). Let $\{\sigma_r\}_{r=0}^{\infty}$ ($\sigma_r := (S_1^-)^r \sigma_0$, $r = 1, 2, \cdots$) be a Markov chain on S_n starting at $\sigma_0 = id$. Then the $(-b, n, 1)$-carries process $\{\kappa_r^-\}_{r=0}^{\infty}$ with $\kappa_0^- = 0$ has the same distribution to that of the descent $\{d(\sigma_r)\}_{i=0}^{\infty}$ of $\{\sigma_r\}_{r=0}^{\infty}$.

Theorem 6.2 *[7] For any $i_1, \cdots, i_N \in \{0, 1, \cdots, n-1\}$, we have*

$$\begin{aligned}&\mathbf{P}\left(\kappa_1^- = i_1,\ \kappa_2^- = i_2, \cdots, \kappa_N^- = i_N \,|\, \kappa_0^- = 0\right)\\&= \mathbf{P}\left(d(\sigma_1) = i_1,\ d(\sigma_2) = i_2, \cdots, d(\sigma_N) = i_N \,|\, \sigma_0 = id\right).\end{aligned}$$

Similar result also holds for $p = 2$ case. Let $p = 2$, b : odd. Let

$$(R_2\sigma)(i, 0) := (n + 1 - \sigma(i), r_i + 1), \quad S_2^- := R_2 \circ ((b, n, 2)\text{-shuffle})$$

where we set $(\sigma(i), r_i) := \sigma(i, 0)$, $i = 1, \cdots, n$. Let $\{\sigma_r\}_{r=0}^{\infty}$ ($\sigma_r := (S_2^-)^r \sigma_0$, $r = 1, 2, \cdots$) be a Markov chain on $G_{2,n}$ starting at $\sigma_0 = id$. Then we have the following relation between the $(-b, n, 2)$-carries process and the riffle shuffles.

Theorem 6.3 *[7] For any $N \in \mathbf{N}$ and any $j \in \{0, 1, \cdots, n\}$, we have*

$$\mathbf{P}\left(\kappa_N^- = j \,|\, \kappa_0^- = 0\right) = \mathbf{P}\left(d(\sigma_N) = j \,|\, \sigma_0 = id\right).$$

So far we discussed properties of the $(\pm b, n, p)$-process and the relation to the riffle shuffle. There are still some unsolved problems :

(1) Theorem 6.1 does not imply that the $(+b, n, p)$-carries process ($p \in \mathbf{N}$) and the descent of $(+b, n, p)$-shuffles have the same distribution. Since it is true for $p = 1$ [1], we expect that it is also true for $p \geq 2$.

(2) If $p \in \mathbf{N}$, the $(+b, n, p)$-carries process is related to the descent statistics(Remark 3.2), the Foulkes character(Remark 3.3), and the $(+b, n, p)$-shuffles on $\mathbf{Z}_p \wr S_n$ (Theorem 6.1). It is desirable to have the corresponding results for $p \notin \mathbf{N}$.

(3) It is desirable to construct a $(-b, n, p)$-shuffle for $p \neq 1, 2$.

References

[1] Diaconis, P., Fulman, J., Carries, shuffling, and an amazing matrix. The American Mathematical Monthly, **116**(9)(2009), p.780-803.

[2] Diaconis, P., Fulman, J., Carries, shuffling, and symmetric functions Advances in Applied mathematics, **43**(2)(2009), p.176-196

[3] Diaconis, P., Fulman, J., Foulkes characters, Eulerian idempotents, and an amazing matrix, J. Alg. Comb. **36**(3)(2012), 425-440.

[4] Holte, J. Carries, combinatorics, and an amazing matrix The American Mathematical Monthly, **104**(2)(1997), p.138-149

[5] Miller, A., Foulkes characters for complex reflection groups, to appear in Proc. A.M.S.

[6] Nakano, F., and Sadahiro, T., A generalization of carries process and Eulerian numbers, Advances in Applied Mathematics, **53** (2014) 28 - 43.

[7] Nakano, F., and Sadahiro, T., A generalization of carries process and a relation to riffle shuffles, arXiv:1403.5822.

[8] Steingrímsson, E. Permutation statistics of indexed permutations, Europ. J. Combin. **15**(1994) 187-205.