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Kronecker coefficients: the tensor square conjecture and unimodality

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Abstract. We consider two aspects of Kronecker coefficients in the directions of representation theory and combinatorics. We consider a conjecture of Jan Saxl stating that the tensor square of the S_n-irreducible representation indexed by the staircase partition contains every irreducible representation of S_n. We present a sufficient condition allowing to determine whether an irreducible representation is a constituent of a tensor square and using this result together with some analytic statements on partitions we prove Saxl conjecture for several partition classes. We also use Kronecker coefficients to give a new proof and a generalization of the unimodality of Gaussian (q-binomial) coefficients as polynomials in q, and extend this to strict unimodality.

Résumé. Nous considérons deux aspects des coéfficients de Kronecker dans le domaine de la Théorie des Représentations et le domaine Combinatoire. Nous considérons la conjecture suivante de Jan Saxl: le tenseur au carré de la représentation irréductible du groupe S_n indexée par la partition $(k, k-1, ..., 1)$ contient toutes les représentations irréductibles du groupe S_n ($n = \binom{k+1}{2}$). Nous présentons une condition suffisante qui permet de déterminer si une représentation irréductible est une constituante d'un tenseur au carré. En utilisant ce résultat avec des résultats analytiques sur les partitions, nous prouvons la conjecture de Saxl pour plusieurs classes de partitions. Nous utilisons aussi les coéfficients de Kronecker pour donner une nouvelle preuve et une généralisation de l'unimodalité des coéfficients de Gauss (q-binomiaux) comme polynômes en q et nous étendons cela à l'unimodalité stricte.

Keywords: Kronecker coefficients, symmetric group, irreducible representations, tensor square, unimodality, integer partitions, Gaussian coefficients

1 Introduction

The *Kronecker product problem* is a problem of computing the multiplicities, called Kronecker coefficients,

$$g(\lambda, \mu, \nu) = \langle \chi^\lambda, \chi^\mu \otimes \chi^\nu \rangle$$

of an irreducible character of S_n in the tensor product of two others. These coefficients and the problem were introduced 75 years ago by Murnaghan following the discovery of the Littlewood-Richardson rule. It

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is often referred as "classic", and "one of the last major open problems" in algebraic combinatorics [BWZ, Reg]. The significance of Kronecker coefficients has recently found a new meaning in Complexity Theory via a program designed to prove the "P vs NP" problem (see e.g. [Mul, MNS, MS]). Despite a large body of work on the Kronecker coefficients, both classical and very recent (see e.g. [BO2, Bla, BOR, Ike, Reg, Rem, RW, Val1, Val2] and references therein), it is universally agreed that "frustratingly little is known about them" [Bür]. Most results are limited to partitions of very specific shapes (when one of the partitions λ, μ, ν is a hook or two row with further restrictions) and are far from answering the natural questions that arise. We now present two problems, one of representation theoretic and another of combinatorial nature, where we develop some tools and consider new aspects of this old problem. The work presented in this abstract is based on the results from [PPV], [PP-u], [PP-s], and partially on [Val3].

1.1 The tensor square conjecture

Motivated by *John Thompson's conjecture* and *Passman's problem*, Heide, Saxl, Tiep and Zalesski recently proved that with a few known exceptions, every irreducible character of a simple group of Lie type is a constituent of the tensor square of the Steinberg character [HSTZ]. They conjecture that for every $n \geq 5$, there is an irreducible character χ of A_n whose tensor square $\chi \otimes \chi$ contains every irreducible character as a constituent.(i) Here is the symmetric group analogue of this conjecture:

Conjecture 1.1 (Tensor square conjecture) *For every $n \geq 3$, $n \neq 4, 9$, there is a partition $\mu \vdash n$, such that tensor square of the irreducible character χ^μ of S_n contains every irreducible character as a constituent.*

During a talk at UCLA, Jan Saxl made the following conjecture, somewhat refining the tensor square conjecture.(ii)

Conjecture 1.2 (Saxl conjecture) *Denote by $\rho_k = (k, k-1, \ldots, 2, 1) \vdash n$, where $n = \binom{k+1}{2}$. Then for every $k \geq 1$, the tensor square $\chi^{\rho_k} \otimes \chi^{\rho_k}$ contains every irreducible character of S_n as a constituent.*

Andrew Soffer checked the validity of this conjecture for $k \leq 8$.(iii) While we believe the conjecture, we also realize that it is beyond the reach of current technology. More importantly, we develop a new tool, aimed specifically at the Saxl conjecture, which gives a positivity criterion for Kronecker coefficients:

Lemma 1.3 (Main Lemma) *Let $\mu = \mu'$ be a self-conjugate partition of n, and let $\widehat{\mu} = (2\mu_1 - 1, 2\mu_2 - 3, 2\mu_3 - 5, \ldots) \vdash n$ be the partition whose parts are lengths of the principal hooks of μ. Suppose $\chi^\lambda[\widehat{\mu}] \neq 0$ for some $\lambda \vdash n$. Then χ^λ is a constituent of $\chi^\mu \otimes \chi^\mu$.*

We use this lemma together with analytic results on the growth of certain partition functions to obtain the following technical results (among others) towards Saxl Conjecture.

Theorem 1.4 *There is a universal constant L, such that for every $k \geq L$, the tensor square $\chi^{\rho_k} \otimes \chi^{\rho_k}$ contains characters χ^λ as constituents, for all*

$$\lambda = (n-\ell, \ell), \quad 0 \leq \ell \leq n/2, \qquad \lambda = (n-r, 1^r), \quad 0 \leq r \leq n-1,$$

$$\lambda = (n-\ell-m, \ell, m), \quad m \in \{1,3,5,7,8,9\}, \ L \leq \ell + m \leq n/2,$$

(i) Authors of [HSTZ] report that this conjecture was checked by Eamonn O'Brien for $n \leq 17$.
(ii) UCLA Combinatorics Seminar, Los Angeles, March 20, 2012.
(iii) Personal communication.

$$\textit{or} \quad \lambda = (n-r-m, m, 1^r), \quad 1 \le m \le 10,\ L \le r < n/2 - 5.$$

These results are based on the work in [PPV], we present some of them here.

1.2 Unimodality via Kronecker coefficients

A sequence $(a_1, a_2, \ldots, a_n)$ is called *unimodal*, if for some k we have

$$a_1 \,\le\, a_2 \,\le\, \ldots \,\le\, a_k \,\ge\, a_{k+1} \,\ge\, \ldots \,\ge\, a_n .$$

The study of unimodality of combinatorial sequences is a classical subject going back to Newton, and has intensified in recent decades. There is a remarkable diversity of applicable tools, ranging from analytic to topological, and from representation theory to probabilistic analysis. The results have a number of application, but are also important in their own right. We refer to [B1, B2, Sta1] for a broad overview of the subject.

We present two extensions of the following classical unimodality result. The *q-binomial* (Gaussian) *coefficients* are defined as:

$$\binom{m+\ell}{m}_q \;=\; \frac{(q^{m+1}-1)\,\cdots\,(q^{m+\ell}-1)}{(q-1)\,\cdots\,(q^{\ell}-1)} \;=\; \sum_{n=0}^{\ell m} p_n(\ell,m)\, q^n .$$

Recall that $p_n(\ell,m) = \#\mathcal{P}_n(\ell,m)$, where $\mathcal{P}_n(\ell,m)$ is the set of partitions $\alpha \vdash n$, such that $\alpha_1 \le m$ and $\alpha_1' \le \ell$. The unimodality of a sequence

$$p_0(\ell,m),\, p_1(\ell,m),\, \ldots\,,\, p_{\ell m}(\ell,m)$$

is a celebrated result first conjectured by Cayley in 1856, and proved by Sylvester in 1878 [Syl] (see also [S1]). Historically, it has been a starting point of many investigations and various generalizations, both of combinatorial and algebraic nature. The original proof of this result by Sylvester relies on an intricate use of Lie algebras. Subsequent proofs were given by Stanley [S2] and Proctor [Pro] which are also algebraic. The only known combinatorial proof of of the unimodality of q-binomial coefficients is given by O'Hara in [O'H] (see also [SZ, Zei]). However, O'Hara's construction does not give a symmetric chain decomposition of the poset $L(\ell,m)$ of partitions which fit the $\ell \times m$ rectangle (in other words, the difference between successive partitions is not always a corner). Existence of such decompositions remains an open problem (see e.g. [S2, Wen] and references therein).

Denote by $v(\alpha)$ the number of distinct part sizes in the partition α. The sequence $(a_1, \ldots, a_n)$ is called *symmetric* if $a_i = a_{n+1-i}$, for all $i \le i \le n$.

Using Kronecker coefficients we prove in [PP-u] the following extension of Sylvester's theorem.

Theorem 1.5 *Let*

$$p_n(\ell,m,r) \;=\; \sum_{\alpha \in \mathcal{P}_n(\ell,m)} \binom{v(\alpha)}{r}.$$

Then the sequence

$$p_r(\ell,m,r), p_{r+1}(\ell,m,r), \ldots, p_{\ell m}(\ell,m,r)$$

is symmetric and unimodal.

Note that $p_n(\ell,m,r)=0$ for $n<\binom{r+1}{2}$ or $n>\ell m-\binom{r}{2}$, and that $v(\alpha)$ can be viewed as the *number of corners* of the corresponding Young diagram $[\alpha]$. Moreover, $p_n(\ell,m,0)=p_n(\ell,m)$ and therefore, for $r=0$, Theorem 1.5 gives the unimodality of q-binomial coefficients, and hence a new (up to our knowledge) proof of Sylvester's theorem. Moreover, the other known algebraic proofs do not imply such results. The chain construction argument from O'Hara's combinatorial proof also does not seem to imply Theorem 1.5 even in the case $r=1$. Indeed, the value of $v(\alpha)$ is not unimodal on the chains. For example, the fourth chain on p. 50 in [O'H] is

$$(2^2)\to(32)\to(42)\to(43)\to(43)\to(4^2)\to(4^21)\to(4^22)\to(4^23)\to(4^3),$$

and the number of corners dips in the middle. It would be interesting to find a combinatorial proof of the Theorem 1.5, which might lead to a way of finding a symmetric chain decomposition of the poset $L(m,\ell)$ mentioned above. (iv)

We also use the recently established *semigroup property* of Kronecker coefficients to prove in [PP-s] *strict unimodality* of q-binomial coefficients:

Theorem 1.6 *For all $\ell,m\geq 8$, we have the following strict inequalities:*

$$(\circ)\qquad p_1(\ell,m)\,<\,\ldots\,<\,p_{\lfloor \ell m/2\rfloor}(\ell,m)\,=\,p_{\lceil \ell m/2\rceil}(\ell,m)\,>\,\ldots\,>\,p_{\ell m-1}(\ell,m).$$

The fact that strict unimodality of q-binomial coefficients was open until now is perhaps a reflection on the lack of analytic proof of Sylvester's theorem, as all known proofs are either algebraic or combinatorial (see [Pro, Sta1]).(v)

2 A positivity criterion for $g(\mu,\mu,\lambda)$

Lemma 1.3 (Main Lemma) *Let $\lambda,\mu\vdash n$, such that $\mu=\mu'$ and $\chi^\lambda[\widehat{\mu}]\neq 0$, where $\widehat{\mu}=(2\mu_1-1,2\mu_2-3,\ldots)$. Then $g(\mu,\mu,\lambda)>0$.*

Proof: Let

$$\varepsilon_\mu\,=\,\chi^\mu[\widehat{\mu}]\,=\,(-1)^{(n-\mathrm{d}(\mu))/2}.$$

The second equality follows from the Murnaghan-Nakayama rule, realizing that there is a unique way of fitting ribbons of size $2\mu_1-1,\ldots$ in the Young diagram of μ. Recall also that the S_n conjugacy class of cycle type ζ, when ζ is a partition into distinct odd parts, splits into two conjugacy classes in the alternating group A_n, which we denote by ζ^1 and ζ^2. There are two kinds of irreducible characters of A_n. For each partition ν of n such that $\nu=\nu'$ there are two irreducible characters associated to ν, which we denote by $\alpha^{\nu+}$ and $\alpha^{\nu-}$; and for each partition ν of n such that $\nu\neq\nu'$ there is an irreducible character associated to the pair ν,ν', which we denote by α^ν. These characters are related to irreducible characters of S_n as indicated below. We will need the following standard results (see e.g. [JK, Section 2.5]).

1. If $\nu\neq\nu'$, then

$$\mathsf{Res}^{S_n}_{A_n}(\chi^\nu)\,=\,\mathsf{Res}^{S_n}_{A_n}(\chi^{\nu'})\,=\,\alpha^\nu,$$

(iv) Since then, a combinatorial proof relying on Sylvester's original result was given by J. Shareshian (personal communication), but without further insight into the structure of $L(m,\ell)$.

(v) New proofs using elementary methods like the KOH identity ([Zei]) have been found since [PP-s], see [Dh, Zan].

is an irreducible character of A_n.

2. If $\nu = \nu'$, then

$$\mathsf{Res}^{S_n}_{A_n}(\chi^\nu) \,=\, \alpha^{\nu+} \,+\, \alpha^{\nu-}\,,$$

is the sum of two different irreducible characters of A_n. Moreover, both characters are conjugate, that is, for any $\sigma \in A_n$ we have

$$\alpha^{\nu+}\big[(12)\sigma(12)\big] \,=\, \alpha^{\nu-}[\sigma].$$

3. The characters α^ν, $\nu \neq \nu'$ and $\alpha^{\nu+}$, $\alpha^{\nu-}$, where $\nu = \nu'$ are all different and form a complete set of irreducible characters of A_n.

4. If $\nu = \nu'$, and γ is a conjugacy class of A_n different from $\widehat{\nu}^1$ or $\widehat{\nu}^2$, then

$$\alpha^{\nu+}[\gamma] \,=\, \alpha^{\nu-}[\gamma] \,=\, \frac{1}{2}\,\chi^\nu[\gamma].$$

We also have

$$\alpha^{\nu+}\big[\widehat{\nu}^1\big] \,=\, \alpha^{\nu-}\big[\widehat{\nu}^2\big] \,=\, \frac{1}{2}\left(\varepsilon_\nu + \sqrt{\varepsilon_\nu \prod_i \widehat{\nu}_i}\right),$$

$$\alpha^{\nu+}\big[\widehat{\nu}^2\big] \,=\, \alpha^{\nu-}\big[\widehat{\nu}^1\big] \,=\, \frac{1}{2}\left(\varepsilon_\nu - \sqrt{\varepsilon_\nu \prod_i \widehat{\nu}_i}\right).$$

In other words, for any self-conjugate partition ν, the only irreducible characters of A_n that differ on the classes $\widehat{\nu}^1$ and $\widehat{\nu}^2$ are precisely $\alpha^{\nu\pm}$.

There are two cases to consider with respect to whether λ is self-conjugate or not.

First, assume that $\lambda \neq \lambda'$. Then α^λ is an irreducible character of A_n. Since

$$\alpha^\lambda[\widehat{\mu}^1] \,=\, \alpha^\lambda[\widehat{\mu}^2] \,=\, \chi^\lambda[\widehat{\mu}],$$

we obtain:

$$\begin{aligned}\big(\alpha^{\mu+} \otimes \alpha^\lambda\big)[\widehat{\mu}^1] - \big(\alpha^{\mu+} \otimes \alpha^\lambda\big)[\widehat{\mu}^2] \,&=\, \big(\alpha^{\mu+}[\widehat{\mu}^1] - \alpha^{\mu+}[\widehat{\mu}^2]\big)\cdot \chi^\lambda[\widehat{\mu}] \\ &=\, \left(\sqrt{\varepsilon_\mu \prod_i \widehat{\mu}_i}\right)\cdot \chi^\lambda(\widehat{\mu}) \,\neq\, 0.\end{aligned}$$

Therefore, either $\alpha^{\mu+}$ or $\alpha^{\mu-}$ is a component of $\alpha^{\mu+} \otimes \alpha^\lambda$. In other words,

$$\text{either} \quad \langle \alpha^{\mu+} \otimes \alpha^{\,\lambda}, \alpha^{\mu+}\rangle \neq 0 \quad \text{or} \quad \langle \alpha^{\mu+} \otimes \alpha^{\,\lambda}, \alpha^{\mu-}\rangle \neq 0.$$

We claim that the terms in these product can be interchanged. Formally, we claim that:

$$(\star) \qquad \text{either} \quad \langle \alpha^{\mu+} \otimes \alpha^{\mu+}, \alpha^{\lambda}\rangle \neq 0 \quad \text{or} \quad \langle \alpha^{\mu+} \otimes \alpha^{\mu-}, \alpha^{\lambda}\rangle \neq 0.$$

There are two cases. If $\varepsilon_\mu = 1$, then both $\alpha^{\mu+}$ and $\alpha^{\mu-}$ take real values. Thus

$$\langle \alpha^{\mu+} \otimes \alpha^{\mu\pm}, \alpha^{\lambda}\rangle \,=\, \langle \alpha^{\mu+} \otimes \alpha^{\lambda}, \alpha^{\mu\pm}\rangle \,\neq\, 0\,,$$

which implies $(\star)$ in this case.

If $\varepsilon_\mu = -1$, then

$$\mathrm{Im}\left(\alpha^{\mu+}[\widehat{\mu}^1]\right) \,=\, -\mathrm{Im}\left(\alpha^{\mu-}[\widehat{\mu}^1]\right) \quad \text{and} \quad \mathrm{Im}\left(\alpha^{\mu+}[\widehat{\mu}^2]\right) \,=\, -\mathrm{Im}\left(\alpha^{\mu-}[\widehat{\mu}^2]\right).$$

Therefore, $\overline{\alpha^{\mu+}} = \alpha^{\mu-}$, since all other character values are real. Thus,

$$\langle \alpha^{\mu+} \otimes \alpha^{\mu\pm}, \alpha^\lambda \rangle \,=\, \langle \alpha^{\mu+} \otimes \alpha^\lambda, \alpha^{\mu\mp} \rangle \,\neq\, 0,$$

which implies $(\star)$ in this case.

In summary, we have both cases in $(\star)$ imply that α^λ is a component of $\mathrm{Res}^{S_n}_{A_n}(\chi^\mu \otimes \chi^\mu)$. Therefore, either χ^λ or $\chi^{\lambda'}$ is a component of $\chi^\mu \otimes \chi^\mu$. Since $\mu = \mu'$, we have,since $g(\mu,\mu,\lambda) = g(\mu,\mu,\lambda')$, that χ^λ and $\chi^{\lambda'}$ are components of $\chi^\mu \otimes \chi^\mu$, as desired. This completes the proof of the $\lambda \neq \lambda'$ case.

Now, suppose $\lambda = \lambda'$. The case $\lambda = \mu$ is given in a Theorem in [BB], whose proof inspired the current result. If $\lambda \neq \mu$, then

$$\alpha^{\lambda\pm}[\widehat{\mu}^1] \,=\, \alpha^{\lambda\pm}[\widehat{\mu}^2] \,=\, \frac{1}{2}\chi^\lambda(\widehat{\mu}) \neq 0.$$

By a similar argument as above applied to $\lambda+$ and $\lambda-$ in place of λ, we have the following analogue of $(\star)$:

$$\begin{aligned} \text{either} \quad & \langle \alpha^{\mu+} \otimes \alpha^{\mu+}, \alpha^{\lambda+} \rangle \,=\, \langle \alpha^{\mu+} \otimes \alpha^{\mu+}, \alpha^{\lambda-} \rangle \,\neq\, 0, \\ \text{or} \quad & \langle \alpha^{\mu+} \otimes \alpha^{\mu-}, \alpha^{\lambda+} \rangle \,=\, \langle \alpha^{\mu+} \otimes \alpha^{\mu-}, \alpha^{\lambda-} \rangle \,\neq\, 0. \end{aligned}$$

This implies that $\alpha^{\lambda+}$ and $\alpha^{\lambda-}$ are components of $\mathrm{Res}^{S_n}_{A_n}(\chi^\mu \otimes \chi^\mu)$. Therefore, χ^λ is a component of $\chi^\mu \otimes \chi^\mu$, as desired. This completes the proof of the $\lambda = \lambda'$ case, and finishes the proof of the lemma. □

3 Proof of Saxl conjecture for “near”-hooks and “near”-two-rows

We apply Lemma 1.3 to prove the partial cases of Saxl conjecture, by computing the characters $\chi^\lambda[\widehat{\rho}_k]$. Computing these characters requires some results on partitions into certain part sizes, as stated below.

3.1 Partitions into finite arithmetic progressions

Denote by $R = R(a,m,k) = \{a, a+m, a+2m, \ldots, a+km\}$ a finite arithmetic progression, with $a, m \geq 1$, such that $\gcd(a,m) = 1$ as above. Denote by π'_R the coefficients in

$$\sum_{n=0}^{N} \pi'_R(n) t^n \,=\, \prod_{r=0}^{k} \left(1 + t^{a+rm}\right),$$

where $N = (k+1)a + \binom{k+1}{2}m$ is the largest degree with a nonzero coefficient. Note that the sequence $\{\pi'_R(n)\}$ is symmetric:

$$\pi'_R(n) \,=\, \pi'_R(N-n).$$

The following special case of a general result by Odlyzko and Richmond [OR] is the key tool we use throughout the paper.

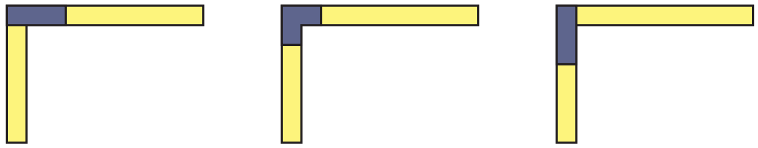

Fig. 1: Three ways to place a rim-hook of length 3 into a hook diagram.

Theorem 3.1 ([OR]) *For every a and m with $\gcd(a, m) = 1$, there exists a constant $L = L(a, m)$, such that for every $R = R(a, m, k)$ as above,*

$$\pi'_R(n+1) \;>\; \pi'_R(n) \;>\; 0\,, \quad \textit{for all } L \le n < \lfloor N/2 \rfloor\,.$$

3.2 Proof [sketch] of Theorem 1.4

Let $n = \binom{k+1}{2}$, so that $\rho_k \vdash n$. Note that $\widehat{\rho}_k = (2k-1, 2k-5, 2k-9, \ldots)$.

Lemma 3.2 *There exists a constant L, such that $g(\rho_k, \rho_k, (n-\ell, 1^\ell)) > 0$, for all $L \le \ell < n/2$.*

When $\ell < k$ we use a particular simple construction with Blasiak's combinatorial interpretation [Bla] for $g(\lambda, \mu, \nu)$ when ν is a hook, and obtain the full result for hooks:

Corollary 3.3 *For k large enough, we have $g(\rho_k, \rho_k, (n-\ell, 1^\ell)) > 0$, for all $1 \le \ell \le n-1$.*

Proof of Lemma 3.2: Let $\lambda = (n-\ell, 1^\ell)$. There are two different cases: odd k and even k, which correspond to the smallest principal hooks $(\ldots, 9, 5, 1)$ and $(\ldots, 11, 7, 3)$, respectively.

Let first k be even, then $\widehat{\rho}_k = \{2k-1, \ldots, 7, 3\}$. We evaluate the character by the well-known Murnagahn-Nakayama rule (see e.g. [Sta2, §7.17]). There are three ways to fit a 3-rim hook in the hook λ, as Figure 3.2 indicates, so

$$\chi^\lambda[\widehat{\rho}_k] = \pi'_R(\ell) - \pi'_R(\ell-1) + \pi'_R(\ell-2),$$

where $R = \{2k-1, \ldots, 7, 3\}$. By Theorem 3.1 we have $\pi'_R(\ell) - \pi'_R(\ell-1) > 0$, so the character is nonzero.

The odd k case is even easier: hook 1 can be placed in $[\lambda]$ in a unique way, after which we get $\chi^\lambda[\widehat{\rho}_k] = \pi'_R(\ell)$, which is the number of partitions into distinct parts $R = \{5, 9, 13, \ldots, 2k-1\}$. This is again nonzero by Theorem 3.1.

In both cases $\chi^\lambda[\widehat{\rho}_k] > 0$ and the Lemma 1.3 now implies the result. $\square$

Lemma 3.4 *There exists a constant L, such that for all $L < \ell \le n/2$ we have $g(\rho_k, \rho_k, (n-\ell, \ell)) > 0$.*

Proof:

Recall the Frobenius formula

$$\chi^{(n-\ell,\ell)} \;=\; \chi^{(n-\ell)\circ(\ell)} \;-\; \chi^{(n-\ell+1)\circ(\ell-1)}.$$

By the Murnaghan–Nakayama rule for skew shapes, we have:

$$\chi^{(n-m)\circ(m)}[\widehat{\rho}_k] \;=\; \pi'_R(m)\,,$$

where $R = \{2k-1, 2k-5, \ldots\}$. Therefore, for $L \le \ell \le n/2$, by Theorem 3.1 we have

$$\chi^{n-\ell,\ell}[\widehat{\eta}_k] \,=\, {\pi'}_R(\ell) \,-\, {\pi'}_R(\ell-1) \,>\, 0.$$

Now Lemma 1.3 implies the result. □

For small values of ℓ (specifically $\ell < k$) there is a combinatorial interpretation in [BO2], which can be applied in this case, and complete the result to

Corollary 3.5 *For k large enough, we have $g(\rho_k, \rho_k, (n-\ell, \ell)) > 0$, for all $0 \le \ell \le n/2$.*

When λ "near-hook" and "near-two-row" shapes the proofs use Gambieli and Frobenius formulas to evaluate the characters $\chi^\lambda[\widehat{\rho}_k]$ as certain differences of partition functions ${\pi'}_R(\ell)$, similar to the proofs above. Again, Theorem 3.1 can be applied to show that these characters are nonzero. However, in these cases there are no other interpretations that can be used to decide the positivity of $g(\rho_k, \rho_k, \lambda)$ for $\lambda < L$. The full proofs and expressions can be found in [PPV]. Together these imply Theorem 1.4.

4 Extensions of Sylvester's theorem

For every two partitions of size n, define

$$a_k(\lambda, \mu) \,=\, \sum_{\alpha \vdash k,\, \beta \vdash n-k} c^\lambda_{\alpha\,\beta}\, c^\mu_{\alpha\,\beta}\,,$$

where $c^\nu_{\pi\theta}$ are the *Littlewood–Richardson coefficients*.

Lemma 4.1 *For any two partitions $\lambda, \mu \vdash n$, the sequence*

$$a_0(\lambda, \mu), \ldots, a_n(\lambda, \mu)$$

is symmetric and unimodal.

Proof: We use the language of symmetric functions. Recall that the inner product $*$ in the ring of symmetric functions is defined on the basis of Schur functions as

$$s_\lambda(x) * s_\mu(x) = \sum_{\nu \vdash n} g(\lambda, \mu, \nu) s_\nu(x).$$

We start with *Littlewood's identity*:

$$(\circ) \qquad s_\lambda * (s_\pi\, s_\theta) \,=\, \sum_{\alpha \vdash k\,,\, \beta \vdash n-k} c^\lambda_{\alpha\,\beta} (s_\alpha * s_\pi)(s_\beta * s_\theta)\,,$$

where $\lambda \vdash n$, $\pi \vdash k$ and $\theta \vdash n-k$ (see [Lit]).

Since s_a corresponds to the trivial representation, we have $s_\nu * s_a = s_\nu$, for all $\nu \vdash a$. For $\pi = (k)$ and $\theta = (n-k)$, we obtain:

$$s_\lambda * (s_k\, s_{n-k}) \,=\, \sum_{\alpha \vdash k,\, \beta \vdash n-k} c^\lambda_{\alpha\,\beta}\, s_\alpha\, s_\beta \,=\, \sum_{\alpha \vdash k,\, \beta \vdash n-k,\, \nu \vdash n} c^\lambda_{\alpha\,\beta}\, c^\nu_{\alpha\,\beta}\, s_\nu\,.$$

Now let $\tau = (n-k, k)$, where $k \le n/2$. By the Jacobi–Trudi formula, we have:

$$s_\tau \,=\, s_k\, s_{n-k} \,-\, s_{k-1}\, s_{n-k+1}\,.$$

We obtain:

$$s_\lambda * s_\tau \,=\, s_\lambda * (s_k\, s_{n-k}) \,-\, s_\lambda * (s_{k-1}\, s_{n-k+1}) \,=\, \sum_\nu a_k(\lambda,\nu)\, s_\nu \,-\, a_{k-1}(\lambda,\nu)\, s_\nu\,.$$

Therefore, the Kronecker coefficient $g(\lambda,\mu,\tau)$ is equal to the coefficient at s_μ in the expansion of $s_\lambda * s_\tau$ in terms of Schur functions:

$$g(\lambda,\mu,\tau) \,=\, a_k(\lambda,\mu) \,-\, a_{k-1}(\lambda,\mu)\,.$$

Since $g(\lambda,\mu,\tau) \ge 0$, the unimodality follows. The symmetry is clear from the definition and the symmetry of the LR coefficients. $\square$

Proof (sketch) of Theorem 1.5: Apply Lemma 4.1 with $\lambda = (m^\ell, 1^r)$ and $\mu = (m+r, m^{\ell-1})$. Denote by $\bar\beta$ the skew partition $(m^\ell)/(m^\ell-\beta)$, where $m^\ell - \beta$ is the partition $(m-\beta_\ell, m-\beta_{\ell-1}, \ldots, m-\beta_1)$. Using combinatorics of Littlewood-Richardson coefficients, we obtain (see [PP-u]) that

$$c^\lambda_{\alpha\beta} c^\mu_{\alpha\beta} = \begin{cases} 1 & \text{if } \bar\beta \subset \alpha \text{ and } \alpha/\bar\beta \sim 1^{\circ r} \\ 0 & \text{otherwise,} \end{cases}$$

where $\sim 1^{\circ r}$ means that the skew shape consists of r disconnected boxes. This implies that

$$a_k(\lambda,\mu) = p_k(\ell, m, r)$$

and the unimodality follows from Lemma 4.1. $\square$

5 Strict unimodality of Gaussian coefficients

The following result lies in the heart of the proof of the theorem. Although stated combinatorially, the only proof we know is algebraic.

Lemma 5.1 (Additivity Lemma) *Suppose inequalities* $(\circ)$ *as in Theorem 1.6 hold for pairs* (ℓ, m_1) *and* (ℓ, m_2)*, with at least one of* ℓ, m_1, m_2 *even and at least one* ≥ 3*. Then* $(\circ)$ *holds for* (ℓ, m_1+m_2).

The proof relies on the following results, conjectured by Klyachko in 2004, and recently proved in [CHM] and [Man]:

Theorem 5.2 (Semigroup property) *Suppose* $\lambda,\mu,\nu,\alpha,\beta,\gamma$ *are partitions of* n*, such that* $g(\lambda,\mu,\nu) > 0$ *and* $g(\alpha,\beta,\gamma) > 0$*. Then* $g(\lambda+\alpha, \mu+\beta, \nu+\gamma) > 0$.

Proof of Lemma 5.1: By Lemma 4.1 and the proof of Theorem 1.5 with $r = 0$ we have that

$$g((m^\ell),(m^\ell),(n-k,k)) = p_k(m,\ell) - p_{k-1}(m,\ell).$$

Let $\lambda = \mu = (m_1^\ell)$, $\alpha = \beta = (m_2^\ell)$, $\nu = (\ell m_1 - r, r)$, $\gamma = (\ell m_2 - s, s)$. By the strict unimodality assumption for (ℓ, m_1) and (ℓ, m_2) , we have

$$g(m_1^\ell, m_1^\ell, \nu) \,>\, 0, \qquad g(m_2^\ell, m_2^\ell, \gamma) \,>\, 0,$$

for all $r, s \geq 0, \neq 1$. Apply Theorem 5.2 to the fixed partitions above. Now, for all $k = r + s$, we then have

$$g\big((m_1+m_2)^\ell, (m_1+m_2)^\ell, \tau_k\big) \,=\, g(m_1^\ell + m_2^\ell, m_1^\ell + m_2^\ell, \nu + \gamma) \,>\, 0,$$

where $n = (m_1 + m_2)\ell$ and $\tau_k = (n - k, k)$ as before. For $k \leq 3$ we can choose $(r, s) = (0, k)$ or $(k, 0)$, as at it is implicit that $\ell, m_1, m_2 \geq 2$ and one of them is ≥ 3. For $3 < k \leq \lfloor n/2 \rfloor - 1$ we have that $k \leq \lfloor \ell m_1/2 \rfloor + \lfloor \ell m_2/2 \rfloor$, so there are values $r, s \geq 2$, $r \leq \lfloor \ell m_1/2 \rfloor$ and $s \leq \lfloor \ell m_2/2 \rfloor$, such that $k = r + s$. Finally, when $k = \lfloor n/2 \rfloor$, by the parity conditions we have that at least one of ℓm_1, ℓm_2 is even, so we can choose $(r, s) = (\ell m_1/2, \lfloor \ell m_2/2 \rfloor)$ or $(\lfloor \ell m_1/2 \rfloor, \ell m_2/2)$. □

Theorem 5.3 *Let $m, \ell \geq 2$. Strict unimodality $(\circ)$ as in Theorem 1.6 holds if and only if $\ell = m = 2$ or $\ell, m \geq 5$ with the exception of (assuming $\ell \leq m$)*

$$(\ell, m) = (5,6), (5,10), (5,14), (6,6), (6,7), (6,9), (6,11), (6,13), (7,10).$$

Proof: A direct calculation gives strict unimodality for each $\ell \in \{8, \ldots, 15\}$, and $8 \leq m < 16$. For each fixed $\ell \in \{8, \ldots, 15\}$ and $m \geq 16$, we have that $m = 8a + b$ for $a \geq 1$ and $8 \leq b < 16$. Applying the additivity lemma successively with $\ell, m_1 = 8k + b, m_2 = 8$ for $k = 0, 1, \ldots, a - 1$, shows that $(\circ)$ holds for all $\ell \in \{8, \ldots, 15\}$ and $m \geq 16$.

Fixing any $m \geq 8$ and applying the additivity lemma in the direction of ℓ the same way by expressing $\ell = 8a' + b'$, shows that $(\circ)$ holds for all $m, \ell \geq 8$.

A direct calculation also gives strict unimodality for all values of $\ell \in \{5, 6, 7\}$ and $5 \leq m \leq 20$ with the exception of the listed cases, where the middle three coefficients of the expansion of $\binom{\ell+m}{m}_q$ are equal. Now we apply the additivity lemma for each value of $\ell = 5, 6, 7$ and $m = 10a + b$ where $10 \leq b \leq 19$ and induct over a with the values $m_1 = 10(a-1) + b$ and $m_2 = 10$. The cases $\ell > m$ follow from the symmetry.

Now, case $\ell = 2$ is straightforward, since $p_{2i}(2, m) = p_{2i+1}(2, m)$ for all $i < n/4$. On the other hand, cases $\ell = 3, 4$ have been studied in [Lin, W] using an explicit symmetric chain decomposition. Since all chain lengths there are ≥ 3, we obtain equalities for the middle coefficients. □

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(vi) Specifically, the references in the answers to this `MathOverflow` question proved very useful: `http://mathoverflow.net/questions/111507/`

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