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Factorization of the Characteristic Polynomial of a Lattice using Quotient Posets

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Abstract. We introduce a new method for showing that the roots of the characteristic polynomial of a finite lattice are all nonnegative integers. Our method gives two simple conditions under which the characteristic polynomial factors. We will see that Stanley's Supersolvability Theorem is a corollary of this result. We can also use this method to demonstrate a new result in graph theory and give new proofs of some classic results concerning the Möbius function.

Résumé. Nous donnons une nouvelle méthode pour démontrer que les racines du polynôme caractéristique d'un treillis fini sont tous les entiers nonnégatifs. Notre méthode donne deux conditions simples pour une telle décomposition. Nous voyons que le théorème de Stanley sur les treillis supersolubles est un corollaire. Nous considerons une application à la théorie des graphes et nous pouvons donner des nouvelles démonstrations de plusieurs résultats classiques concernant la fonction de Möbius.

Keywords: characterstic polynomial, lattice, Möbius function, quotient, supersolvability

1 Introduction

For the entirety of this paper let us assume that all our posets are finite and contain a $\hat{0}$. Recall the one-variable Möbius function of a poset, $\mu : P \to \mathbb{Z}$, is defined recursively by

$$\sum_{y \leq x} \mu(y) = \delta_{\hat{0},x}$$

where $\delta_{\hat{0},x}$ is the Kronecker delta.

We say that a poset P, is *ranked* if, for each $x \in P$, every saturated $\hat{0} - x$ chain has the same length. Given a ranked poset, we get a *rank function* $\rho : P \to \mathbb{N}$ defined by setting $\rho(x)$ to be the length of a $\hat{0} - x$ saturated chain. We define the *rank* of a ranked poset P to be

$$\rho(P) = \max_{x \in P} \rho(x).$$

When P is ranked, the generating function for μ is called the *characteristic polynomial* and is given by

$$\chi(P, t) = \sum_{x \in P} \mu(x) t^{\rho(P) - \rho(x)}.$$

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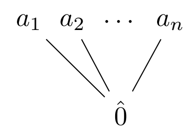

Fig. 1: Claw with n atoms

We are interested in identifying lattices which have characteristic polynomials with only nonnegative integer roots. In this case, we also wish to show that the roots are the cardinalities of sets of atoms of the lattice. We begin with a simple, well-known lemma.

Lemma 1 *Let P and Q be finite ranked posets with minimum elements. Then we have the following.*

1. $\chi(P \times Q, t) = \chi(P, t)\chi(Q, t)$.
2. *If $P \cong Q$, then $\chi(P, t) = \chi(Q, t)$.*

Now let us investigate a family of lattices whose characteristic polynomial has only nonnegative integer roots. We will often refer back to this example in the sequel. The *partition lattice*, Π_n, is the lattice whose elements are the set partitions of $\{1, 2, \ldots, n\}$ under the refinement ordering. It is well-known that the characteristic polynomial is given by

$$\chi(\Pi_n, t) = (t-1)(t-2)\cdots(t-n+1).$$

Note that the characteristic polynomial of the partition lattice can be written as the product of linear factors over $\mathbb{Z}_{\geq 0}$. Motivated by this fact, we consider what posets have these linear factors as their characteristic polynomial.

Definition 2 *The* claw *with n atoms is the ranked poset with a $\hat{0}$, n atoms and no other elements. It will be denoted CL_n and is the poset which has Hasse diagram depicted in Figure 1. Clearly,*

$$\chi(CL_n, t) = t - n.$$

Now let us look at the special case of Π_3. We wish to show that

$$\chi(\Pi_3, t) = (t-1)(t-2).$$

Since the roots of $\chi(\Pi_3, t)$ are 1 and 2, we consider $CL_1 \times CL_2$ which has the same characteristic polynomial. Unfortunately, these two posets are not isomorphic since one contains a maximum element and the other does not. We now wish to modify $CL_1 \times CL_2$ without changing its characteristic polynomial and in such a way that the resulting poset will be isomorphic to Π_3. This will verify that

$$\chi(\Pi_3, t) = \chi(CL_1 \times CL_2) = (t-1)(t-2).$$

Let CL_1 have its atom labeled by a and let CL_2 have its two atoms labeled by b and c. Now suppose that we identify (a, b) and (a, c) in $CL_1 \times CL_2$ and call this new element d. After this collapse, we get a

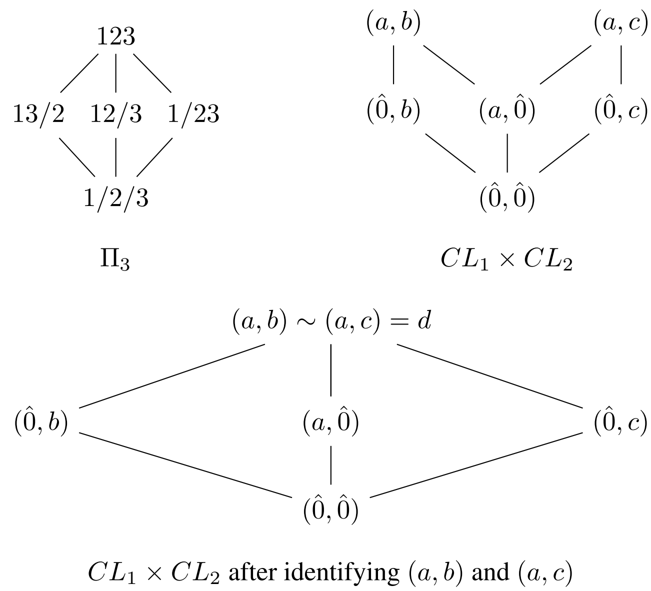

Fig. 2: Hasse diagrams for the partition lattice example

poset isomorphic to Π_3 as can be seen in Figure 2. Note that performing this collapse did not change the characteristic polynomial since $\mu(d) = \mu((a,b)) + \mu((a,c))$ and $\rho(d) = \rho((a,b)) = \rho((a,c))$. Thus we have fulfilled our goal.

It turns out that we can use this technique of collapsing elements to find the roots of a characteristic polynomial in a wide array of lattices. The basic idea is that it is trivial to calculate the characteristic polynomial of a product of claws. Moreover, under certain conditions which we will see later, we are able to identify elements of the product and form a new poset without changing the characteristic polynomial. If we can show the product with identifications made is isomorphic to P, then we will have succeeded in showing $\chi(P,t)$ only has nonnegative integer roots.

Before we continue, let us mention some previous work done by others on the factorization of the characteristic polynomial. For a more complete overview, we suggest reading the survey paper by Sagan [Sag99]. In [Sta72], Stanley showed that the characteristic polynomial of a supersolvable semimodular lattice always has nonnnegative integer roots. Additionally, he showed these roots where counted by the sizes of blocks in a partition of the atom set of the lattice. Blass and Sagan [BS97] extended this result to LL lattices. In [Zas82], Zaslavsky generalized the concept of coloring of graphs to coloring of signed graphs and showed how these colorings were related to the characteristic polynomial of certain hyperplane arrangements. Saito [Sai80] and Terao [Ter81] studied a module of derivations associated with a hyperplane arrangement. When this module is free, the characteristic polynomial has roots which are the degrees of its basis elements.

In the next section, we formally define what we mean by identifying elements of a poset P as well as give conditions under which making these identifications will not change the characteristic polynomial.

In Section 3, we discuss a canonical way to put an equivalence relation on P when it is a lattice and give three simple conditions which together imply that $\chi(P,t)$ has nonnegative integral roots. Section 4 contains a generalization of the notion of a claw. This enables us to remove one of the conditions needed to prove factorization. Section 5 is concerned with partitions of the atom set of P induced by a chain. When P is a semimodular lattice, this permits us to give three conditions which are equivalent to $\chi(P,t)$ having the sizes of the blocks of the partition as roots. This result will imply Stanley's Supersolvability Theorem [Sta72]. Additionally, we will use our result to prove a new theorem in graph theory. Finally, Section 6 contains a discussion of some other applications of quotient posets.

2 Quotients of Posets

We begin this section by defining, in a rigorous way, what we mean by collapsing elements in a Hasse diagram of a poset. We do so by putting an equivalence relation on the poset and then ordering the equivalence classes.

Definition 3 *Let P be a poset and let $\sim$ be an equivalence relation on P. We define $P/\sim$ to be the set of the equivalence classes with the binary relation $\leq$ defined by $X \leq Y$ in $P/\sim$ if and only if $x \leq y$ in P for some $x \in X$ and some $y \in Y$.*

Note that this binary relation on $P/\sim$ is reflexive and transitive, but is not necessarily antisymmetric. For example, let P be the chain with elements $0 < 1 < 2$ and take $X = \{0, 2\}$ and $Y = \{1\}$. Then in $P/\sim$ we have that $X \leq Y$ and $Y \leq X$, but $X \neq Y$. Since we want a poset, it is necessary to require two more properties of our equivalence relation.

Definition 4 *Let P be a poset and let $\sim$ be an equivalence relation on P. Order the equivalence classes as in the previous definition. We say the poset $P/\sim$ is a* homogeneous quotient *if*

- $\hat{0}$ *is in an equivalence class by itself, and*
- *if $X \leq Y$ in $P/\sim$ and $x \in X$, then there is a $y \in Y$ such that $x \leq y$.*

When P is finite, the second condition forces the binary relation to be antisymmetric. We note that while we chose to insist that for every $x \in X$ there is a $y \in Y$ with $x \leq y$, this is not the only way to get a quotient which is a poset. It turns out that this condition, however, does allow us to determine the Möbius values in the quotient poset.

Since we would like to use quotient posets to find characteristic polynomials, it would be quite helpful if the Möbius value of an equivalence class was the sum of the Möbius values of the elements of the equivalence class. This is not always the case when using homogeneous quotients, however, we only need one simple requirement on the equivalence classes so that this does occur. Note the similarity of the hypothesis in the next result to the definition of the Möbius function.

Theorem 5 *Let $P/\sim$ be a homogeneous quotient poset. Suppose that for all $X \in P/\sim$*

$$\sum_{y \in L(X)} \mu(y) = \delta_{\hat{0},X}$$

where $L(X)$ is the lower order ideal generated by X. Then, for all equivalence classes X

$$\mu(X) = \sum_{x \in X} \mu(x).$$

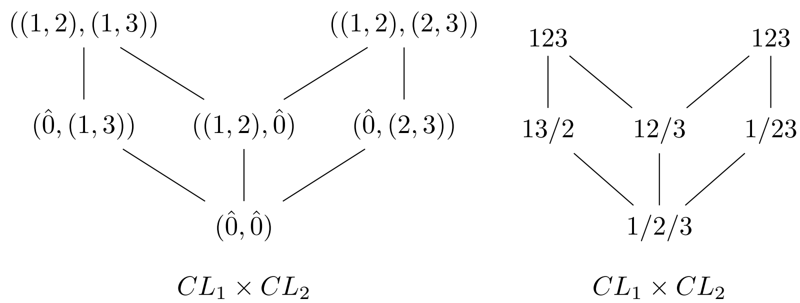

Fig. 3: Hasse diagrams for partition lattice example with new labelings

We now know how the Möbius values behave when taking quotients under certain circumstances. We also need to know how the rank behaves under quotients. It turns out that by forcing all the elements in each equivalence class to have the same rank, we get a quotient which is ranked. Moreover, under this assumption, if $X \in P/\sim$, then $\rho(X) = \rho(x)$ for any $x \in X$. Thus, we get the following corollary.

Corollary 6 *Let P be a ranked poset and let $\sim$ be an equivalence relation on P such that $P/\sim$ is a homogeneous quotient. If the condition of Theorem 5 is satisfied and $x \sim y$ implies $\rho(x) = \rho(y)$, then*

$$\chi(P/\sim, t) = \chi(P, t).$$

We now have some conditions under which the characteristic polynomial does not change when taking a quotient. However, the previous results do not tell us how to choose an appropriate equivalence relation for a given poset. It turns out that when the poset is a lattice, there is a canonical choice for $\sim$, as we will see in the next section.

3 The Standard Equivalence Relation

Once again, let us look at the partition lattice example and give new labelings to $CL_1 \times CL_2$ which will be helpful in determining an equivalence relation. First, we set up some notation for the atoms of the partition lattice. For $i < j$, let (i, j) be the notation for the atom which has i and j in one block and everything else in singleton blocks. Let CL_1 have its atom labeled by $(1, 2)$ and CL_2 have its atoms labeled by $(1, 3)$ and $(2, 3)$. In both of the claws, label the minimum element by $\hat{0}$. The poset on the left in Figure 3 shows the induced labeling on $CL_1 \times CL_2$.

Now relabel $CL_1 \times CL_2$ by taking the join in Π_3 of the two elements in each pair. The poset on the right in Figure 3 shows this step. Finally, identify elements which have the same label. In this case, this means identifying the top two elements as we did before. Upon doing this, we get a poset which is isomorphic to Π_3 and has the same labeling as Π_3.

In order to generalize the previous example, we will be putting an equivalence relation on the product of claws whose atom sets come from partitioning the atoms of the original lattice. We need some terminology before we can define our equivalence relation.

Suppose that L is a lattice and $(A_1, A_2, \ldots, A_n)$ is an ordered partition of the atoms of L. We will use CL_{A_i} to denote the claw whose atom set is A_i and whose minimum element is labeled by $\hat{0}_L$. The elements of $\prod_{i=1}^n CL_{A_i}$ will be called *atomic transversals* and written in boldface. Since the rank of an element in the product of claws is just the number of nonzero elements in the tuple, it will be useful to have a name for this number.

Definition 7 *Let L be a lattice and let $(A_1, A_2, \ldots, A_n)$ be an ordered partition of the atoms of L. For $\boldsymbol{t} \in \prod_{i=1}^n CL_{A_i}$ define the* support *of $\boldsymbol{t}$ as the set of nonzero elements in the tuple $\boldsymbol{t}$. We will denote it by* $\operatorname{supp}(\boldsymbol{t})$.

In what follows, we will need to take the join of the elements in an atomic transversal. For $\boldsymbol{t} \in \prod_{i=1}^n CL_{A_i}$ with $\boldsymbol{t} = (t_1, t_2, \ldots, t_n)$ we will define

$$\bigvee \boldsymbol{t} = t_1 \vee t_2 \cdots \vee t_n.$$

With this new terminology we are now in a position to define a natural equivalence relation on the product of the claws. Since we are trying to show that the characteristic polynomial of a lattice has certain roots, we will need to show the quotient of the product of claws is isomorphic to the lattice. Therefore it is reasonable to define the equivalence relation by identifying two elements of the product of claws if their joins are the same in L.

Definition 8 *Let L be a lattice and let $(A_1, A_2, \ldots, A_n)$ be an ordered partition of the atoms of L. The* standard equivalence relation *on $\prod_{i=1}^n CL_{A_i}$ is defined as*

$$\boldsymbol{s} \sim \boldsymbol{t} \text{ in } \prod_{i=1}^n CL_{A_i} \iff \bigvee \boldsymbol{s} = \bigvee \boldsymbol{t} \text{ in } L.$$

We will use the notation

$$\mathcal{T}_x^a = \left\{ \boldsymbol{t} \in \prod_{i=1}^n CL_{A_i} \mid \bigvee \boldsymbol{t} = x \right\}$$

and call the elements of this set *atomic transversals of x*. Therefore, the equivalence classes of the quotient $(\prod_{i=1}^n CL_{A_i}) / \sim$ are of the form $\mathcal{T}_x^a$ for some $x \in L$. To be able to use any of the theorems from the previous section, we need to make sure that the standard equivalence relation gives us a homogeneous quotient. This will happen if the elements $\boldsymbol{t}$ which we identify all have the same rank, which in the product of claws is given by $|\operatorname{supp}(\boldsymbol{t})|$.

Note that by requiring all the elements of a fixed equivalence class to have the same rank, we have satisfied the second hypothesis of Corollary 6. To get the first hypothesis, we require for each nonzero element x of the lattice there is a block in the partition of atoms containing exactly one atom below x.

Now that we know the product of claws and its quotient have the same characteristic polynomial, we need to show there is an isomorphism between L and this quotient. The isomorphism is given by sending an $x \in L$ to $\mathcal{T}_x^a \in (\prod_{i=1}^n CL_{A_i}) / \sim$. Of course for this to make sense we need $\mathcal{T}_x^a \neq \emptyset$.

Taking all that we have discussed previously together, we get the following theorem.

Theorem 9 *Let L be a ranked lattice and let $(A_1, A_2, \ldots, A_n)$ be an ordered partition of the atoms of L. Let $\sim$ be the standard equivalence relation on $\prod_{i=1}^n CL_{A_i}$. Suppose the following hold:*

(1) If $\boldsymbol{t} \in \mathcal{T}_x^a$, then $|\operatorname{supp}(\boldsymbol{t})| = \rho(x)$.

(2) *For all nonzero $x \in L$, there is an index i such that there is a unique atom below x in A_i.*

(3) *For all $x \in L$, $\mathcal{T}_x^a \neq \emptyset$.*

Then we have

- *For all $x \in L$, $\mu(x) = (-1)^{\rho(x)}|\mathcal{T}_x^a|$*
- $\chi(L, t) = \prod_{i=1}^{n}(t - |A_i|)$

Let us make a few comments about this theorem. It turns out that assumption (1) implies that, for each $x \in L$, all the atomic transversals for x have the same support set. Using this fact and assumption (2) we get that any atomic transversal for x must contain the atom from the block containing the unique atom below x. This implies that the assumptions of Theorem 5 are satisfied. Therefore, we get the Möbius value of the equivalence class is the sum of the Möbius values of its elements. Since all the equivalence classes have elements coming from the product of claws, all the elements have Möbius value -1 or 1. By assumption (1) all the elements have the same rank and so have the same Möbius value. This implies that

$$\mu(x) = (-1)^{\rho(x)}|\mathcal{T}_x^a|.$$

Now Corollary 6 and the isomorphism between L and the quotient imply that χ factors.

Let us return to the partition lattice and see how we can apply Theorem 9. Again label the atoms of Π_n by (i, j) for $i < j$ where (i, j) is the unique partition with i and j in a block and the rest of the blocks singletons. Partition the atoms as $(A_1, A_2, \dots, A_{n-1})$ where $A_j = \{(i, j+1) \mid i < j+1\}$. With each atomic transversal $\boldsymbol{t}$ we will associate a graph, $G_{\boldsymbol{t}}$ on n vertices such that there is an edge between vertex i and vertex j if and only if (i, j) is in $\boldsymbol{t}$. We will use the graph to verify the assumptions of Theorem 9 are satisfied for Π_n.

First, we show that assumption (1) holds. We claim that if $\boldsymbol{t} \in \mathcal{T}_\pi^a$ then $G_{\boldsymbol{t}}$ is a forest. To see why, suppose that there was a cycle and let c be the largest vertex in the cycle. Then c must be adjacent to two vertices a and b. Since c is the largest (a, c) and (b, c) must be in $\boldsymbol{t}$. This is impossible since both come from A_{c-1}.

Since $G_{\boldsymbol{t}}$ is forest, if $G_{\boldsymbol{t}}$ has k components then the number of edges in $G_{\boldsymbol{t}}$ is $n-k$. It is not hard to see that i and j are in the same block in $\vee\boldsymbol{t}$ if and only if i and j are in the same component of $G_{\boldsymbol{t}}$. Moreover, it is well known that if $\pi \in \Pi_n$ and π has k blocks then $\rho(\pi) = n - k$. It follows that if $\boldsymbol{t} \in \mathcal{T}_\pi^a$ and π has k blocks then $|\operatorname{supp}(\boldsymbol{t})| = |E(G_{\boldsymbol{t}})| = n - k = \rho(\pi)$. We conclude that assumption (1) holds.

To verify assumption (2), let $\pi \in \Pi_n$ with $\pi \neq \hat{0}$. Then π contains a nontrivial block. Let i be the second smallest number in this block. We claim that there is only one atom in A_{i-1} below π. Suppose this was not the case and let $(a, i), (b, i) \in A_{i-1}$ with $(a, i), (b, i) \leq \pi$. Then $(a, i) \vee (b, i) \leq \pi$ and so a, b and i are all in the same block in π which is impossible since $a, b < i$ but i was chosen to be the second smallest in its block.

Finally, let us show assumption (3) holds. First we will consider the case when there is a single block $B = \{b_1 < b_2 < \cdots < b_m\}$. Then the elements $(b_1, b_2), \dots, (b_{m-1}, b_m)$ form an atomic transversal and their join is B. Now to get the elements which have more than one nontrivial block do the same for each block and take the union of the atomic transversals. It follows every element has an atomic transversal.

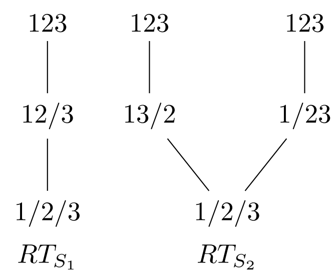

Fig. 4: Hasse diagrams for rooted trees

Now applying the theorem we get that

$$\chi(\Pi_n, t) = (t-1)(t-2)\cdots(t-n+1)$$

since $|A_i| = i$ for $1 \leq i \leq n-1$.

One of the drawbacks of Theorem 9 is that assumption (3) requires that every element of the lattice have an atomic transversal. This forces L to be atomic. However, by generalizing the notion of a claw, we will be able to remove assumption (3). This will allow us to apply Theorem 9 to a wider class of lattices. We give this generalization in the next section.

4 Rooted Trees

In this section, we generalize claws which will allow us to consider lattices which are not atomic. We begin with a definition.

Definition 10 *Let L be a lattice and S be a subset of L containing $\hat{0}$. Let $\mathcal{C}$ be the collection of saturated chains of P which start at $\hat{0}$ and use only elements of S. The* rooted tree with respect to S *is the poset obtained by ordering $\mathcal{C}$ by inclusion and will be denoted by RT_S.*

Strictly speaking the elements of RT_S are chains of L. However, it will be useful to think of the elements of RT_S as elements of L where we associate a chain with the top element of that chain. Let us consider an example in Π_3. As before, partition the atom set as $A_1 = \{12/3\}$ and $A_2 = \{13/2, 1/23\}$. Let S_1, S_2 be the upper ideals of A_1, A_2, respectively, together with $\hat{0}$. Then we we get RT_{S_1} and RT_{S_2} as in Figure 4. Note that we label the chains $\hat{0} < 12/3 < 123$, $\hat{0} < 13/2 < 123$ and $\hat{0} < 1/23 < 123$ all by 123 since they all terminate at 123.

As we can see in Figure 4 the poset we obtain contains a $\hat{0}$ and has no cycles in the Hasse diagram. In fact, given any subset S of a lattice containing $\hat{0}$, RT_S always has these two properties and thus it is deserving of the name rooted tree.

In the previous sections, we used a partition of the atom set to form claws. In this section, we will use the partition of the atom set to form rooted trees. Given a partition of the atoms of a lattice $(A_1, A_2, \ldots, A_n)$ for each i we form the rooted tree $RT_{\hat{U}(A_i)}$ where $\hat{U}(A_i)$ is the upper order ideal generated by A_i together with $\hat{0}$. Note that since $(A_1, A_2, \ldots, A_n)$ is a partition of the atoms, every element of the lattice appears in an $RT_{\hat{U}(A_i)}$ for some i.

Given $(A_1, A_2, \dots, A_n)$, an ordered partition of the atoms of L, we call $\boldsymbol{t} \in \prod_{i=1}^{n} RT_{\hat{U}(A_i)}$ a *transversal*. If $\boldsymbol{t}$ consists of only atoms of L or $\hat{0}$ then $\boldsymbol{t}$ is called an *atomic transversal*. This agrees with the terminology we used for claws. The set of atomic transversals for x will be denoted $\mathcal{T}_x^a$ as before

There is very little change in the approach of using rooted trees as opposed to claws. As before, given a partition $(A_1, A_2, \dots, A_n)$ of the atom set of L, we will put the standard equivalence relation on $\prod_{i=1}^{n} RT_{\hat{U}(A_i)}$. Note that one can take the join using all the elements of a chain or just the top element as the results will be equal. Since we are using rooted trees, the natural map from $\left(\prod_{i=1}^{n} RT_{\hat{U}(A_i)}\right) / \sim$ to L is automatically surjective. In other words, we can remove the condition that every element of L has an atomic transversal. Since the Hasse diagram of $RT_{\hat{U}(A_i)}$ is a rooted tree, the only elements with nonzero Möbius value are $\hat{0}$ and the atoms. It follows that $\chi(RT_{\hat{U}(A_i)}, t)$ has roots $|A_i|$ with multiplicty one and 0 with multiplicity $\rho(RT_{\hat{U}(A_i)}) - 1$. We also get that in the product of the trees, the Möbius value of any transversal which is not atomic is zero and so does not affect the characteristic polynomial of the product of the trees. Therefore, we get the following improvement on Theorem 9

Theorem 11 *Let L be a ranked lattice and let $(A_1, A_2, \dots, A_n)$ be an ordered partition of the atoms of L. Let $\sim$ be the standard equivalence relation on $\prod_{i=1}^{n} RT_{\hat{U}(A_i)}$. Suppose the following hold:*

(1) If $\boldsymbol{t} \in \mathcal{T}_x^a$, then $|\operatorname{supp}(\boldsymbol{t})| = \rho(x)$.

(2) For all nonzero $x \in L$, there is an index i such that there is a unique atom below x in A_i.

Then we have

- *For all $x \in L$, $\mu(x) = (-1)^{\rho(x)} |\mathcal{T}_x^a|$*
- $\chi(L, t) = \displaystyle\prod_{i=1}^{n} (t - |A_i|)$

It turns out that under certain circumstances we can show that assumption (2) of Theorem 11 and factorization of the characteristic polynomial are equivalent. To be able to prove this equivalence, we will not be able to take an arbitrary partition of the atoms, but rather we will need the partition to be induced by a chain in the lattice. We explore this idea in the next section.

5 Partitions Induced by a Chain

If L is a lattice and $C : \hat{0} = x_0 < x_1 < \cdots < x_n = \hat{1}$ is a $\hat{0} - \hat{1}$ chain of L, we get a partition $(A_1, A_2, \dots, A_n)$ of the atoms of L by defining the set A_i as

$$A_i = \{a \in A(L) \mid \ a \leq x_i \text{ and } a \not\leq x_{i-1}\}.$$

In this case we say $(A_1, A_2, \dots, A_n)$ is *induced* by the chain C. Partitions induced by chains have several nice properties. The first property will apply to any ranked lattice, but for the second we will need the lattice to be semimodular.

Suppose that we have a partition of the atoms of a lattice L induced by a chain such that for each $x \in L$, we have that $\boldsymbol{t} \in \mathcal{T}_x^a$ implies that $|\operatorname{supp}(\boldsymbol{t})| = \rho(x)$. We know, by Theorem 11 that if for all nonzero $x \in L$, there is an index i with a unique atom below x in A_i then the roots of $\chi(L,t)$ are exactly $|A_1|, |A_2|, \ldots, |A_n|$. It turns out that when using a partition to create a product of rooted trees that the conditions about the unique index is equivalent to the condition of Theorem 5. Moreover, Theorem 5 can be modified to show that if the original condition of the theorem is not satisfied and the partition of atoms is induced by a chain, then the roots of the characteristic polynomial cannot be the same as the roots of the product of the trees. This implies that if the index condition is not satisfied for some x in the lattice, the characteristic polynomial cannot be $t^{\rho(L)-n}\prod_{i=1}^{n}(t-|A_i|)$.

Now it is not hard to show that if L is a lattice and $(A_1, A_2, \ldots, A_n)$ is a partition of the atoms then for every nonzero $x \in L$ there is an index i with a unique atom below x in A_i if and only if there is such an index for each nonzero atomic element of L. Using this fact and the previous paragraph we get the next theorem.

Theorem 12 *Let L be a ranked lattice and let $(A_1, A_2, \ldots, A_n)$ be induced by a chain. Moreover, suppose that if $\boldsymbol{t} \in \mathcal{T}_x^a$, then $|\operatorname{supp}(\boldsymbol{t})| = \rho(x)$. Under these conditions, every nonzero atomic $x \in L$ has an index i such that there is a unique atom below x in A_i if and only if*

$$\chi(L,t) = t^{\rho(L)-n}\prod_{i=1}^{n}(t-|A_i|).$$

Let us note that we need the factor of $t^{\rho(L)-n}$ on the outside of the product since we do not require that the chain be saturated and so n need not be the rank of L. It would be nice if all the atomic transversals had the correct support size when using a partition induced by a chain. Unfortunately this does not always occur, but if L is semimodular it does. Therefore, we can remove the assumption about support size in the previous theorem at the expense of assuming L is semimodular.

We have seen that when a partition of the atoms is induced by a $\hat{0}-\hat{1}$ chain in a semimodular lattice we only need to check one more condition to ensure factorization. It turns out that, for any lattice, this condition is equivalent to having a chain which satisfies the following condition.

Definition 13 *Let L be a lattice and let $C : \hat{0} = x_0 < x_1 < x_2 < \cdots < x_n = \hat{1}$ be a chain. For atomic $x \in L$, x neither $\hat{0}$ nor an atom, let i be the index such that $x \le x_i$ but $x \not\le x_{i-1}$. We say that C satisfies the* meet condition *if for each such x we have $x \wedge x_{i-1} \neq \hat{0}$.*

If L is a lattice and $(A_1, A_2, \ldots, A_n)$ is induced by a chain C, one can show that every atomic element of L having an atomic transversal guarantees that C satisfies the meet condition. On the other hand if L is semimodular and C satisfies the meet condition, we get that the Möbius value of an element is the number of atomic transversals of x. If L is semimodular, it is well known that every atomic element has nonzero Möbius value. Thus if C satisfies the meet condition then every atomic element must have an atomic transversal. Therefore, we get the following list of conditions equivalent to factorization.

Theorem 14 *Let L be a semimodular lattice and let $(A_1, A_2, \ldots, A_n)$ be induced by a $\hat{0}-\hat{1}$ chain, C. The following are equivalent:*

1. *For every nonzero atomic $x \in L$, there is an index i such that there is a unique atom below x in A_i.*
2. *For every atomic $x \in L$, $\mathcal{T}_x^a \neq \emptyset$.*
3. *C satisfies the meet condition.*

4. *We have*

$$\chi(L,t) = t^{\rho(L)-n}\prod_{i=1}^{n}(t-|A_i|).$$

Let us now consider semimodular supersolvable lattices. Recall that every supersolvable semimodular lattice contains a saturated $\hat{0}-\hat{1}$ left-modular chain. It turns out that saturated $\hat{0}-\hat{1}$ left-modular chains satisfy the meet condition. Thus, we get Stanley's Supersolvability Theorem as a corollary of our previous result.

Theorem 15 (Stanley's Supersolvability Theorem [Sta72]) *Let L be a semimodular lattice with partition of the atoms $(A_1, A_2, \ldots, A_n)$ induced by a saturated $\hat{0}-\hat{1}$ left-modular chain. Then*

$$\chi(L,t) = \prod_{i=1}^{n}(t-|A_i|).$$

We will now consider an application of Theorem 14 to graph theory. We start with a definition.

Definition 16 *Let G be a graph with a total ordering of the vertices given by $v_1 < v_2 < \cdots < v_n$. Let f_i be the number of spanning forests with i edges in G which are increasing with respect to the ordering. The* increasing spanning forest generating function *is given by*

$$IF(G,t) = \sum_{i=0}^{n-1}(-1)^i f_i t^{n-i}.$$

To see what the roots of $IF(G,t)$ are, we will need a partition of the edge set which is given by the ordering on the vertices.

Definition 17 *Let G be a graph with a total ordering of the vertices given by $v_1 < v_2 < \cdots < v_n$. Label the edge v_iv_j by (i,j) where $i<j$. The ordered partition $(E_1, E_2, \ldots, E_{n-1})$ of $E(G)$* induced by the total ordering *is the one with blocks*

$$E_j = \{(i,j+1):\ (i,j+1)\in E(G)\}$$

It turns out that the sizes of the blocks in the partition and 0 are exactly the roots of $IF(G,t)$. Additionally, this partition is induced by a chain in the bond lattice of G. Using this fact and Theorem 14 we get the following theorem.

Theorem 18 *Let G be a graph with the partition $(E_1, E_2, \ldots, E_{n-1})$ induced by the total ordering $v_1 < v_2 < \cdots < v_n$. Then*

$$IF(G,t) = t\prod_{i=1}^{n-1}(t-|E_i|).$$

Moreover, denoting the chromatic polynomial of G as $P(G,t)$ we have

$$P(G,t) = IF(G,t) = t\prod_{i=1}^{n-1}(t-|E_i|)$$

if and only if $v_1 < v_2 < \cdots < v_n$ is a perfect elimination ordering, i.e., for each i, the neighbors of v_i coming before v_i in the ordering form a clique of G.

6 Other Applications of Quotient Posets

Thus far we have assumed that the posets we are dealing with are ranked. However, one can generalize the notion of rank and, using similar methods, we can prove Blass and Sagan's result [BS97] concerning factorization of characteristic polynomials of LL lattices.

We now mention some applications of quotient posets which do not explicitly involve the characteristic polynomial. The main idea is that if a poset has a $\hat{0}$ and a $\hat{1}$, then we can form a very simple homogeneous quotient by identifying a coatom and $\hat{1}$. Additionally, this quotient has nice properties as we see in the next lemma.

Lemma 19 *Let P be a poset with a $\hat{0}$ and $\hat{1}$ and $|P| \geq 3$. Let c be a coatom and define $\sim$ to be the equivalence where $c \sim \hat{1}$ and $x \sim x$ for all $x \in P$. Then $P/\sim$ is a homogeneous quotient and*

$$\mu([x]) = \begin{cases} \mu(x) & \textit{if } [x] \neq [\hat{1}] \\ \mu(c) + \mu(\hat{1}), & \textit{if } [x] = [\hat{1}]. \end{cases}$$

Moreover, if P is a lattice, then $P/\sim$ is a lattice with $[x] \vee [y] = [x \vee y]$ for all $[x], [y] \in P/\sim$, and $[x] \wedge [y] = [x \wedge y]$, provided $[x], [y] \neq [\hat{1}]$.

In the interest of saving space, we just give a list of theorems which are corollaries of this lemma. We can use this lemma to prove Hall's Theorem [Hal36], Rota's Crosscut Theorem [Rot64] and Weisner's Theorem [Wei35].

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