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The purity of set-systems related to Grassmann necklaces

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1 Introduction

Studying the problem of quasi-commuting quantum minors, Leclerc and Zelevinsky [3] introduced the notion of weakly separated sets in $[n] := \{1, \dots, n\}$. They raised conjectures on the purity, with respect to this weakly separation relation, of certain set-systems. In particular, a conjecture on the purity of the whole Boolean cube $2^{[n]}$; equivalently: on the max-clique purity of the graph on $2^{[n]}$ whose edges are generated by this relation. (Here a finite graph G is called *pure* if all (inclusion-wise) maximal cliques in it have the same cardinality.) Answering those conjectures, [1] proved the purity of $2^{[n]}$ and some other set-systems, including the discrete Grassmannian $\binom{[n]}{r}$. In [5] the purity was proved for a certain weakly separated set-system inside the so-called *positroid* defined by a *Grassmann necklace* $\mathcal{N}$. We denote this set-system as $\mathcal{I}nt(\mathcal{N})$. It is a special collection in the discrete Grassmannian, and the whole discrete Grassmannian is obtained by considering the 'largest' necklace.

In this paper we give an alternative (and shorter) proof of the purity of $\mathcal{I}nt(\mathcal{N})$ and present a stronger result. More precisely, we introduce a set-system $\mathcal{O}ut(\mathcal{N})$ complementary to $\mathcal{I}nt(\mathcal{N})$, in a sense, and establish its purity. This is a consequence of the result (Theorem 3) that the set-systems $\mathcal{I}nt(\mathcal{N})$ and $\mathcal{O}ut(\mathcal{N})$ are weakly separated from each other, which means that any element of the former is weakly separated from any element of the latter. To prove this, we use a technique of plabic tilings from [5]. As one more consequence of Theorem 3, we also demonstrate the purity of set-systems related to pairs of weakly separated necklaces (Proposition 4 and Corollaries 1 and 2). Finally, we introduce a notion of *generalized necklaces* and claim the purity of the corresponding interior and exterior set-systems related to them; this extends the statement of Theorem 3. (This claim and a further generalization were recently shown in [2]. Note that results in [2] and [6] give rise to a cluster algebra structure on the interior and exterior of a generalized necklace. However, the natural questions on a positroid concerning the purity and the existence of a cluster algebra structure are still open.)

2 Preliminaries

For a natural number n, we denote by $\binom{[n]}{r}$ the set of r-element subsets in $[n] := \{1, \dots, n\}$ (the discrete Grassmannian). Subsets of $\binom{[n]}{r}$ are called *(set-)systems* and we use calligraphic letters for them.

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It will be convenient for us to think of $[n]$ as being $\mathbb{Z}$ modulo n. We consider the cyclically shifted orders $<_i$ on $[n]$, $i = 1, \dots, n$, defined by $i <_i (i+1) <_i \dots <_i n <_i 1 <_i \dots <_i (i-1)$. A sequence $i_1, \dots, i_k$ is called *cyclically ordered* if $i_1 <_i i_2 <_i \dots <_i i_k$ for some i.

We denote by $\ll_i$ the following binary relation on $\binom{[n]}{r}$. For two sets X and Y of cardinality r, we write $X \ll_i Y$ if for any $x \in X - Y$ and $y \in Y - X$, one holds $x <_i y$.

Definition. Two subsets $X, Y \subset [n]$ of the same cardinality[(i)] are called weakly *separated* (denoted as $X \| Y$) if $X \ll_j Y$ holds for some $j \in [n]$.

In general, the relation $\ll_i$ is not transitive. Nevertheless, the following assertion is valid.

Lemma 1. [3, Lemma 3.6] *Let $X \ll_i Y \ll_i Z$, where X, Y, Z have the same cardinality, and X and Z are weakly separated. Then $X \ll_i Z$.*

The notion of weak separation has proved its usefulness in the study of Plücker coordinates on Grassmannians. Since we never deal with the strong separation in this paper, we will use the term 'separation' instead of 'weak separation' for short.

It is easy to see that $X \ll_i Y$ for some $i \in [n]$ if and only if $Y \ll_j X$ for some j. Therefore, the separation relation $\|$ on $\binom{[n]}{r}$ is symmetric and reflexive. We say that two set-systems $\mathcal{X}$ and $\mathcal{Y}$ from $\binom{[n]}{r}$ are *separated from each other* (and write $\mathcal{X} \| \mathcal{Y}$) if $X \| Y$ for any $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$. A system $\mathcal{X}$ is called *separated* if $\mathcal{X} \| \mathcal{X}$. A system $\mathcal{D} \subset \binom{[n]}{r}$ is called *pure* if all maximal separated subsystems in $\mathcal{D}$ are of the same size; this size is called the *rank* of $\mathcal{D}$ and denoted by $rk(\mathcal{D})$.

We will essentially use the following important fact.

Theorem 1. *The Grassmannian $\binom{[n]}{r}$ is a pure system of rank $r(n-r)+1$.*

This assertion was conjectured in [3, 6]. Its validity follows from the purity of the Boolean cube $2^{[n]}$ shown in [1], using the argument of Leclerc and Zelevinsky in [3] that the purity of $2^{[n]}$ would imply the purity of $\binom{[n]}{r}$.

In [5, theorem 4.7] the purity was shown for some systems of a more general character in $\binom{[n]}{r}$; they are produced from the so-called Grassmann necklaces.

In the next section we recall necessary definitions. Throughout the paper, symbol $\subset$ stands for non-strict inclusion (admitting equality).

3 Necklaces and related set-systems

Definition. [4] A (Grassmann) *necklace* in $\binom{[n]}{r}$ is a family $\mathcal{N} = (N_1, \dots, N_n)$ of sets from $\binom{[n]}{r}$ such that N_{i+1} contains $N_i - \{i\}$ for each i (hereinafter the indices are taken modulo n).

In particular, if $i \notin N_i$ then $N_{i+1} = N_i$ and $i \notin N_j$ for all j. We will assume for simplicity (see Remark 2 below) that this is not the case, and that any $i \in [n]$ satisfies $i \in N_i$.

The necklaces are closely related to permutations on $[n]$. The set N_{i+1} is obtained from N_i by deleting i and adding some element $\pi(i)$ (which may coincide with i). Thus, the necklace $\mathcal{N}$ defines the corresponding map $\pi : [n] \to [n]$. This π is bijective. (Indeed, suppose that some element j is not used. Then

(i) The definition of weak separability can be given for arbitrary subsets in $[n]$; see [3, 6, 1, 5]. But in this paper we deal only with the above-mentioned case.

it occurs either in no N_i (which contradicts $j \in N_j$) or in all N_i (yielding $j = \pi(j)$).) Therefore, π is indeed a permutation on $[n]$.

Conversely, let π be a permutation on $[n]$. We can associate to it the following family of sets $\mathcal{N} = \mathcal{N}_\pi = (N_1, \dots, N_n)$ by the rule

$$N_i = \{j \in [n], j \leq_i \pi^{-1}(j)\}.$$

It is easy to see that $\mathcal{N}$ is a necklace in $\binom{[n]}{r}$, where the number r is defined to be the 'average clockwise rotation' by π of the elements of $[n]$.

Example 1. Let a permutation π send every i to $i + r$ ('rotation' by r positions). Then $N_i = \{i, i+1, \dots, i+r-1\} = [i, i+r)$ is a cyclic interval of length r beginning at i. The corresponding necklace is called the *largest* one; this terminology will be justified later.

An important property of necklaces is given in the following

Lemma 2. ([5, Lemma 4.4]) For all i and j, it holds that $N_i - N_j \subset [i, j) = \{i, i+1, \dots, j-1\}$.

Symmetrically, $N_j - N_i \subset [j, i)$. As a corollary, we obtain that $N_i \lll_i N_j$ for any i and j. In particular, all sets in a necklace $\mathcal{N}$ are separated from each other.

For a necklace $\mathcal{N}$, let us call the *interior* of $\mathcal{N}$ the following set-system

$$\mathcal{I}nt(\mathcal{N}) = \{X \in \binom{[n]}{r},\ N_i \lll_i X \text{ for every } i\}.$$

Obviously, $\mathcal{N} \subset \mathcal{I}nt(\mathcal{N})$ and $\mathcal{N} \| \mathcal{I}nt(\mathcal{N})$.

A supplement to Example 1. Let $\mathcal{N}$ be the largest necklace consisting of cyclic intervals (see Example 1). Since $[i, i+r) \lll_i X$ for any r-element set X, we obtain that the interior of $\mathcal{N}$ is the discrete Grassmannian, $\mathcal{I}nt(\mathcal{N}) = \binom{[n]}{r}$. This justifies the term 'largest': this necklace has the largest interior.

Theorem 1 asserts that the interior of the largest necklace is a pure system. This is generalized as follows.

Theorem 2. *For every Grassmann necklace $\mathcal{N}$, the set-system $\mathcal{I}nt(\mathcal{N})$ is pure.*

Remark 1. This result is obtained in [5]. Strictly speaking, [5] considers another system, a *positroid* $Pos(\mathcal{N})$, and the purity is proved only with respect to the weakly separated systems $\mathcal{C} \subset Pos(\mathcal{N})$ containing $\mathcal{N}$. It is shown that such systems are exactly weakly separated systems in $\mathcal{I}nt(\mathcal{N})$. Therefore, Theorem 2 is equivalent to Theorem 4.7 in [5]. A question on the purity of the positroid $Pos(\mathcal{N})$ (without the additional condition $\mathcal{N} \subset \mathcal{C}$) is open.

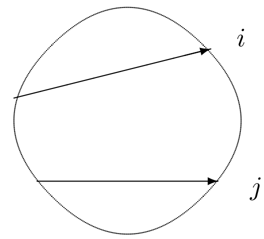
Remark 2. Suppose that $i \notin N_i$ for some i. Then $i \notin X$ for every $X \in \mathcal{I}nt(\mathcal{N})$. Indeed, supposing $i \in X$, we obtain a contradiction to $N_i \lll_i X$. Deleting such dummy i's, we may assume that $i \in N_i$ for any $i \in [n]$.

We give an alternative proof of Theorem 2 in the next section.

4 Alignments and extensions of necklaces

To prove Theorem 2, it is convenient to consider another description for the system $\mathcal{I}nt(\mathcal{N})$, given in terms of alignments of the permutation $\pi = \pi(\mathcal{N})$. We use the notion of an alignment introduced by

Postnikov [4]. Let π be a permutation of $[n]$. A pair (i,j) is said to be an *alignment* for π (and denoted by $i \Rightarrow_\pi j$) if the quadruple $\pi^{-1}(i), i, j, \pi^{-1}(j)$ occurs in this cyclical order (the case $j = \pi(j)$ is admitted, whereas $i = \pi(i)$ is not). Roughly speaking, the 'arrows' entering i and j, go parallel (do not cross) and in the same direction. See the picture.



Notation $i \Rightarrow_\pi j$ for the alignment is justified as follows. Let $Y \in \mathcal{I}nt(\mathcal{N})$. If $i \in Y$ satisfies the relation $i \Rightarrow_\pi j$, then $j \in Y$. We call this property of Y the π-*chamberness*. Indeed, without loss of generality, one may assume that $i = \pi(1)$. Then i does not belong to $\mathcal{N}_1$, whereas $j \in \mathcal{N}_1$. Now suppose that $i \in Y$ and $j \notin Y$. Then $i \in Y - \mathcal{N}_1$ and $j \in \mathcal{N}_1 - Y$. Due to the relation $N_1 \ll_1 Y$ we obtain $j <_1 i$, which contradicts $i < j$.

The converse property takes place as well.

Proposition 1. *For a set $Y \subset [n]$ of size r, the following statements are equivalent:*
1) $Y \in \mathcal{I}nt(\mathcal{N}(\pi))$,
2) Y is π-chamber set.

The implication $1) \Rightarrow 2)$ has been proved. To see the implication $2) \Rightarrow 1)$, we show that 2) implies $N_i \ll_i Y$ for any i. Without loss of generality we may assume that $i = 1$; so we have to prove that $N_1 \ll_1 Y$. Suppose this is not so, i.e., there exist $j \in \mathcal{N}_1 - Y$ and $i \in Y - \mathcal{N}_1$ such that $i < j$. Then $j \in \mathcal{N}_1$ means that $\pi^{-1}(j) > j$; and $i \notin \mathcal{N}_1$ means that $\pi^{-1}(i) < i$. This together with the inequality $i < j$ means that the pair (i,j) is an alignment. But then the chamberness of Y implies that $j \in Y$ (since $i \in Y$). A contradiction. $\square$

In what follows we write $\mathcal{I}nt(\pi)$ for $\mathcal{I}nt(\mathcal{N})$.

We prove Theorem 2 by induction on the number of alignments of the permutation π corresponding to a necklace $\mathcal{N}$.

1. *A base of induction*: there are no alignments. In this case the permutation π sends each i to $i + r$. Indeed, let π send i to $i+k(i)$, $0 < k(i) \le n$. Choose i with $k(i)$ minimum. Then in case $\pi(i-1) > \pi(i)$, we have $i - 1 \Rightarrow_\pi i$. This is impossible; so $\pi(i-1) < \pi(i)$. Hence, $k(i-1) \le k(i)$. The minimality of $k(i)$ gives $k(i-1) = k(i)$. Repeating this procedure, we obtain that $k(\cdot)$ is constant (and equal to r).

Hence, the necklace with no alignments is the largest necklace and the proposition follows from Theorem 1.

2. *A step of induction.* Suppose that the permutation π has an alignment. Then there exists a 'simple' alignment $i \Rightarrow_\pi j$, in the sense that $\pi^{-1}(i)$ and $\pi^{-1}(j)$ are (cyclically) consecutive numbers. Without loss of generality, we may assume that the first number is 1 and the second one is n, so that $i = \pi(1)$ and $j = \pi(n)$.

Now we consider the permutation π' which coincides with π everywhere except for the elements 1 and n. More precisely, $\pi'(1) = j$ and $\pi'(n) = i$. If for the permutation π, the arrows going from 1 and from n do not cross (and therefore give a simple alignment $i \Rightarrow j$), then similar arrows for π' do cross (and the alignment $i \Rightarrow j$ vanishes). All other alignments preserve. Thus, the set of alignments for π' is obtained from that of π by deleting one alignment $i \Rightarrow_\pi j$. By induction the set-system $\mathcal{I}nt(\pi')$ is pure (and contains $\mathcal{I}nt(\pi)$, as follows from Proposition 1). Now Theorem 2 follows from the following

Proposition 2. *Let π and π' be as above, let X be a set in $\mathcal{I}nt(\pi')$ which is separated from N_1 and such that $X \neq N_1'$. Then $X \in \mathcal{I}nt(\pi)$.*

Indeed, let $\mathcal{C}$ be a maximal separated subsystem in $\mathcal{I}nt(\pi)$. Then the system $\mathcal{C} \cup \{N_1'\}$ is contained in $\mathcal{I}nt(\pi')$ and is weakly separated. We assert that it is a maximal separated system in $\mathcal{I}nt(\pi')$. For if this is not so, we can add some X to this system. Then, due to Proposition 2, X belongs to $\mathcal{I}nt(\pi)$, which contradicts the maximality of $\mathcal{C}$ in $\mathcal{I}nt(\pi)$. Thus, $\mathcal{I}nt(\pi)$ is pure and the rank of $\mathcal{I}nt(\pi)$ is less by 1 than the rank of $\mathcal{I}nt(\pi')$. By the induction, we conclude that the rank of $\mathcal{I}nt(\pi)$ is equals to $k(n-k)+1$ minus the number of alignments for π. This gives Theorem 2.

Proof of Proposition 2. Let X be as in Proposition 2. We assert that X belongs to $\mathcal{I}nt(\pi)$. Suppose, for a contradiction, that X is not a π-chamber set. Since X is a π'-chamber set and π has exactly one additional alignment $i \Rightarrow j$ compared with π', we have $i \in X$ and $j \notin X$. The set N_1' also contains i but not j. (Recall that N_1' differs from N_1 by swapping the roles of i and j: N_1 contains j and does not contain i.) Our aim is to prove that X coincides with N_1'.

We have $N_n \ll_n X$. This means that any element of $X - N_n$ is greater by $>_n$ than any element of $N_n - X$. Since i belongs to X and does not belong to N_n (as i appears only in N_2), we conclude that any element of $N_n - X$ is $<_n i$. Hence, besides n, any element of $N_n - X$ is $< i$. In other words, within the interval $I = (i, n)$ we have the inclusion $N_n \subset X$. In this I the sets N_n and N_1' coincide; so within I we have the inclusion $N_1' \subset X$. Since $n \notin N_1'$ (as n is replaced by i under changing $N_n' = N_n$ to N_1'), the set N_1' is contained in X within $(i, n]$.

Similarly, using the relation $N_2 \ll_2 X$, we obtain that $X \subset N_1'$ on the interval $[1, j)$. In particular, within (i, j) (and even within $[i, j]$) the sets X and N_1' coincide.

If the inclusion $X \cap [1, i) \subset N_1' \cap [1, i)$ is strict, then the inclusion $N_1' \cap (j, n] \subset X \cap (j, n]$ is also strict. Hence, there are an element $i' < i$ belonging to $N_1' - X$ and an element $j' > j$ belonging to $X - N_1'$. Since N_1' and N_1 coincide outside $\{i, j\}$, the element i' belongs to $N_1 - X$, and j' belongs to $X - N_1$. Recall also that $i \in X - N_1$ and $j \in N_1 - X$. These relations together with the inequalities $i' < i < j < j'$ imply that the sets N_1 and X are not weakly separated. This contradiction completes the proof of Proposition 2. $\square$

5 Exterior of a necklace

In this section we show the purity of the so-called exterior of a necklace. Denote by $\mathcal{S}(\mathcal{N})$ the system of sets weakly separated from the necklace $\mathcal{N}$:

$$\mathcal{S}(\mathcal{N}) := \{X \in \binom{[n]}{r},\ X \| N_i,\ \forall\, i \in [n]\}.$$

We know that $\mathcal{I}nt(\mathcal{N})$ is a subset of $\mathcal{S}(\mathcal{N})$. The *exterior* of a necklace $\mathcal{N}$, denoted as $\mathcal{O}ut(\mathcal{N})$, is the complement to $\mathcal{I}nt(\mathcal{N})$ in $\mathcal{S}(\mathcal{N})$, that is $\mathcal{O}ut(\mathcal{N}) = \mathcal{S}(\mathcal{N}) \setminus \mathcal{I}nt(\mathcal{N})$.

The purity of the exterior of a necklace is a consequence of the following main result of the paper.

Theorem 3. *Let $\mathcal{N}$ be a Grassmann necklace in $\binom{[n]}{r}$, $X \in \mathcal{O}ut(\mathcal{N})$, and $Y \in \mathcal{I}nt(\mathcal{N})$. Then X and Y are separated, $X \| Y$.*

We prove this theorem in the next section. Now we establish an important corollary from it.

Proposition 3. *Let $\mathcal{N}$ be a neklace. Then the exterior $\mathcal{O}ut(\mathcal{N})$ of $\mathcal{N}$ is a pure system; its rank is equal to the number of alignments of the corresponding permutation $\pi(\mathcal{N})$.*

Proof of Proposition 3. Let $\mathcal{C}$ be a maximal separated system in $\mathcal{O}ut(\mathcal{N})$ and let $\mathcal{D}$ be a maximal separated system in $\mathcal{I}nt(\mathcal{N})$. Obviously, $\mathcal{N} \subset \mathcal{D}$.

We claim that the union $\mathcal{C} \cup \mathcal{D}$ is a maximal separated system in the Grassmannian $\binom{[n]}{r}$. Indeed, due to Theorem 3, the union is separated. To see the maximality, suppose that the union can be extended by an additional set Z of cardinality r. Since Z is separated from $\mathcal{N}$, it belongs to $\mathcal{S}(\mathcal{N})$. Hence Z belongs either to $\mathcal{I}nt(\mathcal{N})$ or to $\mathcal{O}ut(\mathcal{N})$, which contradicts the maximality of $\mathcal{D}$ or $\mathcal{C}$.

By Theorem 1, the size of $\mathcal{C} \cup \mathcal{D}$ does not depend of a choice of $\mathcal{C}$ and $\mathcal{D}$ (implying the same property for each of $\mathcal{C}$ and $\mathcal{D}$). This proves the purity of $\mathcal{I}nt(\mathcal{N})$ and $\mathcal{O}ut(\mathcal{N})$. The assertion on the rank of $\mathcal{O}ut(\mathcal{N})$ follows from the fact that the rank of $\mathcal{I}nt(\mathcal{N})$ is equal to $k(n-k)+1$ minus the number of alignments for π. $\square$

Remark 3. It may seem that the above reasonings lead to a new proof of the purity of $\mathcal{I}nt$. However, they rely on Theorem 3, and the proof of the latter given in Section 6 uses arguments from [5].

Proposition 3 can be generalized for the case of two (or more) necklaces. To formulate such generalizations, we use a shorter notation. Namely, considering two necklaces $\mathcal{N}_1, \mathcal{N}_2$, we will write $\mathcal{I}_k$ for $\mathcal{I}nt(\mathcal{N}_k)$, and write $\mathcal{O}_k$ for $\mathcal{O}ut(\mathcal{N}_k)$, $k = 1, 2$.

Proposition 4. *Suppose that necklaces $\mathcal{N}_1$ and $\mathcal{N}_2$ are separated from each other. Then the following four systems are pure: $\mathcal{I}_1 \cap \mathcal{I}_2$, $\mathcal{I}_1 \cap \mathcal{O}_2$, $\mathcal{O}_1 \cap \mathcal{I}_2$, and $\mathcal{O}_1 \cap \mathcal{O}_2$. The sum of their ranks is equal to $r(n-r)+1$.*

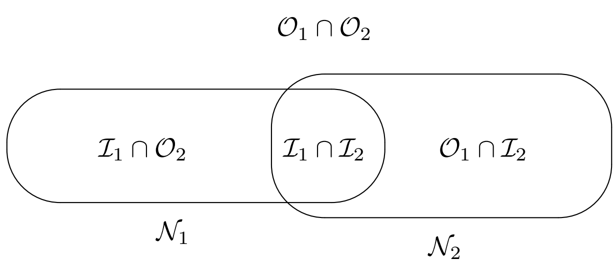


Figure 1. Necklaces from Proposition 4.

Proof. Let $\mathcal{A}$ be a maximal separated system in $\mathcal{I}_1 \cap \mathcal{I}_2$. Let $X \in \mathcal{N}_1 \cap \mathcal{I}_2$. Then $X \in \mathcal{I}_1 \cap \mathcal{I}_2$, X is separated from $\mathcal{I}_1$ and, moreover, X is separated from $\mathcal{A}$. By the maximality of $\mathcal{A}$, X belongs to $\mathcal{A}$. Therefore,

a) $\mathcal{N}_1 \cap \mathcal{I}_2$ (as well as $\mathcal{I}_1 \cap \mathcal{N}_2$) is contained in $\mathcal{A}$.

Let $\mathcal{B}_1$ be a maximal separated system in $\mathcal{I}_1 \cap \mathcal{O}_2$. By similar reasonings,

b1) $\mathcal{N}_1 \cap \mathcal{O}_2$ is contained in $\mathcal{B}_1$.

Similarly, if B_2 is a maximal separated system in $O_1 \cap I_2$, then

b2) $\mathcal{N}_2 \cap \mathcal{O}_1$ is contained in $\mathcal{B}_2$.

Finally, let $\mathcal{C}$ be a maximal separated system in $\mathcal{O}_1 \cap \mathcal{O}_2$. We assert that the union $\mathcal{A} \cup \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{C}$ is a maximal separated system in $\binom{[n]}{r}$. Indeed:

First, by Theorem 3, this union is a separated system.

Second, since $\mathcal{N}_1$ is separated from $\mathcal{N}_2$, we have $\mathcal{N}_1 = (\mathcal{N}_1 \cap \mathcal{I}_2) \cup (\mathcal{N}_1 \cap \mathcal{O}_2)$. Hence, due to a) and b1), $\mathcal{N}_1$ is contained in $\mathcal{A} \cup \mathcal{B}_1$. Similarly, $\mathcal{N}_2$ is contained in $\mathcal{A} \cup \mathcal{B}_2$. Therefore, $\mathcal{N}_1$ and $\mathcal{N}_2$ are contained in $\mathcal{A} \cup \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{C}$.

Third, let a set X be separated from $\mathcal{A} \cup \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{C}$. Since $\mathcal{N}_1$ and $\mathcal{N}_2$ are contained in the union $\mathcal{A} \cup \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{C}$, the set X is separated from $\mathcal{N}_1$ and from $\mathcal{N}_2$. Hence, X belongs to one of the systems $\mathcal{I}_1 \cap \mathcal{I}_2$, $\mathcal{I}_1 \cap \mathcal{O}_2$, $\mathcal{O}_1 \cap \mathcal{I}_2$, $\mathcal{O}_1 \cap \mathcal{O}_2$. If X belongs to $\mathcal{I}_1 \cap \mathcal{I}_2$, then it is separated from $\mathcal{A}$. By the maximality of $\mathcal{A}$ in $\mathcal{I}_1 \cap \mathcal{I}_2$, X belongs to $\mathcal{A}$. In a similar way, we obtain $X \in \mathcal{A}$ in the other cases. Thus, the maximality of the union is proven.

Now by Theorem 1, the size of the union $\mathcal{A} \cup \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{C}$ does not depend on the choice of $\mathcal{A}$ in $\mathcal{I}_1 \cap \mathcal{I}_2$. This proves the purity of $\mathcal{I}_1 \cap \mathcal{I}_2$. Similarly, we obtain the purity for the other cases. $\square$

There are two interesting special cases of necklaces $\mathcal{N}_1, \mathcal{N}_2$ in Proposition 4. The first case is when one necklace is 'less' than the other.

Definition. We say that $\mathcal{N}_1$ is *less* than $\mathcal{N}_2$ if $\mathcal{I}nt(\mathcal{N}_1) \subset \mathcal{I}nt(\mathcal{N}_2)$.

In this case, obviously, $\mathcal{N}_1 \| \mathcal{N}_2$ and $\mathcal{O}_2 \subset \mathcal{O}_1$. We have the following criterion:

Lemma 3. A necklace $\mathcal{N}_1$ *is less than a necklace* $\mathcal{N}_2$ *if and only if* $\mathcal{N}_1 \subset \mathcal{I}nt(\mathcal{N}_2)$.

Proof. The part 'only if' is trivial because $\mathcal{N}_1 \subset \mathcal{I}nt(\mathcal{N}_1)$. Let us prove the converse: if $\mathcal{N}_1 \subset \mathcal{I}nt(\mathcal{N}_2)$, then $\mathcal{I}nt(\mathcal{N}_1) \subset \mathcal{I}nt(\mathcal{N}_2)$.

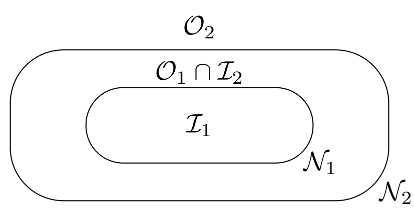
Let $X \in \mathcal{I}nt(\mathcal{N}_1)$. We have to show that $(\mathcal{N}_2)_i \ll_i X$ for any i, where $(\mathcal{N}_2)_i$ denotes i-th set of the necklace $\mathcal{N}_2$. Since $(\mathcal{N}_2)_i \| \mathcal{N}_1$ holds for any i, the set $(\mathcal{N}_2)_i$ belongs either to $\mathcal{I}nt(\mathcal{N}_1)$ or to $\mathcal{O}ut(\mathcal{N}_1)$.

In the first case, we have $(\mathcal{N}_1)_i \ll_i (\mathcal{N}_2)_i$ and $(\mathcal{N}_2)_i \ll_i (\mathcal{N}_1)_i$, implying $(\mathcal{N}_1)_i = (\mathcal{N}_2)_i$. Hence $(\mathcal{N}_2)_i = (\mathcal{N}_1)_i \ll_i X$.

In the second case, $(\mathcal{N}_2)_i$ belongs to $\mathcal{O}ut(\mathcal{N}_1)$. Then, by Theorem 3, $(\mathcal{N}_2)_i$ is separated from X. Moreover, it holds that $(\mathcal{N}_2)_i \ll_i (\mathcal{N}_1)_i$ (because $(\mathcal{N}_1)_i$ belongs to $\mathcal{I}nt(\mathcal{N}_2)$) and $(\mathcal{N}_1)_i \ll_i X$ (because X belongs to $Int(\mathcal{N}_1)$). Thus, due to Lemma 1, we obtain $(\mathcal{N}_2)_i \ll_i X$. $\square$

The first special case is exposed in the following

Corollary 1. *Let* $\mathcal{N}_1$ *and* $\mathcal{N}_2$ *be two necklaces. Suppose that* $\mathcal{N}_1$ *is less than* $\mathcal{N}_2$, $\mathcal{N}_1 \subset \mathcal{I}_2$. *Then the system* $\mathcal{I}_2 \cap \mathcal{O}_1$ *(the 'ring' between* $\mathcal{N}_2$ *and* $\mathcal{N}_1$*) is pure and its rank is equal to* $r(n-r) + 1 - rk(\mathcal{I}_1) - rk(\mathcal{O}_2)$.



Proof. By Lemma 3, $\mathcal{I}_1 \subset \mathcal{I}_2$ and $\mathcal{O}_1 \supset \mathcal{O}_2$. Therefore, $I_1 \cap I_2 = I_1$, $I_1 \cap O_2 = \emptyset$, and $O_1 \cap O_2 = O_2$. Since $\mathcal{N}_1 \subset \mathcal{I}_2$, we have $\mathcal{N}_1 \| \mathcal{N}_2$. Now the result follows from Proposition 4. □

The second special case strengthens the condition $\mathcal{N}_1 \| \mathcal{N}_2$.

Corollary 2. *Let $\mathcal{N}_1$ and $\mathcal{N}_2$ be two necklaces. Suppose that $\mathcal{I}_1 \| \mathcal{N}_2$ and $\mathcal{I}_2 \| \mathcal{N}_1$. Then $\mathcal{I}_1 \cup \mathcal{I}_2$ is a pure system.*

Proof. The condition $\mathcal{I}_1 \| \mathcal{N}_2$ (or $\mathcal{I}_2 \| \mathcal{N}_1$) implies $\mathcal{N}_1 \| \mathcal{N}_2$. Thus, we can apply Proposition 4. Moreover, the relation $\mathcal{I}_1 \| \mathcal{N}_2$ gives the partitions $\mathcal{I}_1 = (\mathcal{I}_1 \cap \mathcal{I}_2) \sqcup (\mathcal{I}_1 \cap \mathcal{O}_2)$ and $\mathcal{I}_2 = (\mathcal{I}_2 \cap \mathcal{I}_1) \sqcup (\mathcal{I}_2 \cap \mathcal{O}_1)$. Therefore, we have the partition

$$\mathcal{I}_1 \cup \mathcal{I}_2 = (\mathcal{I}_1 \cap \mathcal{I}_2) \sqcup (\mathcal{I}_1 \cap \mathcal{O}_2) \sqcup (\mathcal{I}_2 \cap \mathcal{O}_1).$$

Let $\mathcal{C}$ be a maximal separated system in $\mathcal{I}_1 \cup \mathcal{I}_2$. Consider the intersection of $\mathcal{C}$ with each of $\mathcal{I}_1 \cap \mathcal{I}_2$, $\mathcal{I}_1 \cap \mathcal{O}_2$, and $\mathcal{I}_2 \cap \mathcal{O}_1$. We assert that $\mathcal{C} \cap (\mathcal{I}_1 \cap \mathcal{I}_2)$ is a maximal separated system in $\mathcal{I}_1 \cap \mathcal{I}_2$. Indeed, suppose that one can extend it by adding a new set $X \in \mathcal{I}_1 \cap \mathcal{I}_2$. Since X is separated from $\mathcal{I}_1 \cap \mathcal{O}_2$ and from $\mathcal{I}_2 \cap \mathcal{O}_1$, X is separated from $\mathcal{C}$. A contradiction. Similarly, $\mathcal{C} \cap (\mathcal{I}_1 \cap \mathcal{O}_2)$ is maximal in $\mathcal{I}_1 \cap \mathcal{O}_2$, and $\mathcal{C} \cap (\mathcal{I}_2 \cap \mathcal{O}_1)$ is maximal in $\mathcal{I}_2 \cap \mathcal{O}_1$. By Proposition 4, $|\mathcal{C}| = rk(\mathcal{I}_1 \cap \mathcal{I}_2) + rk(\mathcal{I}_1 \cap \mathcal{O}_2) + rk(\mathcal{I}_2 \cap \mathcal{O}_1)$. □

Note that if, in addition to the hypotheses in Corollary 2, we require that $\mathcal{I}_1$ and $\mathcal{I}_2$ are disjoint, then it follows that $rk(\mathcal{I}_1 \cup \mathcal{I}_2) = rk(\mathcal{I}_1) + rk(\mathcal{I}_2)$.

6 Proof of Theorem 3

Theorem 3 can be reformulated in the following equivalent form.

Theorem 3′. *Let $\mathcal{N}$ be a necklace. Suppose that $\mathcal{C}$ is a maximal separated system in the Grassmannian $\binom{[n]}{r}$, containing $\mathcal{N}$, and $\mathcal{C}' = \mathcal{C} \cap \mathcal{I}nt(N)$. Then $\mathcal{C}'$ is a maximal separated system in $\mathcal{I}nt(N)$.*

Indeed, let X be a set in $\mathcal{I}nt(N)$ which is separated from $\mathcal{C}'$. Then, due to Theorem 3, X is separated from $\mathcal{C} - \mathcal{C}'$. Therefore, X is separated from $\mathcal{C}$. By the maximality of $\mathcal{C}$, X belongs to $\mathcal{C}$ and, hence, belongs to $\mathcal{C}'$.

To prove the converse, we notice that Theorem $3'$ can be regarded as a generalization of the following

Theorem 4. [6, Theorem 3], see also [5, Proposition 3.2] *Let A be a subset in $\binom{[n]}{r-2}$, and let i, j, k, l be a cyclically ordered quadruple of elements of $[n] - A$. Suppose that $\mathcal{C}$ is a maximal separated system in the Grassmannian $\binom{[n]}{r}$ containing the sets Aij, Ajk, Akl, Ali. Then $\mathcal{C}$ contains either Aik or Ajl.*

Here we can interpret the quadruple Aij, Ajk, Akl, Ali as a 'small necklace' whose interior consists of the quadruple plus the sets Aik and Ajl. There are two maximal separated systems in the interior of this necklace, one containing Aik and the other containing Ajl. Moving from one of such systems to the other is called a *mutation*.

Let us deduce Theorem 3 from Theorem $3'$. Let $\mathcal{N}, X, Y$ be as in the hypotheses of Theorem 3. Consider a maximal separated system $\mathcal{C}$ in the Grassmannian containing X and $\mathcal{N}$. Due to Theorem $3'$, its restriction $\mathcal{C}' = \mathcal{C} \cap \mathcal{I}nt(\mathcal{N})$ is a maximal separated system in $\mathcal{I}nt(\mathcal{N})$. Let $\mathcal{C}''$ be a maximal separated system in $\mathcal{I}nt(\mathcal{N})$ which contains Y. Due to Postnikov's theorem ([4, Theorem 13.4], see also

[5, theorem 4.7]), the systems $\mathcal{C}'$ and $\mathcal{C}''$ can be connected by a sequence of mutations. Each mutation preserves the separation from X (Theorem 4). Therefore, X is separated from $\mathcal{C}''$, and we get $X\|Y$. $\square$

Thus, it remains to prove Theorem $3'$. Using a decomposition of the necklace along with the corresponding permutation and the interior of the necklace into connected components [5, Sec. 5], one may assume that the necklace $\mathcal{N}$ is *connected*, that is the sets N_i, $i \in [n]$, are distinct. The proof will use a technique of plabic tilings developed in [5, Sec. 9]. Let us recall this notion and details.

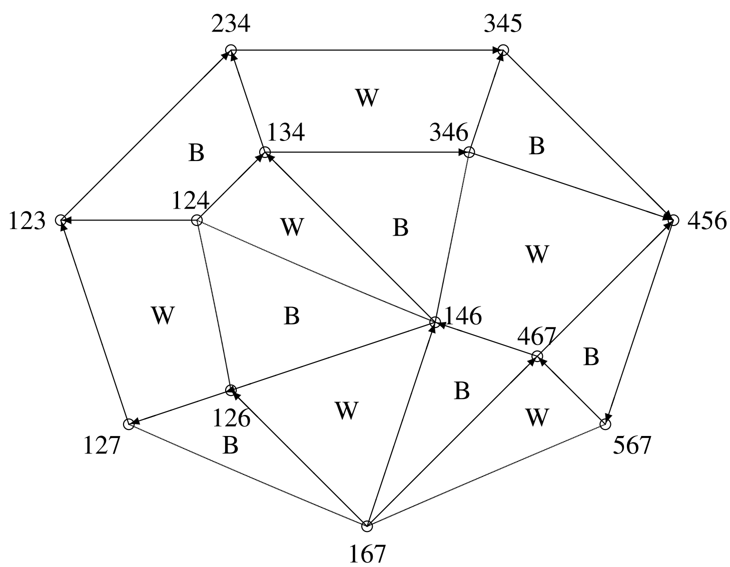
Plabic tilings. Suppose that $\mathcal{C}$ is a separated system in the Grassmannian $\binom{[n]}{r}$. Then it is possible to construct a planar bicolored (plabic) polygonal complex $\Sigma(\mathcal{C})$, with a chessboard coloring of its two-dimensional cells. In the beginning, we take n vectors $\xi_1, \ldots, \xi_n$ in the plane $\mathbb{R}^2$, being the clockwise ordered roots of 1 of degree n (identifying the plane with the set $\mathbb{C}$ of complex numbers).

Then one can assign to every set $X \subset [n]$ the vector (point) $\xi(X) = \sum_{i \in X} \xi_i$.

The set (structure) $\Sigma(\mathcal{C})$ consists of 0-dimensional cells (points), 1-dimensional cells (edges) and 2-dimensional cells (polygons), which form a polygonal complex (where the nonempty intersection of two cells is again a cell and is the common face of these two cells).

Here the 0-dimensional cells (vertices) are the points of the form $\xi(X)$ for $X \in \mathcal{C}$. One can check (using the separability) that these points are distinct.

Two-dimensional cells are colored black and white. More precisely, let K be an $(r-1)$-element subset of $[n]$. The *white clique* $\mathcal{W}(K) = \mathcal{W}_{\mathcal{C}}(K)$ consists of those sets $X \in \mathcal{C}$ that contain K, $K \subset X$. Thus, $\mathcal{W}(K)$ consists of sets $Ka_1, Ka_2, \ldots, Ka_k$, where the elements $a_1, \ldots, a_k$ are taken in cyclic order. A white clique is *nontrivial* if it has at least three elements. For a nontrivial white clique $W(K)$, the convex hull of the points $\xi(X)$, $X \in \mathcal{W}(K)$, is a white-colored cell of the complex $\Sigma(\mathcal{C})$.



Similarly, for a set L of the size $r+1$, the *black clique* $\mathcal{B}(L)$ is constituted from those sets $X \in \mathcal{C}$ that are contained in L. A nontrivial black clique $\mathcal{B}(L)$ generates the black-colored two-dimensional cell to be the convex hull of points $\xi(X)$, where X runs over the elements of $\mathcal{B}(L)$.

The set of one-dimensional cells (edges) consists of the edges of its two-dimensional cells and the segments joining vertices $\xi(X)$ and $\xi(Y)$ such that $\mathcal{W}(X \cap Y) = \mathcal{B}(X \cup Y) = \{X, Y\}$.

Let us notice that in the complex $\Sigma(\mathcal{C})$, only neighbors can be joined by edge, where sets X and Y (of the same size) are called *neighbors* if the symmetric difference $(X - Y) \cup (Y - X)$ consists of exactly two elements.

The picture above illustrates the plabic tiling for a certain set-system; here the sets of the system are indicated at the vertices and the letters on tiles indicate their colors. (A more sophisticated example of a plabic tilings is given in [5, Fig. 9].)

Proposition 9.4 of [5] asserts that $\Sigma(\mathcal{C})$ is a complex. In particular, the following holds.

Fact. *Let X and Y be neighbors of a separated system $\mathcal{C}$. If the segment $[\xi(X), \xi(Y)]$ and a cell C of $\Sigma(\mathcal{C})$ have more than one common point, then the points $\xi(X)$ and $\xi(Y)$ are vertices of C.*

The tiling $\Sigma(\mathcal{C})$ in the above picture fills in the regular n-gon. This is not by coincidence, but is caused by the maximality of the system $\mathcal{C}$.

Now let $\mathcal{N} = (N_1, \ldots, N_n)$ be a connected necklace. Let $\xi(\mathcal{N})$ be the closed polygonal curve (in the 1-dimensional subcomplex) joining the points $\xi(N_1)$, $\xi(N_2), \ldots, , \xi(N_n)$, $\xi(N_1)$ in this order. An important fact (cf. [5, Proposition 8.8]) is that $\xi(\mathcal{N})$ is a *simple closed curve*. Therefore, it divides the plane into the *inside* and the *outside* w.r.t. $\xi(\mathcal{N})$, where the former is homeomorphic to a disk and is denoted by $in(\mathcal{N})$.

We reformulate Proposition 9.10 from [5] as follows.

Proposition 5. *Let $X \in \binom{[n]}{r}$ be separated from a (connected) necklace $\mathcal{N}$. Then X belongs to $\mathcal{I}nt(\mathcal{N})$ if and only if $\xi(X)$ belongs to $in(\mathcal{N})$.* □

There is the following important characterization for the maximality of a separated system established in [5].

Proposition 6. *Let $\mathcal{N}$ be a connected necklace, and let $\mathcal{C}$ be a separated system in $\mathcal{I}nt(\mathcal{N})$. The system $\mathcal{C}$ is maximal in $\mathcal{I}nt(\mathcal{N})$ if and only if the complex $\Sigma(\mathcal{C})$ fill in the polygon $in(\mathcal{N})$.*

(One implication, namely, that the maximality of $\mathcal{C}$ implies filling-in is stated in [5, Proposition 11.2]. For the converse implication, let $\Sigma(\mathcal{C})$ fill in $in(\mathcal{N})$. Then the dual graph G to $\Sigma(\mathcal{C})$ is a reduced plabic graph (see the proof of [5, Proposition 11.2]) and $\mathcal{F}(G) = \mathcal{C}$. Now from [5, Theorem 9.16] it follows that $\mathcal{F}(G)$ is a maximal separated system in $\mathcal{I}nt(\mathcal{N})$.)

In particular, if $\mathcal{C}$ is a separated system in $\mathcal{I}nt(\mathcal{N})$, then the complex $\Sigma(\mathcal{C})$ is located in the polygon $in(\mathcal{N})$.

Now we are ready to prove the theorem.

Proof of Theorem $3'$. Let $\mathcal{C}$ be a maximal separated system in $\binom{[n]}{r}$. Then the complex $\Sigma(\mathcal{C})$ fills in the regular n-gon. Let $\mathcal{N}$ be a connected necklace, $\xi(\mathcal{N})$ the corresponding simple closed polygonal curve, and $in(\mathcal{N})$ the inside of the curve.

The intersection $\mathcal{C}' = \mathcal{C} \cap \mathcal{I}nt(\mathcal{N})$ is a separated system in $\mathcal{I}nt(\mathcal{N})$.

Hence, the complex $\Sigma(\mathcal{C}')$ is located inside the curve $\xi(\mathcal{N})$, $\Sigma(\mathcal{C}) \subset in(\mathcal{N})$. We assert that this complex fills in the polygon $in(\mathcal{N})$. Indeed, let P be a point in $in(\mathcal{N})$. Since $\Sigma(\mathcal{C})$ fills in the regular n-gon, the point P lies in some two-dimensional cell C of $\Sigma(\mathcal{C})$. Let for definiteness C be a white-colored cell corresponding to a white clique $\mathcal{W}_{\mathcal{C}}(K)$. Consider the intersection of C with the polygon $in(\mathcal{N})$. The edges of the polygonal curve $\xi(\mathcal{N})$ passing inside the cell C are some (non-intersecting) diagonals of the convex polygon C; see the Fact above and Figure 2.

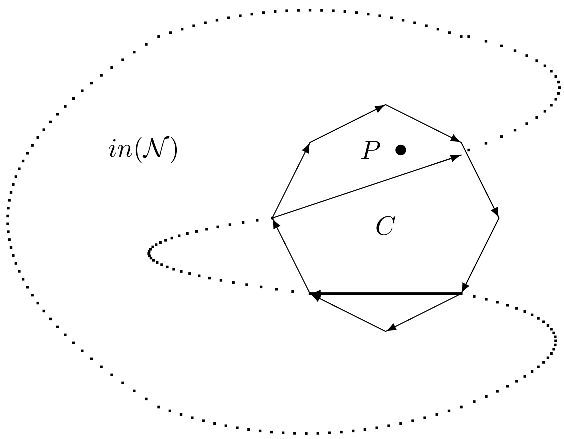


Figure 2. Such a picture is impossible: the intersection of C and $in(\mathcal{N})$ is a convex polygon.

Therefore, the intersection of C with the polygon $in(\mathcal{N})$ is the union of convex polygons with vertices of the form $\xi(X)$, where $X \in \mathcal{W}_{\mathcal{C}}(K) \cap \mathcal{C}'$. Hence, P lies in the convex hull of points $\xi(X)$, while X runs over the set $\mathcal{C}' \cap \mathcal{W}_{\mathcal{C}}(K) = \mathcal{W}_{\mathcal{C}'}(K)$. But then P lies in the white cell of the complex $\Sigma(\mathcal{C}')$, corresponding to the white clique $\mathcal{W}_{\mathcal{C}'}(K)$.

Thus, the complex $\Sigma(\mathcal{C}')$ fills in the polygon $in(\mathcal{N})$ and, by Proposition 6, the separated system $\mathcal{C}'$ is maximal in $\mathcal{I}nt(\mathcal{N})$. Theorem $3'$ is proven. $\square$

Remark 4. Below we formulate a generalization of Theorem 3 (or $3'$) and Corollary 2. This is proved in [2] using a technique of combined tilings introduced there. Let $\mathcal{K} = (K_1, \dots, K_m)$ be a sequence of elements of the discrete Grassmannian which satisfies the following three conditions:

1. K_i and K_{i+1} are neighbors for any $i = 1, \dots, m$ (where $K_{m+1} = K_1$);
2. $\mathcal{K}$ is a separeted set-system;
3. the closed curve $\xi(\mathcal{K})$ is simple (without self-intersections).

We call such a $\mathcal{K}$ a *generalized necklace*. The inside $in(\mathcal{K})$ of the curve $\xi(\mathcal{K})$ is defined as before. Define the interior of the generalized necklace as follows:

$$\mathcal{I}nt(\mathcal{K}) = \{X \in \binom{[n]}{r}, X \| \mathcal{K} \text{ and } \xi(X) \in in(\mathcal{K})\}.$$

The exterior $\mathcal{O}ut(\mathcal{K})$ is defined to be the complement to $\mathcal{I}nt(\mathcal{K})$ in $\mathcal{S}(\mathcal{K})$: $\mathcal{O}ut(\mathcal{K}) = \mathcal{S}(\mathcal{K}) \setminus \mathcal{I}nt(\mathcal{K})$.

Theorem 4 [2] The following properties are valid:

(i) *both $\mathcal{I}nt(\mathcal{K})$ and $\mathcal{O}ut(\mathcal{K})$ are pure systems;*

(ii) *If C is a maximal separated system in the Grassmannian, then the intersections $C \cap \mathcal{I}nt(K)$ and $C \cap \mathcal{O}ut(K)$ are maximal separated systems in $\mathcal{I}nt(K)$ and $\mathcal{O}ut(K)$, respectively;*

(iii) $\mathcal{I}nt(K) \| \mathcal{O}ut(K)$.

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