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Super quasi-symmetric functions via Young diagrams

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Abstract. We consider the multivariate generating series F_P of P-partitions in infinitely many variables $x_1, x_2, \dots$. For some family of ranked posets P, it is natural to consider an analog N_P with two infinite alphabets. When we collapse these two alphabets, we trivially recover F_P. Our main result is the converse, that is, the explicit construction of a map sending back F_P onto N_P. We also give a noncommutative analog of the latter. An application is the construction of a basis of $\mathbf{WQSym}$ with a non-negative multiplication table, which lifts a basis of $QSym$ introduced by K. Luoto.

Résumé. Nous considérons la série génératrice multi-variée F_P des P-partitions en un ensemble infini de variables $x_1, x_2, \dots$. Pour une certaine famille d'ensembles ordonnés P, on peut considérer un analogue N_P en deux ensembles de variables. En égalant les deux alphabets, on retrouve évidemment F_P. Notre résultat principal est la réciproque de cela : nous montrons qu'il existe une opération retournant N_P à partir de F_P. Nous donnons aussi un analogue non-commutatif de cette opération. Nous obtenons ainsi une nouvelle base de $\mathbf{WQSym}$, base qui relève une base de K. Luoto et dont les coefficients de structure sont positifs.

Keywords: P-partitions, quasi-symmetric functions, Hopf algebras, Young diagrams

This article is an extended abstract of [1] in the sense that most results and proofs given here are also in [1]. Nevertheless, the focus here is very different from the focus in [1].

1 Introduction

Consider a ranked poset P on n elements. We consider non-decreasing functions r from P to the set $\mathbb{N}$ of positive integers, with the additional condition that $r(x) < r(y)$ whenever $x <_P y$ and x has odd height in P (we say that such functions satisfy the *order condition*). These correspond to P-partition for some labeling of the elements of P and thus fit in the general context studied by R. Stanley in [11].

Following I. Gessel [5], we consider the multivariate generating series

$$F_P(x_1, x_2, \cdots) = \sum_r \prod_{i \in P} x_{r(i)}, \tag{1}$$

where the sum runs over functions r from P to $\mathbb{N}$ satisfying the order condition. This series in infinitely many variables turns out to be a quasi-symmetric function (in fact, quasi-symmetric functions were origi-

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nally introduced by I. Gessel in [5] to give an algebraic framework to these multivariate generating series of P-partitions).

Notice that, in our framework, elements of even and odd heights in the poset P play different roles. Therefore it is also natural to consider generating series in two alphabets $p_1, p_2, \cdots$ and $q_1, q_2, \ldots$ (i) :

$$N_P \begin{pmatrix} p_1 & p_2 & \cdots \\ q_1 & q_2 & \cdots \end{pmatrix} = \sum_r \left(\prod_{v_0 \in V_0} p_{r(v_0)} \prod_{v_1 \in V_1} q_{r(v_1)} \right), \tag{2}$$

where V_0, resp. V_1, denotes the elements of P of even, resp. odd, heights (we shall use this notation throughout the paper) and where the sum runs over functions r from P to $\mathbb{N}$ satisfying the order condition. Then N_P is a quasi-symmetric in two sets of variables: by analogy with the work of Stembridge [12], we call these functions *super quasi-symmetric functions*. An example of F_P and N_P is given in Section 4. Clearly, setting $p_i = q_i = x_i$ in N_P allows us to recover F_P.

Our main result is a converse of this simple remark. More precisely, we construct explicitly a map from quasi-symmetric functions to series in two infinite alphabets which sends F_P onto N_P. Our construction is naturally stated in the language of Hopf algebras calculus, as explained in Section 2.2. Fix some integer m. Define the virtual alphabet

$$\mathbb{X}_m = \ominus(x_1) \oplus (x_2) \ominus (x_3) \cdots \ominus (x_{2m+1}), \tag{3}$$

where the x_i are the following linear combinations of p_i and q_i

$$\begin{cases} x_{2i+1} = q_{i+1} + \cdots + q_m + p_{i+1} + \cdots + p_{m+1}; \\ x_{2i} = q_i + \cdots + q_m + p_{i+1} + \cdots + p_{m+1}. \end{cases} \tag{4}$$

Our main theorem is the following :

Theorem 1.1 *Let P be a ranked poset and m a non-negative integer. With the notations above,*

$$N_P \begin{pmatrix} p_1 & \ldots & p_m & p_{m+1} & 0 & \ldots \\ q_1 & \ldots & q_m & 0 & 0 & \ldots \end{pmatrix} = (-1)^{|V_0|} F_P(\mathbb{X}_m). \tag{5}$$

Note that the left-hand side for all values of m determines the series N_P, so, as claimed, the theorem allows to reconstruct N_P explicitly from F_P. We prove it at the end of Section 4.

Let us say a word about the proof. Notably, it does not involve any computation, but relies on the structure of the spaces of solutions of two functional equations presented in Section 3. These functional equations come from the analysis of smooth functions on Young diagrams (and formula (4) has a transparent interpretation in terms of Young diagrams).

Another remarkable feature of our result is that it readily extends to a noncommutative framework. In Section 5, we state a noncommutative analog of Theorem 1.1.

We end the article by an application of our noncommutative result. In [6], K. Luoto introduced a basis of $QSym$ with interesting applications to matroid theory. Here, we consider the natural noncommutative lift of his basis in $\mathbf{WQSym}$, the natural noncommutative analog of $QSym$. Thanks to our result, we are able to show that this family is linearly independent and hence, a basis of $\mathbf{WQSym}$. This new basis has nice properties, as the fact that its multiplication table contains only nonnegative integers.

The linear independence of this new basis is easily proved using our theorem involving Hopf algebra calculus, while we have not been able to find an elementary proof of it.

(i) In the case of posets of height 1, this generating series in two variables appears in representation theory of symmetric groups, see [2, Definition 2.2.1].

2 Definitions and notations

2.1 Stable polynomials

A *stable* homogeneous polynomial of degree d is a sequence $R = (R_n(x_1, \dots, x_n))_{n\geq 0}$ of homogeneous polynomials of degree d such that $R_{n+1}(x_1, \dots, x_n, 0) = R_n(x_1, \dots, x_n)$. Intuitively, it is nothing else than a polynomial in infinitely many variables $x_1, x_2, \cdots$. Their set will be denoted by $\mathbb{Q}[X]$, where X is the infinite variable set $X = \{x_1, x_2, \dots\}$ (which should not be confused with the virtual alphabet $\mathbb{X}_m$ defined by (3)).

In our framework, it is more natural to define stable polynomials by a sequence of polynomials in an odd number of variables $(R_{2m+1})_{m\geq 0}$ such that $R_{2m+1}(x_1, \dots, x_{2m-1}, 0, 0) = R_{2m-1}(x_1, \dots, x_{2m-1})$. This is not an issue, as such a sequence can be extended in a unique way to a stable sequence $(R_n)_{n\geq 0}$ by setting $R_{2m}(x_1, \dots, x_{2m}) = R_{2m+1}(x_1, \dots, x_{2m}, 0)$.

In the same spirit, we define an element of $\mathbb{Q}[\boldsymbol{p}, \boldsymbol{q}]$ as a sequence $(h_m)_{m\geq 0}$, where each h_m is a polynomial in the $2m+1$ variables $p_1, \dots, p_m, p_{m+1}, q_1, \dots, q_m$ satisfying the stability property

$$h_{m+1}\begin{pmatrix} p_1 & \dots & p_m & p_{m+1} & 0 \\ q_1 & \dots & q_m & 0 & \end{pmatrix} = h_m\begin{pmatrix} p_1 & \dots & p_m & p_{m+1} \\ q_1 & \dots & q_m & \end{pmatrix}. \tag{6}$$

Clearly, N_P can be seen as an element of $\mathbb{Q}[\boldsymbol{p}, \boldsymbol{q}]$.

2.2 Quasi-symmetric functions and Hopf algebra calculus

Quasi-symmetric functions were introduced by I. Gessel in relation with multivariate generating series of P-partition [5] and may be seen as a generalization of the notion of symmetric functions. A comprehensive survey can be found in [7].

A *composition* of n is a sequence $I = (i_1, i_2, \dots, i_r)$ of positive integers, whose sum is equal to n. We denote by $\mathcal{C}$ the set of all compositions (that is all compositions of all integers n).

The algebra $QSym$ of quasi-symmetric functions is a subalgebra of the algebra $\mathbb{Q}[X]$ of polynomials in the totally ordered commutative alphabet $X = \{x_1, x_2, \dots\}$. A basis of $QSym$ is given by the *monomial quasi-symmetric functions* M indexed by compositions $I = (i_1, \dots, i_r)$, where

$$M_I = \sum_{a_1 < \cdots < a_r} x_{a_1}^{i_1} \cdots x_{a_r}^{i_r}. \tag{7}$$

By convention, $M_{()} = 1$, where $()$ is the empty composition. Note that the dimension of $QSym$ in degree n is the number of compositions of n, that is 2^{n-1}.

An important property for Hopf algebras calculus is the duality with the algebra $\mathbf{Sym}$ of *noncommutative symmetric functions*. For a totally ordered alphabet $A = \{a_i | i \geq 1\}$ of noncommuting variables, and t an indeterminate, one sets

$$\sigma_t(A) = \overrightarrow{\prod_{i\geq 1}}(1 - ta_i)^{-1} = \sum_{n\geq 0} S_n(A)t^n\,. \tag{8}$$

The functions $S_n(A)$ generate a free associative algebra, which is by definition $\mathbf{Sym}(A)$. One denotes by $S^I(A) := S_{i_1} S_{i_2} \cdots S_{i_r}$ its natural basis. Actually, $\mathbf{Sym}(A)$ is the graded dual of $QSym$. This can be deduced from the noncommutative Cauchy formula

$$\prod_{i\geq 1}^{\rightarrow} \sigma_{x_i}(A) = \prod_{i\geq 1}^{\rightarrow}\prod_{j\geq 1}^{\rightarrow} (1 - x_i a_j)^{-1} = \sum_{I\in\mathcal{C}} M_I(X) S^I(A) \tag{9}$$

which allows to identify M_I with the dual basis of S^I [4, 8].

Now, the evaluation of M_I on the virtual alphabet $\mathbb{X}_m$ from the introduction is implicitly defined by :

$$\sum_{I\in\mathcal{C}} M_I(\mathbb{X}_m) S^I(A) = \prod_{1\leq i\leq m+1}^{\rightarrow} \sigma_{x_i}(A)^{(-1)^i}. \tag{10}$$

Then, we extend this definition to $F(\mathbb{X}_m)$, for any F in $QSym$ by linearity.

Example 2.1 *Here are the functions $M_I(\mathbb{X})$ for compositions I of length at most 3:*

$$M_{(k)}(\mathbb{X}) = -x_1^k + x_2^k - x_3^k + \cdots + x_{2m}^k - x_{2m+1}^k; \tag{11}$$

$$M_{(k,\ell)}(\mathbb{X}) = \sum_{i=1}^{2m+1} x_{2i+1}^{k+\ell} + \sum_{1\leq i<j\leq 2m+1} (-1)^{i+j} x_i^k x_j^\ell; \tag{12}$$

$$\begin{aligned} M_{(k,\ell,m)}(\mathbb{X}) = &- \sum_{i=1}^{2m+1} x_{2i+1}^{k+\ell+m} + \sum_{\substack{i,j\\ i<2j+1}} (-1)^i x_i^k x_{2j+1}^{\ell+m} \\ &+ \sum_{\substack{i,j\\ 1\leq 2i+1<j\leq 2m+1}} (-1)^j x_{2i+1}^{k+\ell} x_j^m + \sum_{1\leq h<i<j\leq 2m+1} (-1)^{h+i+j} x_h^k x_i^\ell x_j^m \end{aligned} \tag{13}$$

Remark 2.2 The definition of $M_I(\mathbb{X}_m)$ fits in the more general theory of Hopf algebras calculus, which allows to define $F(\mathbb{X})$, where $\mathbb{X}$ is written as a finite sum/difference of ordered alphabets. In general $F \mapsto F(\mathbb{X})$ is an algebra morphism, but we do not need this property here.

3 Two equivalent equations

3.1 The functional equations

The first functional equation that we consider is the following:

Definition 3.1 *Let $\mathcal{S}_x$ be the space of stable polynomials $f = (f_{2m+1})_{m\geq 0}$ in $\mathbb{Q}[X]$ such that, for each $m \geq 1$ and each $1 \leq i \leq 2m$, one has:*

$$f_{2m+1}(x_1, \ldots, x_{2m+1})\big|_{x_{i+1}=x_i} = f_{2m-1}(x_1, \ldots, x_{i-1}, x_{i+2}, \ldots, x_{2m+1}). \tag{14}$$

Note that the left-hand side means that we substitute x_{i+1} by x_i. Then the equality must be understood as an equality between polynomials in $x_1, \ldots, x_{i-1}, x_{i+2}, \ldots, x_n$. In particular, the left-hand side must be independent of x_i.

The second functional equation we are interested in is the following:

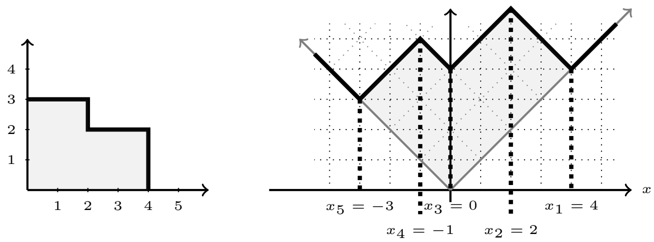

Fig. 1: Young diagram $\lambda = (4,4,2)$, the graph of the associated function ω_λ and a non-centered version of it.

Definition 3.2 *Let $\mathcal{S}_{pq}$ be the subspace of $\mathbb{Q}[\boldsymbol{p},\boldsymbol{q}]$ of elements $h = (h_m)_{m\geq 0}$ such that, for all $m \geq 1$ and positive integer $i \leq m$,*

$$h_m\begin{pmatrix} p_1 & \dots & p_m & p_{m+1} \\ q_1 & \dots & q_m & \end{pmatrix}\Bigg|_{q_i=0} = h_{m-1}\begin{pmatrix} p_1 & \dots & p_{i-1} & p_i+p_{i+1} & \dots & p_m & p_{m+1} \\ q_1 & \dots & q_{i-1} & q_{i+1} & \dots & q_m & \end{pmatrix}; \tag{15}$$

$$h_m\begin{pmatrix} p_1 & \dots & p_m & p_{m+1} \\ q_1 & \dots & q_m & \end{pmatrix}\Bigg|_{p_i=0} = h_{m-1}\begin{pmatrix} p_1 & \dots & p_{i-1} & p_{i+1} & \dots & p_m & p_{m+1} \\ q_1 & \dots & q_{i-1}+q_i & q_{i+1} & \dots & q_m & \end{pmatrix}. \tag{16}$$

By convention, in the second equation for $i = 1$, one should forget the column containing the undefined variables p_0 and q_0.

3.2 Origin: interlacing and multi-rectangular coordinates of Young diagrams

This section explains where our two equations come from. It is written in an informal way and can be safely skipped by a reader only interested in the proof of our theorem.

Consider a Young diagram λ drawn with the Russian convention, (*i.e.*, draw it with the French convention, rotate it counterclockwise by $45°$ and scale it by a factor $\sqrt{2}$). Its border can be interpreted as the graph of a piecewise affine function ω_λ. We denote by $x_1, x_2, \dots, x_{2m+1}$ the abscissas of its local minima and maxima in decreasing order, see Figure 1 (central part).

These numbers $x_1, x_2, \cdots, x_{2m+1}$ are called *(Kerov) interlacing coordinates*, see, *e.g.*, [9, Section 6 with $\theta = 1$]. They are usually labeled with two different alphabets for minima and maxima, but we shall rather use the same alphabet here and distinguish between odd-indexed and even-indexed variables when necessary.

Note that not any decreasing sequence of integers can be obtained in this way, as interlacing coordinates always satisfy the relation $\sum_i (-1)^i x_i = 0$. A way to have independent coordinates is to consider *non-centered* Young diagrams. By definition, the piecewise affine function ω of a non-centered Young diagram is given by $x \mapsto \omega_\lambda(x-c)$ for some integer c and *usual* Young diagram λ – see the right-most part on Figure 1.

A Young diagram can be easily recovered from its Kerov coordinates $x_1, \dots, x_{2m+1}$. Take some decreasing integral sequence $x_1, \dots, x_{2m+1}$. First compute $c = \sum_i (-1)^i x_i$. Then, to obtain its border, first draw the half-line $y = -x + c$ for $x \leqslant x_{2m+1}$, then, without raising the pen, draw line segments of slope

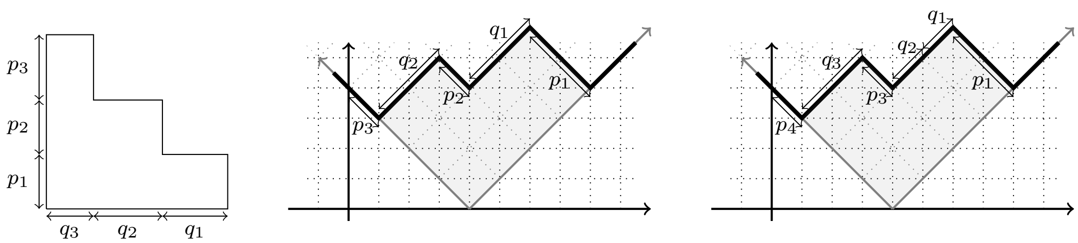

Fig. 2: Multirectangular coordinates of Young diagrams.

alternatively $+1$ and -1 between points of x-coordinates $x_{2m+1}, x_{2m}, \ldots, x_1$ and finally a half-line of slope $+1$ for $x \geq x_1$. This last half-line has equation $y = x - c$ and the resulting broken line is the border of a non-centered Young diagram.

Apply now the same process to a non-increasing sequence $x_1, x_2, \ldots, x_{2m+1}$ such that $x_i = x_{i+1}$. Reaching the x-coordinate $x_i = x_{i+1}$, one has to change twice the sign of the slope, that is, to do nothing. Hence, one obtains the same diagram as for sequence $x_1, \cdots, x_{i-1}, x_{i+2}, \cdots, x_{2m+1}$. Indeed, the same value c is associated with both sequences. Therefore, if one wants to interpret a stable polynomial in $\mathbb{Q}[X]$ as a function of Young diagrams, it is natural to require that it satisfies Equation (14).

Equations (15) and (16) arise in a similar way if we consider multirectangular coordinates of Young diagrams. These coordinates were introduced by R. Stanley(ii) in [10]. Consider two sequences $\boldsymbol{p}$ and $\boldsymbol{q}$ of non-negative integers of the same length m. We associate with these the Young diagram drawn on the left-hand side of Figure 2.

To construct non-centered Young diagrams, we introduce a new coordinate p_{m+1}, which records the distance between the origin and the first corner of the diagram in Russian representation. Unlike other multirectangular coordinates, p_{m+1} can be negative (in fact, p_{m+1} simply corresponds to x_{2m+1}). The central part of Figure 2 shows the multirectangular coordinates of a non-centered Young diagram.

Note that we allow some p_i or some q_i to be zero, so that the same diagram can correspond to several sequences. The right-hand side part of Figure 2 shows a different set of multirectangular (with $p_2 = 0$) associated with the same non-centered Young diagram.

Equations (15) and (16) exactly translate the fact that a polynomial in multirectangular coordinates only depend on the underlying Young diagrams and not on the chosen set of multirectangular coordinates.

To conclude, observe that multirectangular coordinates are related to interlacing coordinates by the following linear changes of variables: for all $i \leq m$,

$$\begin{cases} p_i = x_{2i-1} - x_{2i}; \\ q_i = x_{2i} - x_{2i+1}; \\ p_{m+1} = x_{2m+1}; \end{cases} \qquad \begin{cases} x_{2i+1} = q_{i+1} + \cdots + q_m + p_{i+1} + \cdots + p_{m+1}; \\ x_{2i} = q_i + \cdots + q_m + p_{i+1} + \cdots + p_{m+1}. \end{cases} \tag{17}$$

(ii) In fact, R. Stanley considered coordinates $\boldsymbol{p}'$ and $\boldsymbol{q}'$ related to ours by $p'_i = p_i$ and $q'_i = q_1 + \cdots + q_i$. However, for our purpose, we prefer the more symmetric version presented here.

3.3 Solution of the first equation

Theorem 3.3 *A stable polynomial $f = (f_{2m+1})_{m\geq 0}$ satisfies the functional equation* (14) *if and only if there exists $F \in QSym$ such that $f_{2m+1}(x_1, \cdots, x_{2m+1}) = F(\mathbb{X}_m)$.*

Proof: Note that a polynomial f satisfies Equation (14) if and only if all its homogeneous components do. Therefore it is enough to prove the statement for a homogeneous function f.

Let us first prove that the dimension of the space of homogeneous polynomials in $\mathbb{Q}[X]$ of degree n satisfying (14) is at most equal to 2^{n-1}. We say that a monomial $X^{\mathbf{v}} = x_1^{v_1} x_2^{v_2} \cdots$ in $\mathbb{Q}[X]$ is *packed* if $\mathbf{v}$ can be written as $\mathbf{c}, 0, 0, 0, \dots$ with $\mathbf{c}$ a composition (*i.e.* a vector whose entries are positive integers). Thus, the number of packed monomials of degree n is 2^{n-1}. Let $P = \sum_{\mathbf{v}} c_{\mathbf{v}} X^{\mathbf{v}}$ be a homogeneous polynomial of degree n, which is solution of (14) (here, the sums runs over sequences of non-negative integers of sum n). Associate the integer $\ell(X^{\mathbf{v}}) = \sum_{i\geq 1} i v_i$ to a monomial $X^{\mathbf{v}}$.

Then, we claim that all the coefficients of P are determined by those of packed monomials. To see this, consider a non-packed monomial $X^{\mathbf{w}} = x_1^{w_1} x_2^{w_2} \cdots$ with $w_i = 0$ and $w_{i+1} \neq 0$. We substitute $x_i = x_{i+1} = x$ in P. Looking at the monomial

$$x_1^{w_1} \cdots x_{i-1}^{w_{i-1}} x^{w_{i+1}} x_{i+2}^{w_{i+2}} \cdots$$

that does not appear on the right-hand side of (14), we get a linear relation between $c_{\mathbf{w}}$ and the coefficients $c_{\mathbf{v}}$ of the monomials $X^{\mathbf{v}}$ such that $\ell(X^{\mathbf{v}}) < \ell(X^{\mathbf{w}})$, whence the upper bound on the dimension.

Now, we clearly have:

$$\left. \overrightarrow{\prod_{1\leq j\leq m+1}} \sigma_{x_j}(A)^{(-1)^j} \right|_{x_i = x_{i+1}} = \overrightarrow{\prod_{\substack{1\leq j\leq m+1 \\ j\neq i,\ j\neq i+1}}} \sigma_{x_j}(A)^{(-1)^j}.$$

Looking at Equation (10) and using the fact that $S^I(A)$ is a linear basis of $\mathbf{Sym}(A)$, we get that, for each composition I, the stable polynomial $(M_I(\mathbb{X}_m))_{m\geq 1}$ satisfies Equation (14). Moreover, all $M_I(\mathbb{X})$ are linearly independent since, setting $x_{2i+1} = 0$ in $\mathbb{X}$ transforms $M_I(\mathbb{X})$ into the usual monomial quasi-symmetric functions in even-indexed variables $M_I(x_2, x_4, x_6, \cdots)$. We have found 2^{n-1} linearly independent solutions of Equation (14) in degree n, which finishes the proof. ∎

3.4 Bijection between the spaces of solutions

Proposition 3.4 *The substitution* (17) *defines an algebra isomorphism between $\mathcal{S}_x$ and $\mathcal{S}_{pq}$.*

Proof: Consider an element $f = (f_{2m+1})_{m\geq 0}$ in $\mathcal{S}_x$. Let $m \geq 0$. Replace all variables $x_1, \dots, x_{2m+1}$ in f_{2m+1} according to (17), and set

$$h_m \begin{pmatrix} p_1 & \dots & p_m & p_{m+1} \\ q_1 & \dots & q_m & \end{pmatrix} = f_{2m+1}(x_1, \dots, x_{2m+1}). \tag{18}$$

Clearly, h_m is a polynomial in $p_1, \dots, p_m, p_{m+1}, q_1, \dots, q_m$. Moreover, by definition,

$$h_{m+1} \begin{pmatrix} p_1 & \dots & p_m & p_{m+1} & 0 \\ q_1 & \dots & q_m & 0 & \end{pmatrix} = f_{2m+3}(x_1, \dots, x_{2m+1}, 0, 0).$$

But, as f is a stable polynomial, the right-hand side is equal to $f_{2m+1}(x_1, \dots, x_{2m+1})$. Thus Equation (18) implies that $(h_m)_{m\geq 0}$ is an element of $\mathbb{Q}[\boldsymbol{p}, \boldsymbol{q}]$.

We will now show that h satisfies Equation (15). Let us now consider an integer $m \geq 1$ and variables $p_1, \dots, p_m, p_{m+1}, q_1, \dots, q_m$. Assume additionally that $q_i = 0$ for some i, which implies $x_{2i} = x_{2i+1}$ Thus, as f is an element of $\mathcal{S}_x$, we have

$$f_{2m+1}(x_1, \dots, x_{2m+1}) = f_{2m-1}(x_1, \dots, x_{2i-1}, x_{2i+2}, \dots, x_{2m+1}).$$

Observe that the right-hand side corresponds to the definition of

$$h_{m-1}\begin{pmatrix} p_1 & \dots & p_{i-1} & p_i + p_{i+1} & \dots & p_m & p_{m+1} \\ q_1 & \dots & q_{i-1} & q_{i+1} & \dots & q_m & \end{pmatrix},$$

which ends the proof of Equation (15). Equation (16) can be proved in a similar way.

Finally, from a stable polynomial f in $\mathcal{S}_x$, we have constructed an element $h = (h_m)_{m\geq 0}$ in $\mathcal{S}_{pq}$. This map from $\mathcal{S}_x$ to $\mathcal{S}_{pq}$ is clearly an algebra morphism.

Its inverse can be constructed also by using Equation (17), which proves that it is an isomorphism. ■

Remark 3.5 *In light of the interpretation of $\mathcal{S}_x$ and $\mathcal{S}_{pq}$ in terms of Young diagrams, this isomorphism is not surprising, as both equations are in fact the same, written with different sets of coordinates.*

4 Generating functions of P-partition

Let us come back to our problem of P-partition. If P is a ranked poset, definitions of F_P and N_P were given in the introduction. Let us illustrate these with an example.

Example 4.1 Consider the poset P_{ex} drawn on the left-hand side of Figure 3. Let r be a function from P_{ex} to $\mathbb{N}$. Denote by e and f the images of the leftmost white elements, by g and h the images of the black elements just to their right, then by i the image of the white element to their right and finally by j the image of the rightmost black element. Then, by definition, r satisfies the order condition if and only if $e, f \leq g, h < i \leq j$. Note the alternating large and strict inequalities. Finally, one has

$$F_{P_{\text{ex}}}(x_1, x_2, \dots) = \sum_{e,f\leq g,h<i\leq j} x_e\, x_f\, x_g\, x_h\, x_i\, x_j;$$

$$N_{P_{\text{ex}}}\begin{pmatrix} p_1 & p_2 & \dots \\ q_1 & q_2 & \dots \end{pmatrix} = \sum_{e,f\leq g,h<i\leq j} p_e\, p_f\, p_i\, q_g\, q_h\, q_j.$$

Lemma 4.2 *Let P be a ranked poset. Then the stable polynomial N_P belongs to $\mathcal{S}_{pq}$.*

Proof: Let us check that N_P satisfies Equation (15). We define

$$\begin{cases} p'_j = p_j & \text{if } j < i; \\ p'_i = p_i + p_{i+1}; & \\ p'_j = p_{j+1} & \text{if } j > i; \end{cases} \qquad \begin{cases} q'_j = q_j & \text{if } j < i; \\ q'_j = q_{j+1} & \text{if } j \geq i. \end{cases}$$

Fig. 3: Example of an unlabeled ranked poset P, and a labeled version of it. We represent here the graph of its covering relation that is its *Hasse diagram*. Edges are oriented from left to right. Vertices in V_0 (resp. V_1) are drawn in white (resp. black).

These are the variables in the right-hand side of (15). Consider first the left-hand side:

$$N_P \begin{pmatrix} p_1 & \dots & p_m & p_{m+1} \\ q_1 & \dots & q_m & \end{pmatrix}\Bigg|_{q_i=0} = \sum_r \left(\prod_{v_0 \in V_0} p_{r(v_0)} \prod_{v_1 \in V_1} q_{r(v_1)} \right),$$

where the sum runs over functions $r : P \to \{1, \dots, m+1\}$ satisfying the order condition (and such that $r(v_1) \neq m+1$ for $v_1 \in V_1$). Additionnally, one can restrict the sum to functions r such that $r(v_1) \neq i$ for any $v_1 \in V_1$ (we call these i-avoiding functions).

With any function r, we associate a function $r' = \Phi(r) : V \to \{1, \dots, m\}$ defined as follows:

$$r'(v) = r(v) \text{ if } r(v) \leq i \text{ and } r'(v) = r(v) - 1 \text{ if } r(v) > i.$$

It is straightforward to check that, if r is i-avoiding and satisfies the order condition, then r' also satisfies the order condition. Indeed, the only problem which could occur is $r(v_1) = i$ and $r(v_0) = i+1$ for a covering relation $v_1 <_P v_0$ with $v_1 \in V_1$, which cannot happen since we forbid $r(v_1) = i$.

The preimage of a given function r' is obvious: it is the set of functions r such that

$$\begin{cases} r(v) = r'(v) & \text{if } r'(v) < i; \\ r(v) \in \{i; i+1\} & \text{if } r'(v) = i; \\ r(v) = r'(v) + 1 & \text{if } r'(v) > i. \end{cases}$$

If r' satisfies the order condition, all its i-avoiding pre-images r also satisfy the order condition. Once again, the only obstruction to this would be the case $r'(v_0) = r'(v_1) = i$ for some covering relation $v_0 <_P v_1$ with $v_0 \in V_0$. In this case, preimages r with $r(v_0) = i+1$ and $r(v_1) = i$ would not satisfy the order condition but we forbid $r(v_1) = i$. Now, for any function r', one has:

$$\sum_{r \in \Phi^{-1}(r')} \left(\prod_{v_0 \in V_0} p_{r(v_0)} \prod_{v_1 \in V_1} q_{r(v_1)} \right) = \left(\prod_{v_0 \in V_0} p'_{r'(v_0)} \prod_{v_1 \in V_1} q'_{r'(v_1)} \right),$$

Summing over all functions $r' : V \to \{1, \dots, m-1\}$ with the order condition, we get equality (15).

The proof of (16) is similar. Only the case $i = 1$ is slightly different, but it is straightforward to check that every monomial containing q_1 also contains p_1, which proves (16) in the case $i = 1$. ■

Remark 4.3 In [3, Section 1.5], an equivalent definition of N_P as a function on Young diagrams is given in the case of posets of height 1 (which correspond to bipartite graphs). The fact that N_P can be defined using only the Young diagram and not its multirectangular coordinates explains that it belongs to $\mathcal{S}_{pq}$.

We can now prove Theorem 1.1.

Proof of Theorem 1.1: As N_P belongs to $\mathcal{S}_{pq}$, if we express it in terms of the variables $x_1, x_2, \ldots$, we get an element of $\mathcal{S}_x$. By Theorem 3.3, it is equal to $(F(\mathbb{X}_m))_{m\geq 0}$ for some quasi-symmetric function.

To identify F, we shall send all odd-indexed variables x_{2i+1} to 0. This amounts to sending p_i to $-x_{2i}$ (for $1 \leq i \leq m$), p_{m+1} to 0 and q_i to x_{2i}. Therefore, under this substitution,

$$N_P \begin{pmatrix} p_1 & \cdots & p_m & p_{m+1} & 0 \\ q_1 & \cdots & q_m & 0 & \end{pmatrix} = (-1)^{|V_0|} F_P(x_2, x_4, \cdots, x_{2m}).$$

On the other hand, it is straightforward to check from (10) that, under the substitution $x_{2i+1} = 0$

$$F(\mathbb{X}_m) = F(x_2, x_4, \cdots, x_{2m}).$$

As both equations are true for all values of m, we get $F = (-1)^{|V_0|} F_P$ and the theorem follows. ■

5 Noncommutative generalization

5.1 Noncommutative analog of the main theorem

The natural noncommutative analogue of $QSym$ is the algebra of *word quasi symmetric functions*, denoted by $\mathbf{WQSym}$. We refer to the long version [1] for basic facts about $\mathbf{WQSym}$ and for the definition of the evaluation of a function in $\mathbf{WQSym}$ on the virtual alphabet $\mathbb{A}_m$ defined below.

Take as data a ranked poset $\boldsymbol{P}$, whose set of elements $V = V_0 \sqcup V_1$ is equal to $\{1, \ldots, n\}$. Then we define the non-commutative analog $\boldsymbol{F_P}$ of F_P as follows:

$$\boldsymbol{F_P}(a_1, a_2, \ldots) = \sum_{\substack{r:V\to\mathbb{N} \\ \text{with order condition}}} a_{r(1)} a_{r(2)} \ldots a_{r(n)}.$$

Here, the a_i are noncommuting variables. Then $\boldsymbol{F_P}$ is a word quasi-symmetric function.

In the same way, we can define a noncommutative analog $\boldsymbol{N_P}$ of N_P:

$$\boldsymbol{N_P} \begin{pmatrix} b_1 & b_2 & \ldots \\ d_1 & d_2 & \ldots \end{pmatrix} = \sum_{\substack{r:V\to\mathbb{N} \\ \text{with order condition}}} \gamma_{r(1)} \gamma_{r(2)} \cdots \gamma_{r(n)},$$

where we use the shorthand notation $\gamma_{r(i)} = b_{r(i)}$ for $i \in V_0$ and $\gamma_{r(i)} = d_{r(i)}$ for $i \in V_1$.

Example 5.1 Consider the ranked poset $\boldsymbol{P}_{\text{ex}}$ drawn on the right-hand side of Figure 3. This is a labeled version of the poset P_{ex} on the left-hand side of the same Figure.

Let r be a function from its element set, that is $\{1, \cdots, 6\}$ to $\mathbb{N}$. Define

$$e := r(2),\ f := r(3),\ g := r(1),\ h := r(5),\ i := r(6),\ j := r(4).$$

Then, by definition, r satisfies the order condition if and only if $e, f \leq g, h < i \leq j$, so one has

$$\boldsymbol{N}_{\boldsymbol{P}_{\text{ex}}} \begin{pmatrix} b_1 & b_2 & \ldots \\ d_1 & d_2 & \ldots \end{pmatrix} = \sum_{e,f\leq g, h<i\leq j} d_g\, b_e\, b_f\, d_j\, d_h\, b_i,$$

which is a noncommutative version of the function $N_{P_{\text{ex}}}$ given in Example 5.1.

We shall now present a noncommutative analog of our main theorem. Using two alphabets of noncommuting variables b_i and d_i, define the virtual alphabet $\mathbb{A}_m = \ominus(a_1) \oplus (a_2) \ominus (a_3) \cdots \ominus (a_{2m+1})$, where the a_i are the following linear combinations of b_i and d_i:

$$\begin{cases} a_{2i+1} = d_{i+1} + \cdots + d_m + b_{i+1} + \cdots + b_{m+1}; \\ a_{2i} = d_i + \cdots + d_m + b_{i+1} + \cdots + b_{m+1}. \end{cases} \tag{19}$$

Theorem 5.2 *For any labeled ranked poset $\boldsymbol{P}$ with element set $V = V_0 \sqcup V_1 = \{1, \ldots, n\}$ and any integer m*

$$\boldsymbol{N_P}\begin{pmatrix} b_1 & \ldots & b_m & b_{m+1} \\ d_1 & \ldots & d_m & \end{pmatrix} = (-1)^{|V_0|}\big(\boldsymbol{F_P}(\mathbb{A}_m)\big).$$

Proof: Omitted for brevity. The main lines of the proof are the same than in the commutative framework: defining two equivalent functional equations, solving the first one and showing that $\boldsymbol{N_P}$ is a solution of the second one.

A difficulty is that there is no general theory of difference of alphabets in $\mathbf{WQSym}$ (its antipode is not involutive), so we had to find an ad-hoc definition for $\boldsymbol{F_P}(\mathbb{A}_m)$ so that the theorem holds, see [1, Section 5] for details. ∎

5.2 Noncommutative Luoto basis

Consider the following family of ranked poset: let $\boldsymbol{K}$ be a set-composition (that is an ordered set partition) $(K_0, \ldots, K_{\ell-1})$ of $\{1, \ldots, n\}$, then we define the labeled poset $\boldsymbol{P_K}$ by its set of covering relations:

$$(K_0 \times K_1) \sqcup (K_1 \times K_2) \sqcup \cdots \sqcup (K_{\ell-2} \times K_{\ell-1}).$$

Then $\boldsymbol{P_K}$ is ranked and elements of K_i have height i. Thus

$$V_0 = K_0 \sqcup K_2 \sqcup \cdots \text{ and } V_1 = K_1 \sqcup K_3 \sqcup \cdots .$$

Example 5.3 *If $\boldsymbol{K} = \big(\{2,3\}, \{1,5\}, \{6\}, \{4\}\big)$, then $\boldsymbol{P_K}$ is the graph of the right-hand side of Figure 3.*

Proposition 5.4 *The functions $\boldsymbol{F_{P_K}}$, where $\boldsymbol{K}$ runs over set compositions, form a $\mathbb{Z}$-basis of* $\mathbf{WQSym}$.

Proof: By a dimension argument, proving that $\boldsymbol{F_{P_K}}$ is a basis of $\mathbf{WQSym}$ reduces to proving the linearly independence. But, thanks to Theorem 5.2, one can prove instead that $(\boldsymbol{N_{P_K}})$ is linear independent.

With a noncommutative monomial (a word) in b_i and d_i, we can associate its *evaluation*, which we define as the integer sequence (number of b_1, number of d_1, number of b_2, ...). It is immediate to see that the monomial in $\boldsymbol{N_{P_K}}$ with the lexicographically largest evaluation is obtained as follows: it has letters b_1 in positions given by K_0, letters d_1 in position given by K_1, letters b_2 in positions given by K_2, and so on. It follows that the set-composition $\boldsymbol{K}$ can be recovered from the monomial of lexicographically largest evaluation in $\boldsymbol{N_{P_K}}$, which implies the linear independence of the $\boldsymbol{N_{P_K}}$.

Now, if $\boldsymbol{F} \in \mathbf{WQSym}$ has integer coefficients in a, then $\boldsymbol{F}(\mathbb{A}_m)$ has integer coefficients in b and d and the argument above shows that it is an integral linear combination of the $\boldsymbol{N_{P_K}}$. But, as in the proof of Theorem 1.1, substituting $a_{2i+1} = 0$ sends back $\boldsymbol{F}(\mathbb{A}_m)$ to $\boldsymbol{F}$ and N_{P_K} to $(-1)^{|V_0|}\boldsymbol{F_{P_K}}$, showing that the latter is a $\mathbb{Z}$-basis of $QSym$. ∎

This basis $\boldsymbol{F}_{\boldsymbol{P}_K}$ is a natural noncommutative analog of a basis studied by K. Luoto [6]. We have not been able to prove the linear independence of $\boldsymbol{F}_{\boldsymbol{P}_K}$ without using our theorem to duplicate the alphabet. In particular, Luoto's proof to show that his basis is indeed linearly independent does not seem to extend to the noncommutative framework. The following properties illustrate, in our opinion, the relevance of this new basis of $\mathbf{WQSym}$.

Proposition 5.5 *For any labeled rank poset $\boldsymbol{P}$, the function $\boldsymbol{F}_{\boldsymbol{P}}$ expands as a linear combination with* nonnegative integer coefficients *of $\boldsymbol{F}_{\boldsymbol{P}_K}$.*

Sketch of proof. Let r be a function from $\boldsymbol{P}$ to $\mathbb{N}$ satisfying the order condition. For any pair (x, y) in $V_0 \times V_1$ such that x and y are incomparable in P one has either $r(x) \leqslant r(y)$ or $r(y) < r(x)$. Let us split the sum in (1) depending on the set of such pairs (x, y) such that $r(x) \leqslant r(y)$. Then the nonempty sums correspond to some $F_{\boldsymbol{Q}}$, where $\boldsymbol{Q}$ is the ranked poset in which any element of odd height is comparable with any element of even height. Such posets are exactly the $\boldsymbol{P}_{\boldsymbol{K}}$, which conclude the proof. ■

Corollary 5.6 *The multiplication table of the basis $\boldsymbol{F}_{\boldsymbol{P}_K}$ has nonnegative integer entries.*

Proof: The product $\boldsymbol{F}_{\boldsymbol{P}_K} \cdot \boldsymbol{F}_{\boldsymbol{P}_{K'}}$ is simply $\boldsymbol{F}_{\boldsymbol{P}_K \sqcup \boldsymbol{P}_{K'}}$. Proposition 5.5 ends the proof. ■

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