FPSAC 2014, Chicago, USA DMTCS proc. **AT**, 2014, 297–308

Reflection factorizations of Singer cycles

J.B. Lewis and V. Reiner and D. Stanton*

School of Mathematics, University of Minnesota. Minneapolis, MN, USA

Abstract. The number of shortest factorizations into reflections for a Singer cycle in $GL_n(\mathbb{F}_q)$ is shown to be $(q^n - 1)^{n-1}$. Formulas counting factorizations of any length, and counting those with reflections of fixed conjugacy classes are also given.

Résumé. Nous prouvons que le nombre de factorisations de longueur minimale d'un cycle de Singer dans $GL_n(\mathbb{F}_q)$ comme un produit de réflexions est $(q^n - 1)^{n-1}$. Nous présentons aussi des formules donnant le nombre de factorisations de toutes les longueurs ainsi que des formules pour le nombre de factorisations comme produit de réflexions ayant des classes de conjugaison fixes.

Keywords: Singer cycle, Coxeter torus, anisotropic maximal torus, reflection, transvection, factorization, finite general linear group, regular element, q-analogue, higher genus, Coxeter element

1 Introduction and main result

This paper is motivated by two classic results on the number $t(n, \ell)$ of ordered factorizations $(t_1, \ldots, t_\ell)$ of an n-cycle $c = t_1 t_2 \cdots t_\ell$ in the symmetric group $\mathfrak{S}_n$, where each t_i is a transposition.

Theorem (Hurwitz [8], Dénes [2]). *For $n \geq 1$, one has $t(n, n-1) = n^{n-2}$.*

Theorem (Jackson [10, p. 368]). *For $n \geq 1$, more generally $t(n, \ell)$ has explicit formulas*

$$t(n,\ell) = \frac{n^\ell}{n!} \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} \left(\frac{n-1}{2} - k\right)^\ell = \frac{(-n)^\ell}{n!} (-1)^{n-1} \left[\Delta^{n-1}(x^\ell)\right]_{x=\frac{1-n}{2}} . \qquad (1.1)$$

Here Δ is the *difference operator* $\Delta(f)(x) := f(x+1) - f(x)$.

Our goals are q-analogues, replacing the symmetric group $\mathfrak{S}_n$ with the *general linear group* $GL_n(\mathbb{F}_q)$, replacing transpositions with *reflections*, and replacing an n-cycle with a *Singer cycle* c: the image of a generator for the cyclic group $\mathbb{F}_{q^n}^\times \cong \mathbb{Z}/(q^n-1)\mathbb{Z}$ under any embedding $\mathbb{F}_{q^n}^\times \hookrightarrow GL_{\mathbb{F}_q}(\mathbb{F}_{q^n}) \cong GL_n(\mathbb{F}_q)$ that comes from a choice of $\mathbb{F}_q$-vector space isomorphism $\mathbb{F}_{q^n} \cong \mathbb{F}_q^n$. The analogy between Singer cycles in $GL_n(\mathbb{F}_q)$ and n-cycles in $\mathfrak{S}_n$ is reasonably well-established [17, §7], [18, §§8-9]. Fixing such a Singer cycle c, denote by $t_q(n, \ell)$ the number of ordered factorizations $(t_1, \ldots, t_\ell)$ of $c = t_1 t_2 \cdots t_\ell$ in which each t_i is a *reflection* in $GL_n(\mathbb{F}_q)$, that is, the fixed space $(\mathbb{F}_q^n)^{t_i}$ is a hyperplane in $\mathbb{F}_q^n$.

*(jblewis,reiner,stanton)@math.umn.edu

Work partially supported by NSF grants DMS-1148634 and DMS-1001933.

Theorem 1.1. *For $n \geq 2$, one has $t_q(n,n) = (q^n - 1)^{n-1}$.*

Theorem 1.2. *For $n \geq 2$, more generally $t_q(n,\ell)$ has explicit formulas*

$$t_q(n,\ell) = \frac{(-[n]_q)^\ell}{q^{\binom{n}{2}}(q;q)_n}\left((-1)^{n-1}(q;q)_{n-1} + \sum_{k=0}^{n-1}(-1)^{k+n}q^{\binom{k+1}{2}}\begin{bmatrix} n-1 \\ k \end{bmatrix}_q (1+q^{n-k-1}-q^{n-k})^\ell\right) \tag{1.2}$$

$$= (1-q)^{-1}\frac{(-[n]_q)^\ell}{[n]!_q}\left[\Delta_q^{n-1}\left(\frac{1}{x} - \frac{(1+x(1-q))^\ell}{x}\right)\right]_{x=1} \tag{1.3}$$

$$= [n]_q^{\ell-1}\sum_{i=0}^{\ell-n}(-1)^i(q-1)^{\ell-i-1}\binom{\ell}{i}\begin{bmatrix} \ell-i-1 \\ n-1 \end{bmatrix}_q. \tag{1.4}$$

The q-analogues used above and elsewhere in the paper are defined as follows:

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]!_q}{[k]!_q[n-k]!_q}, \quad \text{where } [n]!_q := [1]_q[2]_q\cdots[n]_q \quad \text{and } [n]_q := 1+q+q^2+\cdots+q^{n-1},$$

$$(x;q)_n := (1-x)(1-xq)(1-xq^2)\cdots(1-xq^{n-1}), \quad \text{and}$$

$$\Delta_q(f)(x) := \frac{f(x)-f(qx)}{x-qx}, \text{ so that}$$

$$\Delta_q^n(f)(x) = \frac{1}{q^{\binom{n}{2}}x^n(1-q)^n}\sum_{k=0}^{n}(-1)^{n-k}q^{\binom{k}{2}}\begin{bmatrix} n \\ k \end{bmatrix}_q f(q^{n-k}x). \tag{1.5}$$

In fact, we will prove the following refinement of Theorem 1.2 for $q > 2$, having no counterpart for $\mathfrak{S}_n$. Transpositions are all conjugate within $\mathfrak{S}_n$, but the conjugacy class of a reflection t in $GL_n(\mathbb{F}_q)$ for $q > 2$ varies with its determinant $\det(t)$ in $\mathbb{F}_q^\times$. When $\det(t) = 1$, the reflection t is called a *transvection* [12, XIII §9], while $\det(t) \neq 1$ means that t is a *semisimple reflection*. One can associate to an ordered factorization $(t_1,\ldots,t_\ell)$ of $c = t_1t_2\cdots t_\ell$ the sequence of determinants $(\det(t_1),\ldots,\det(t_\ell))$ in $\mathbb{F}_q^\ell$, having product $\det(c)$.

Theorem 1.3. *Let $q > 2$. Fix a Singer cycle c in $GL_n(\mathbb{F}_q)$ and a sequence $\alpha = (\alpha_i)_{i=1}^\ell$ in $(\mathbb{F}_q^\times)^\ell$ with $\prod_{i=1}^\ell \alpha_i = \det(c)$. Let m be the number of values i such that $\alpha_i = 1$. Then one has $m \leq \ell - 1$, and the number of ordered reflection factorizations $c = t_1\cdots t_\ell$ with $\det(t_i) = \alpha_i$ depends only upon ℓ and m. This quantity $t_q(n,\ell,m)$ is given by these formulas:*

$$t_q(n,\ell,m) = [n]_q^{\ell-1}\sum_{i=0}^{\min(m,\ell-n)}(-1)^i\binom{m}{i}\begin{bmatrix} \ell-i-1 \\ n-1 \end{bmatrix}_q \tag{1.6}$$

$$= \frac{[n]_q^\ell}{[n]!_q}\left[\Delta_q^{n-1}\left((x-1)^m x^{\ell-m-1}\right)\right]_{x=1}. \tag{1.7}$$

In particular, setting $\ell = n$ in (1.6)*, the number of shortest such factorizations is*

$$t_q(n,n,m) = [n]_q^{n-1},$$

which depends neither on the sequence $\alpha = (\det(t_i))_{i=1}^\ell$ nor on the number of transvections m.

Theorems 1.2 and 1.3 are proven via a standard character-theoretic approach. This approach is reviewed quickly in Section 2, followed by an outline of ordinary character theory for $GL_n(\mathbb{F}_q)$ in Section 3. Section 4 either reviews or derives the needed explicit character values for four kinds of conjugacy classes: the identity element, Singer cycles, semisimple reflections, and transvections. Then Section 5 assembles these calculations into the proofs of Theorems 1.2 and 1.3. Many details and proofs are omitted in this extended abstract; see [13] for a complete version of this paper.

Although Theorem 1.3 is stated for $q > 2$, something interesting also occurs for $q = 2$. All reflections in $GL_n(\mathbb{F}_2)$ are transvections, thus one always has $m = \ell$ for $q = 2$. Furthermore, one can see that (1.4), (1.6) give the same answer when both $q = 2$ and $m = \ell$. This reflects a striking dichotomy in our proofs: for $q > 2$ the only contributions to the computation come from irreducible characters of $GL_n(\mathbb{F}_q)$ arising as constituents of parabolic inductions of characters of $GL_1(\mathbb{F}_q)$, while for $q = 2$ the cuspidal characters for $GL_s(\mathbb{F}_q)$ with $s \geq 2$ play a role, miraculously giving the same polynomial $t_q(n, \ell)$ in q evaluated at $q = 2$.

Question 1.4. Can one derive the formulas (1.4) and (1.6) via *inclusion-exclusion* more directly?

Question 1.5. Can one derive Theorem 1.1 *bijectively*, or by an *overcount*, in the spirit of Dénes [2], that counts factorizations of all conjugates of a Singer cycle and then divides by the conjugacy class size?

2 The character theory approach to factorizations

We recall the classical approach to factorization counts, which goes back to work of Frobenius [3].

Definition 2.1. Given a finite group G, let $\mathrm{Irr}(G)$ be the set of its irreducible ordinary (finite-dimensional, complex) representations V. For each V in $\mathrm{Irr}(G)$, denote by $\deg(V)$ the *degree* $\dim_{\mathbb{C}} V$, and let $\chi_V(g) = \mathrm{Tr}(g : V \to V)$ be its *character value* at g, along with $\widetilde{\chi}_V(g) := \frac{\chi_V(g)}{\deg(V)}$ the *normalized character value*. Both functions $\chi_V(-)$ and $\widetilde{\chi}_V(-)$ on G extend by $\mathbb{C}$-linearity to functionals on the *group algebra* $\mathbb{C}G$.

Proposition 2.2 (Frobenius [3]). *Let G be a finite group, and $A_1, \ldots, A_\ell \subset G$ unions of conjugacy classes in G. Then for g in G, the number of ordered factorizations $(t_1, \ldots, t_\ell)$ with $g = t_1 \cdots t_\ell$ and t_i in A_i for $i = 1, 2, \ldots, \ell$ is*

$$\frac{1}{|G|} \sum_{V \in \mathrm{Irr}(G)} \deg(V) \cdot \chi_V(g^{-1}) \cdot \widetilde{\chi}_V(z_1) \cdots \widetilde{\chi}_V(z_\ell). \tag{2.1}$$

where $z_i := \sum_{t \in A_i} t$ in $\mathbb{C}G$.

This lemma was the main tool used by Jackson [9, §2], as well as by Chapuy and Stump [1, §4] in their solution of the analogous question in well-generated complex reflection groups. The proof, which we omit, is a straightforward computation in the group algebra $\mathbb{C}G$ coupled with the isomorphism of G-representations $\mathbb{C}G \cong \bigoplus_{V \in \mathrm{Irr}(G)} V^{\oplus \deg(V)}$.

3 Review of ordinary characters of $GL_n(\mathbb{F}_q)$

The ordinary character theory of $GL_n := GL_n(\mathbb{F}_q)$ was worked out by Green [6], and has been reworked many times. Aside from Green's paper, some useful references are Macdonald [14, Chaps. III, IV] and Zelevinsky [20, §11].

A GL_n-irreducible U is called *cuspidal* if χ_U does not occur as a constituent in any induced[(i)] character $f_1 * f_2$, where the f_i are class functions on GL_{n_i} and (n_1, n_2) is a composition $n = n_1 + n_2$ with $n_1, n_2 > 0$. Denote by Cusp_n the set of all such cuspidal irreducibles U for GL_n, and say that the *weight* $\mathrm{wt}(U)$ of each such U is n. Let Par_n denote the partitions λ of n (that is, $|\lambda| := \sum_i \lambda_i = n$), and define

$$\mathrm{Par} := \bigsqcup_{n\geq 0} \mathrm{Par}_n \qquad \text{and} \qquad \mathrm{Cusp} := \bigsqcup_{n\geq 1} \mathrm{Cusp}_n,$$

the sets of *all* partitions and *all* cuspidal representations for all groups GL_n. Then the GL_n-irreducible characters can be indexed as $\mathrm{Irr}(GL_n) = \{\chi^{\underline{\lambda}}\}$ where $\underline{\lambda}$ runs through the set of all functions $\underline{\lambda} : \mathrm{Cusp} \to \mathrm{Par}$ having the property that

$$\sum_{U\in\mathrm{Cusp}} \mathrm{wt}(U)\, |\underline{\lambda}(U)| = n. \tag{3.1}$$

Although Cusp is infinite, this condition (3.1) implies that $\underline{\lambda}$ can only take on finitely many non-$\varnothing$ values $\underline{\lambda}(U_1), \dots, \underline{\lambda}(U_m)$, and in this case

$$\chi^{\underline{\lambda}} = \chi^{U_1,\underline{\lambda}(U_1)} * \cdots * \chi^{U_m,\underline{\lambda}(U_m)} \tag{3.2}$$

where each $\chi^{U,\lambda}$ is what Green [6, §7] called a *primary irreducible character*. In particular, a cuspidal character U in Cusp_n is the same as the primary irreducible $\chi^{U,(1)}$. One can also show that there are $GL_n(\mathbb{F}_q)$-versions of the Jacobi-Trudi and dual Jacobi-Trudi formulas, i.e., every primary irreducible character $\chi^{U,\lambda}$ can be written as a linear combination of induction products of characters of the form $\chi^{U,(n)}$ and $\chi^{U,(1^n)}$.

The cuspidals Cusp_n are indexed via free orbits $[\varphi] = \{\varphi, \varphi\circ F, \cdots, \varphi \circ F^{n-1}\}$ for the *Frobenius action* $\beta \overset{F}{\longmapsto} \beta^q$ on the *dual group* $\mathrm{Hom}(\mathbb{F}_{q^n}^\times, \mathbb{C}^\times)$. Say that U in Cusp_n is *associated* to the orbit $[\varphi]$ in this indexing.

When $n = 1$, one simply has $\mathrm{Cusp}_1 = \mathrm{Hom}(\mathbb{F}_q^\times, \mathbb{C}^\times)$. In other words, the Frobenius orbits $[\varphi] = \{\varphi\}$ are singletons, and if U is associated to this orbit then $U = \varphi$ considering both as homomorphisms $GL_1(\mathbb{F}_q) = \mathbb{F}_q^\times \to \mathbb{C}^\times$.

Although we will not need Green's full description of the characters $\chi^{U,(m)}$ and $\chi^{U,\lambda}$, we will use (in the proof of Lemma 4.7 below) the following consequence of his discussion surrounding [6, Lemma 7.2].

Proposition 3.1. *For U in Cusp_s, every $\chi^{U,(m)}$, and hence also every primary irreducible character $\chi^{U,\lambda}$, is in the $\mathbb{Q}$-span of characters of the form $\chi_{U_1} * \cdots * \chi_{U_t}$ where U_i is in Cusp_{n_i}, with s dividing n_i for each i.*

4 Some explicit character values

We will eventually wish to apply Proposition 2.2 with g being a Singer cycle, and with the central elements z_i being sums over classes of reflections with fixed determinants. For this one requires explicit character values on four kinds of conjugacy classes of elements in $GL_n(\mathbb{F}_q)$: the identity (giving the character degrees), the Singer cycles, the semisimple reflections, and the transvections. We review known formulas for most of these, and derive others that we will need, in the next four subsections.

(i) The notion of induction used here is *parabolic* or *Harish-Chandra* induction.

It simplifies matters that the character value $\chi^{\underline{\lambda}}(c^{-1})$ vanishes for a Singer cycle c unless $\chi^{\underline{\lambda}} = \chi^{U,\lambda}$ is a primary irreducible character and the partition λ of $\frac{n}{s}$ takes a very special form; see Proposition 4.6 below. (This may be compared with, for example, Chapuy and Stump [1, p. 9 and Lemma 5.5].) Thus we only compute *primary* irreducible character values, sometimes only those of *hook shape*, i.e., of the form $\chi^{U,\left(\frac{n}{s}-k,1^k\right)}$.

4.1 Character values at the identity: the character degrees

Green computed the degrees of the primary irreducible characters $\chi^{U,\lambda}$ as a product formula involving familiar quantities associated to partitions.

Definition 4.1. For a partition λ, recall [14, (1.5)] the quantity $n(\lambda) := \sum_{i\geq 1}(i-1)\lambda_i$. For a cell a in row i and column j of the Ferrers diagram of λ recall the *hooklength* [14, Example I.1]

$$h(a) := h_\lambda(a) := \lambda_i + \lambda'_j - (i+j) + 1.$$

Theorem 4.2 ([6, Theorem 12])**.** *The primary irreducible GL_n-character $\chi^{U,\lambda}$ for a cuspidal character U of $GL_s(\mathbb{F}_q)$ and a partition λ of $\frac{n}{s}$ has degree*

$$\deg(\chi^{U,\lambda}) = (-1)^{n-\frac{n}{s}}(q;q)_n \frac{q^{s\cdot n(\lambda)}}{\prod_{a\in\lambda} 1 - q^{s\cdot h(a)}} = (-1)^{n-\frac{n}{s}}(q;q)_n s_\lambda(1, q^s, q^{2s}, \ldots).$$

Here $s_\lambda(1, q, q^2, \ldots)$ is the *principal specialization* $x_i = q^{i-1}$ of the Schur function $s_\lambda = s_\lambda(x_1, x_2, \ldots)$.

Two special cases of this formula will be useful in the sequel. First, in the case of hook shapes we have

$$\deg(\chi^{U,\left(\frac{n}{s}-k,1^k\right)}) = (-1)^{n-\frac{n}{s}} q^{s\binom{k+1}{2}} \frac{(q;q)_n}{(q^s;q^s)_{\frac{n}{s}}} \begin{bmatrix} \frac{n}{s}-1 \\ k \end{bmatrix}_{q^s}. \tag{4.1}$$

Second, when $s = 1$ and $U = \mathbf{1}$ is the trivial character of $GL_1(\mathbb{F}_q)$, the degree is given by the usual *q-hook formula* [16, §7.21]

$$\deg(\chi^{\mathbf{1},\lambda}) = f^\lambda(q) := (q;q)_n \frac{q^{n(\lambda)}}{\prod_{a\in\lambda} 1 - q^{h(a)}} = (q;q)_n s_\lambda(1, q, q^2, \ldots) = \sum_Q q^{\mathrm{maj}(Q)} \tag{4.2}$$

where the last sum is over all standard Young tableaux Q of shape λ, and $\mathrm{maj}(Q)$ is the sum of the entries i in Q for which $i+1$ lies in a lower row of Q.

4.2 Character values on Singer cycles and regular elliptic elements

Recall from the Introduction that a *Singer cycle* in $GL_n(\mathbb{F}_q)$ is the image of a generator for the cyclic group $\mathbb{F}_{q^n}^\times \cong \mathbb{Z}/(q^n-1)\mathbb{Z}$ under any embedding $\mathbb{F}_{q^n}^\times \hookrightarrow GL_{\mathbb{F}_q}(\mathbb{F}_{q^n}) \cong GL_n(\mathbb{F}_q)$ that comes from a choice of $\mathbb{F}_q$-vector space isomorphism $\mathbb{F}_{q^n} \cong \mathbb{F}_q^n$. (Such an embedded subgroup $\mathbb{F}_{q^n}^\times$ is sometimes called a *Coxeter torus* or an *anisotropic maximal torus*.) Many irreducible GL_n-character values $\chi^{\underline{\lambda}}(c^{-1})$ vanish not only on Singer cycles, but even for a larger class of elements that we introduce in the following proposition.

Proposition 4.3. *The following are equivalent for g in $GL_n(\mathbb{F}_q)$.*

(i) No conjugates of hgh^{-1} of g lie in a proper parabolic subgroup $P_\alpha \subsetneq GL_n$.
(ii) There are no nonzero proper g-stable $\mathbb{F}_q$-subspaces inside $\mathbb{F}_q^n$.
(iii) The characteristic polynomial $\det(xI_n - g)$ is irreducible in $\mathbb{F}_q[x]$.
(iv) The element g is the image of some β in $\mathbb{F}_{q^n}^\times$ satisfying $\mathbb{F}_q(\beta) = \mathbb{F}_{q^n}$ (that is, a primitive element for $\mathbb{F}_{q^n}$) under one of the embeddings $\mathbb{F}_{q^n}^\times \hookrightarrow GL_{\mathbb{F}_q}(\mathbb{F}_{q^n}) \cong GL_n(\mathbb{F}_q)$.

The elements in $GL_n(\mathbb{F}_q)$ satisfying these properties are called the **regular elliptic elements***.*

Proof sketch: The equivalence of (i) and (ii) and the implications from (iv) to (ii) to (iii) are relatively striaghtforward. For the implication from (iii) to (iv), suppose $f(x) := \det(xI_n - g)$ is irreducible in $\mathbb{F}_q[x]$, so $f(x)$ is also the minimal polynomial of g. Thus g has rational canonical form over $\mathbb{F}_q$ equal to the companion matrix for $f(x)$. This is the same as the rational canonical form for the image under one of the above embeddings of any β in $\mathbb{F}_{q^n}^\times$ having minimal polynomial $f(x)$, so that $\mathbb{F}_q(\beta) \cong \mathbb{F}_{q^n}$. Hence g is conjugate to the image of such an element β embedded in $GL_n(\mathbb{F}_q)$, and then equal to such an element, after conjugating the embedding. $\square$

Part (iv) of Proposition 4.3 shows that Singer cycles c in G are always regular elliptic, since they correspond to elements γ for which $\mathbb{F}_{q^n}^\times = \langle\gamma\rangle$, that is, to *primitive roots* in $\mathbb{F}_{q^n}$.

Definition 4.4. Recall that associated to the extension $\mathbb{F}_q \subset \mathbb{F}_{q^n}$ is the *norm map* $N_{\mathbb{F}_{q^n}/\mathbb{F}_q} : \mathbb{F}_{q^n} \to \mathbb{F}_q$ sending $\beta \mapsto \beta \cdot \beta^q \cdot \beta^{q^2} \cdots \beta^{q^{n-1}}$.

The well-known surjectivity of norm maps for finite fields [12, VII Exer. 28] is equivalent to the following.

Proposition 4.5. *If $\mathbb{F}_{q^n}^\times = \langle\gamma\rangle$, then $\mathbb{F}_q^\times = \langle N(\gamma)\rangle$.*

Proposition 4.6. *Let g be a regular elliptic element in $GL_n(\mathbb{F}_q)$ associated to $\beta \in \mathbb{F}_{q^n}$.*
(i) The irreducible character $\chi^{\underline{\lambda}}(g)$ vanishes unless $\chi^{\underline{\lambda}}$ is a primary irreducible character $\chi^{U,\lambda}$, for some s dividing n and some cuspidal character U in Cusp_s and partition λ in $\mathrm{Par}_{\frac{n}{s}}$.
(ii) Furthermore, $\chi^{U,\lambda}(g) = 0$ except for hook-shaped partitions $\lambda = \left(\frac{n}{s} - k, 1^k\right)$.
(iii) More explicitly, if U in Cusp_s is associated to $[\varphi]$ with φ in $\mathrm{Hom}(\mathbb{F}_{q^s}^\times, \mathbb{C}^\times)$, then

$$\begin{aligned}\chi^{U,\left(\frac{n}{s}-k,1^k\right)}(g) &= (-1)^k \chi^{U,\left(\frac{n}{s}\right)}(g) \\ &= (-1)^{\frac{n}{s}-k-1}\chi^{U,\left(1^{\frac{n}{s}}\right)}(g) \\ &= (-1)^{n-\frac{n}{s}-k}\sum_{j=0}^{s-1}\varphi\left(N_{\mathbb{F}_{q^n}/\mathbb{F}_{q^s}}(\beta^{q^j})\right).\end{aligned}$$

(iv) If in addition g is a Singer cycle then

$$\sum_U \chi^{U,\left(\frac{n}{s}-k,1^k\right)}(g) = \begin{cases}(-1)^{n-\frac{n}{s}-k}\mu(s) & \text{if } q=2,\\ 0 & \text{if } q \neq 2.\end{cases}$$

where the sum is over all U in Cusp_s, and $\mu(s)$ is the usual number-theoretic Möbius function of s.

Proof sketch: The key point is Proposition 4.3(i), showing that regular elliptic elements g are the elements whose conjugates hgh^{-1} lie in no proper parabolic subgroup P_α. It follows from the formula for induced character values that any properly induced character will vanish on a regular elliptic element.

Assertion (i) follows immediately, as (3.2) shows non-primary irreducibles are properly induced.

Assertion (ii) follows by applying the Jacobi-Trudi-style formula for GL_n characters and observing that for non-hook shapes, every summand is properly induced.

The first two equalities asserted in (iii) follow from similar analysis of terms in the Jacobi-Trudi- and dual Jacobi-Trudi-style expansions for a hook shape; in this case, only a single summand survives. The last equality in (iii) comes from a result of Silberger and Zink [15, Theorem 6.1], which they deduced by combining various formulas from Green [6].

For assertion (iv), one may use assertion (iii) and the properties of the norm map and Singer cycles to rewrite the sum to be computed as $(-1)^{n-\frac{n}{s}-k}$ times the sum of all z in $\mathbb{C}^\times$ for which $z^{q^s-1} = 1$ but $z^{q^t-1} \neq 1$ for any proper divisor t of s. The result follows. $\square$

4.3 Character values on semisimple reflections

Recall that a semisimple reflection t in $GL_n(\mathbb{F}_q)$ has conjugacy class determined by its non-unit eigenvalue $\det(t)$, which lies in $\mathbb{F}_q^\times \setminus \{1\}$. Recall also the notion of the *content* $c(a) := j - i$ of a cell a lying row i and column j of the Ferrers diagram for a partition λ.

Lemma 4.7. *Let t be a semisimple reflection in $GL_n(\mathbb{F}_q)$.*

(i) Primary irreducible characters $\chi^{U,\lambda}$ vanish on t unless $\mathrm{wt}(U) = 1$, that is, unless U is in Cusp_1.

(ii) For U in Cusp_1, so $\mathbb{F}_q^\times \xrightarrow{U} \mathbb{C}^\times$, and λ in Par_n, the normalized character $\widetilde{\chi}^{U,\lambda}$ has value on t

$$\widetilde{\chi}^{U,\lambda}(t) = U(\det(t)) \cdot \frac{1}{[n]_q} \sum_{a\in\lambda} q^{c(a)}.$$

(iii) In particular, for U in Cusp_1 and hook shapes $\lambda = (n-k, 1^k)$, this simplifies to

$$\widetilde{\chi}^{U,(n-k,1^k)}(t) = U(\det(t)) \cdot q^{-k}.$$

Proof sketch: For assertion (i), we start with the fact proven by Green [6, §5 Example (ii), p. 430] that cuspidal characters for GL_n vanish on *non-primary* conjugacy classes, that is, those for which the characteristic polynomial is divisible by at least two distinct irreducible polynomials in $\mathbb{F}_q[x]$.

This implies cuspidal characters for GL_n with $n \geq 2$ vanish on semisimple reflections t, since such t are non-primary: $\det(xI - t)$ is divisible by both $x - 1$ and $x - \alpha$ where $\alpha = \det(t) \neq 1$.

Next, the formula for induced characters shows that any character of the form $\chi_{U_1} * \cdots * \chi_{U_\ell}$ in which each U_i is a GL_{n_i}-cuspidal with $n_i \geq 2$ will also vanish on all semisimple reflections t: whenever hth^{-1} lies in the parabolic $P_{(n_1,\ldots,n_\ell)}$ and has diagonal blocks $(g_1, \ldots, g_\ell)$, one of the g_{i_0} is also a semisimple reflection, so that $\chi_{U_{i_0}}(g_{i_0}) = 0$ by the above discussion.

Lastly, Lemma 3.1 shows that every primary irreducible $\chi^{U,\lambda}$ with $\mathrm{wt}(U) \geq 2$ will vanish on every semisimple reflection: $\chi^{U,\lambda}$ is in the $\mathbb{Q}$-span of characters $\chi_{U_1} * \cdots * \chi_{U_\ell}$ with each U_i a GL_{n_i}-cuspidal in which $\mathrm{wt}(U)$ divides n_i, so that $n_i \geq 2$.

Assertion (iii) is an easy calculation using assertion (ii), so it only remains to prove (ii). We first claim that one can reduce to the case where character U in Cusp_1 is the trivial character $\mathbb{F}_q^\times \xrightarrow{U=\mathbf{1}} \mathbb{C}^\times$. This is because one has $\chi^{U,(n)} = U = U \otimes \chi^{\mathbf{1},(n)}$ and hence using the Jacobi-Trudi-style identity for $GL_n(\mathbb{F}_q)$ one has

$$\chi^{U,\lambda} = U \otimes \chi^{\mathbf{1},\lambda} \qquad \text{for } \lambda \text{ in } \mathrm{Par}_n \text{ when } U \text{ lies in } \mathrm{Cusp}_1. \tag{4.3}$$

Thus without loss of generality, $U = \mathbf{1}$, and we wish to show

$$\widetilde{\chi}^{\mathbf{1},\lambda}(t) = \frac{1}{[n]_q} \sum_{a \in \lambda} q^{c(a)}. \tag{4.4}$$

Next, we use an application of the Jacobi-Trudi identity for $GL_n(\mathbb{F}_q)$ and explicitly compute character values by counting certain flags in $\mathbb{F}_q^n$ to show that

$$\widetilde{\chi}^{\mathbf{1},\lambda}(t) = \sum_{\substack{\mu \subset \lambda \\ |\mu|=|\lambda|-1}} \frac{f^\mu(q)}{f^\lambda(q)}$$

where $f^\lambda(q)$ is the q-hook formula from (4.2). Thus the desired result follows from either of two results in the literature: it is equivalent[(ii)], after sending $q \mapsto q^{-1}$, to a result of Kerov [11, Theorem 1 and Eqn. (2.2)], and it is also the $t = q^{-1}$ specialization of a result of Garsia and Haiman [4, (I.15), Thm. 2.3]. $\square$

4.4 Character values on transvections

The GL_n-irreducible character values on transvections appear in probabilistic work[(iii)] of M. Hildebrand [7]. For primary irreducible characters, his result is equivalent[(iv)] to the following.

Theorem 4.8 ([7, Theorem 2.1]). *For U in Cusp_s with λ in $\mathrm{Par}_{\frac{n}{s}}$, a transvection t in $GL_n(\mathbb{F}_q)$ has*

$$\widetilde{\chi}^{U,\lambda}(t) = \begin{cases} \dfrac{1}{1-q^{n-1}} \left(1 - q^{n-1} \displaystyle\sum_{\substack{\mu \subset \lambda: \\ |\mu|=|\lambda|-1}} \frac{f^\mu(q)}{f^\lambda(q)} \right) & \text{if } s = 1, \\ \dfrac{1}{1-q^{n-1}} & \text{if } s \geq 2. \end{cases}$$

One can rephrase the $s = 1$ case similarly to Lemma 4.7(ii). In the hook case, this gives the following.

Corollary 4.9. *For U in Cusp_1 and $0 \leq k \leq n-1$, one has*

$$\widetilde{\chi}^{U,(n-k,1^k)}(t) = \frac{1-q^{n-k-1}}{1-q^{n-1}}.$$

5 Proofs of Theorems 1.2 and 1.3.

In proving the main results Theorems 1.2 and 1.3, we take for granted the equivalences between the various formulas that they assert. We assemble the normalized character values on reflection conjugacy class sums in the form needed to apply (2.1). This is then used to prove Theorem 1.3 for $q > 2$, from which we derive Theorem 1.2 for $q > 2$. Lastly we prove Theorem 1.2 for $q = 2$.

(ii) In checking this equivalence, it is useful to bear in mind that $f^{\lambda^t}(q) = q^{\binom{n}{2}} f^\lambda(q^{-1})$, along with the fact that if $\mu \subset \lambda$ with $|\mu| = |\lambda| - 1$ and the unique cell of λ/μ lies in row i and column j, then $n(\lambda) - n(\mu) = i - 1$ and $n(\lambda^t) - n(\mu^t) = j - 1$.

(iii) The authors thank A. Ram and P. Diaconis for pointing them to this work.

(iv) In seeing this equivalence, note that Hildebrand uses Macdonald's indexing [14, p. 278] of GL_n-irreducibles, where partition values are transposed in the functions $\underline{\lambda} : \mathrm{Cusp} \longrightarrow \mathrm{Par}$ relative to our convention in Section 3

5.1 The normalized characters on reflection conjugacy class sums

Definition 5.1. For α in $\mathbb{F}_q^\times$, let $z_\alpha := \sum_{t:\det(t)=\alpha} t$ in $\mathbb{C}GL_n$ be the sum of reflections of determinant α.

Corollary 5.2. *For U in Cusp_s, and k in the range $0 \le k \le \frac{n}{s}$, and any α in $\mathbb{F}_q^\times \setminus \{1\}$, one has*

$$\widetilde{\chi}^{U,\left(\frac{n}{s}-k,1^k\right)}(z_\alpha) = [n]_q \begin{Bmatrix} q^{n-k-1}U(\alpha) & \text{if } s=1 \\ 0 & \text{if } s \ge 2. \end{Bmatrix}, \tag{5.1}$$

$$\widetilde{\chi}^{U,\left(\frac{n}{s}-k,1^k\right)}(z_1) = [n]_q \begin{Bmatrix} q^{n-k-1}-1 & \text{if } s=1, \\ -1 & \text{if } s \ge 2. \end{Bmatrix}. \tag{5.2}$$

5.2 Proof of Theorem 1.3 for $q > 2$.

For a Singer cycle c in $GL_n(\mathbb{F}_q)$, and $\alpha = (\alpha_1, \ldots, \alpha_\ell)$ in $(\mathbb{F}_q^\times)^\ell$ with $\prod_{i=1}^\ell = \det(c)$, Proposition 2.2 counts the reflection factorizations $c = t_1 t_2 \cdots t_\ell$ with $\det(t_i) = \alpha_i$ as

$$\frac{1}{|GL_n|} \sum_{\substack{(s,U): \\ s|n \\ U \in \mathrm{Cusp}_s}} \sum_{k=0}^{\frac{n}{s}-1} \deg(\chi^{U,\left(\frac{n}{s}-k,1^k\right)}) \cdot \chi^{U,\left(\frac{n}{s}-k,1^k\right)}(c^{-1}) \cdot \prod_{i=1}^{\ell} \widetilde{\chi}^{U,\left(\frac{n}{s}-k,1^k\right)}(z_{\alpha_i}). \tag{5.3}$$

There are several simplifications in this formula.

Firstly, note that the outermost sum over pairs (s,U) reduces to the pairs with $s=1$: since $\det(c)$ is a primitive root in $\mathbb{F}_q^\times$ by Proposition 4.5 and $q > 2$, one knows that $\det(c) \neq 1$, so that at least one of the α_i is not 1. Thus its factor $\widetilde{\chi}^{U,\left(\frac{n}{s}-k,1^k\right)}(z_{\alpha_i})$ in the last product will vanish if $s \ge 2$ by (5.1).

Secondly, when $s=1$ then Corollary 5.2 evaluates the product in (5.3) as

$$\prod_{i=1}^{\ell} \widetilde{\chi}^{U,\left(\frac{n}{s}-k,1^k\right)}(z_{\alpha_i}) = [n]_q^\ell \ (q^{n-k-1}-1)^m \ q^{(n-k-1)(\ell-m)} \ U(\det(c)) \tag{5.4}$$

if exactly m of the α_i are equal to 1, that is, if the number of transvections in the factorization is m. This justifies calling it $t_q(n,\ell,m)$ where $m \le \ell - 1$.

Thirdly, for $s=1$ Proposition 4.6(iii) shows[(v)] that $\chi^{U,\left(n-k,1^k\right)}(c^{-1}) = (-1)^k U(\det(c^{-1}))$, so there will be cancellation of the factor $U(\det(c))$ occurring in (5.4) within each summand of (5.3).

Thus plugging in the degree formula from the $s=1$ case of (4.1), one obtains the following formula for (5.3), which we denote by $t_q(n,\ell,m)$, emphasizing its dependence only on ℓ and m, not on α:

$$t_q(n,\ell,m) = \frac{(q-1)[n]_q^\ell}{|GL_n|} \sum_{k=0}^{n-1} q^{\binom{k+1}{2}} \begin{bmatrix} n-1 \\ k \end{bmatrix}_q (-1)^k \, (q^{n-k-1}-1)^m \, q^{(n-k-1)(\ell-m)}.$$

Finally, since $|GL_n| = q^{\binom{n}{2}}(-1)^n (q;q)_n$, this last expression may be rewritten using (1.5) to give (1.7). This completes the proof of Theorem 1.3 for $q > 2$.

(v) Here we use the fact that c^{-1} is also a Singer cycle.

5.3 Proof of Theorem 1.2 when $q > 2$.

For $q > 2$, Theorem 1.2 follows from Theorem 1.3 by a careful computation, summing over all sequences $(\alpha_1, \dots, \alpha_\ell)$ factoring a given non-identity element in $\mathbb{F}_q^\times$.

5.4 Proof of Theorem 1.2 when $q = 2$.

Here all reflections are transvections and (2.1) gives us

$$\begin{aligned} t_q(n,\ell) &= \frac{1}{|GL_n|} \sum_{\chi^{\underline{\lambda}} \in \mathrm{Irr}(GL_n)} \deg(\chi^{\underline{\lambda}}) \cdot \chi^{\underline{\lambda}}(c^{-1}) \cdot \widetilde{\chi}^{\underline{\lambda}}(z_1)^\ell \\ &= \frac{1}{|GL_n|} \sum_{\substack{(s,U): \\ s|n \\ U \in \mathrm{Cusp}_s}} \underbrace{\sum_{k=0}^{\frac{n}{s}-1} \deg(\chi^{U,(\frac{n}{s}-k,1^k)}) \cdot \chi^{U,(\frac{n}{s}-k,1^k)}(c^{-1}) \cdot \widetilde{\chi}^{U,(\frac{n}{s}-k,1^k)}(z_1)^\ell}_{\text{Call this } f(s,U)} \end{aligned}$$

using the vanishing of $\chi^{\underline{\lambda}}(c^{-1})$ from Proposition 4.6(i,ii). We separate the computation into $s = 1$ and $s \geq 2$, and first compute $\sum_{U \in \mathrm{Cusp}_1} f(s,U)$. As $q = 2$ there is only one U in Cusp_1, namely $U = \mathbf{1}$, and hence

$$\begin{aligned} \sum_{U \in \mathrm{Cusp}_1} f(s,U) = f(1,\mathbf{1}) &= \sum_{k=0}^{n-1} \deg(\chi^{\mathbf{1},(n-k,1^k)}) \cdot \chi^{\mathbf{1},(n-k,1^k)}(c^{-1}) \cdot \widetilde{\chi}^{\mathbf{1},(n-k,1^k)}(z)^\ell \\ &= \sum_{k=0}^{n-1} q^{\binom{k+1}{2}} \begin{bmatrix} n-1 \\ k \end{bmatrix}_q \cdot (-1)^k \cdot [n]_q^\ell (q^{n-k} - q^{n-k-1} - 1)^\ell \end{aligned}$$

using the degree formula (4.1) at $s = 1$, the fact that $\chi^{(\mathbf{1},n-k,1^k)}(c^{-1}) = (-1)^k \chi^{\mathbf{1},(n)}(c^{-1}) = (-1)^k$ from Proposition 4.6(iii), and the value $\widetilde{\chi}^{\mathbf{1},(n-k,1^k)}(z_1) = [n]_q (q^{n-k} - q^{n-k-1} - 1)$ from (5.2).

For $s \geq 2$, we compute

$$\begin{aligned} \sum_{\substack{(s,U): \\ s|n, s\geq 2 \\ U \in \mathrm{Cusp}_s}} f(s,U) &= \sum_{\substack{(s,U): \\ s|n, s\geq 2 \\ U \in \mathrm{Cusp}_s}} \sum_{k=0}^{\frac{n}{s}-1} \deg(\chi^{U,(\frac{n}{s}-k,1^k)}) \cdot \chi^{U,(\frac{n}{s}-k,1^k)}(c^{-1}) \cdot \widetilde{\chi}^{U,(\frac{n}{s}-k,1^k)}(z_1)^\ell \\ &= \sum_{\substack{s|n \\ s\geq 2}} \sum_{k=0}^{\frac{n}{s}-1} \frac{(-1)^{n-\frac{n}{s}} q^{s\binom{k+1}{2}} (q;q)_n}{(q^s;q^s)_{\frac{n}{s}}} \begin{bmatrix} \frac{n}{s}-1 \\ k \end{bmatrix}_{q^s} \cdot \left(\sum_{U \in \mathrm{Cusp}_s} \chi^{U,(\frac{n}{s}-k,1^k)}(c^{-1}) \right) \cdot (-[n]_q)^\ell \end{aligned}$$

again via (4.1), Proposition 4.6(iii), and (5.2). The parenthesized sum is $(-1)^{n-\frac{n}{s}-k}\mu(s)$ by Proposition 4.6(iii, iv), so

$$\sum_{\substack{(s,U):\\ s|n, s\geq 2\\ U\in \mathrm{Cusp}_s}} f(s,U) = (-[n]_q)^\ell (q;q)_n \sum_{\substack{s|n\\ s\geq 2}} \frac{1}{(q^s;q^s)_{\frac{n}{s}}} \left(\sum_{k=0}^{\frac{n}{s}-1} (-1)^k q^{s\binom{k+1}{2}} \begin{bmatrix} \frac{n}{s}-1 \\ k \end{bmatrix}_{q^s} \right) \mu(s)$$

$$= (-[n]_q)^\ell (q;q)_{n-1} \sum_{\substack{s|n\\ s\geq 2}} \mu(s)$$

$$= -(-[n]_q)^\ell (q;q)_{n-1},$$

where the second equality uses the q-binomial theorem [5, p. 25, Exer. 1.2(vi)]. Thus one has for $q=2$ that

$$t_q(n,\ell) = \frac{1}{|GL_n|}\left(-(-[n]_q)^\ell (q;q)_{n-1} + \sum_{k=0}^{n-1} q^{\binom{k+1}{2}} \begin{bmatrix} n-1 \\ k \end{bmatrix}_q \cdot (-1)^k \cdot [n]_q^\ell (q^{n-k} - q^{n-k-1} - 1)^\ell \right). \tag{5.5}$$

Since $|GL_n| = (-1)^n q^{\binom{n}{2}} (q;q)_n$, one finds that (5.5) agrees with the expression (1.2) after redistributing the $[n]_q^\ell$ and powers of -1. This completes the proof of Theorem 1.2 for $q=2$.

6 An open problem

Empirical evidence supports the following hypothesis regarding the *regular elliptic elements* of $GL_n(\mathbb{F}_q)$ that appeared in Proposition 4.3.

Conjecture 6.1. *The number of ordered reflection factorizations $g = t_1 t_2 \cdots t_\ell$ is the same for all regular elliptic elements g in $GL_n(\mathbb{F}_q)$, namely the quantity $t_q(n,\ell)$ that appears in Theorem 1.2.*

Conjecture 6.1 has been verified for $n=2$ and $n=3$ using explicit character values [19]. In the case $\det g \neq 1$, only minor modifications are required in our arguments to prove Conjecture 6.1. The spot in our proof that breaks down for regular elliptic elements with $\det g = 1$ is Proposition 4.6(iv). For example, when $s=n=4$ and $q=2$, if one chooses β in $\mathbb{F}_{2^4}^\times$ with $\beta^5 = 1$ (so still one has $\mathbb{F}_{2^4} = \mathbb{F}_2(\beta)$, but $\mathbb{F}_{2^4}^\times \neq \langle\beta\rangle$), then $\sum_U \chi^{U,(4)} = -3 \quad (\neq 0 = \mu(4))$. Nevertheless, in this $GL_4(\mathbb{F}_2)$ example, the regular elliptic g with $g^5 = 1$ have the same number of factorizations into ℓ reflections for all ℓ as does a Singer cycle in $GL_4(\mathbb{F}_2)$.

References

[1] G. Chapuy and C. Stump, Counting factorizations of Coxeter elements into products of reflections, arXiv preprint `arXiv:1211.2789`.

[2] J. Dénes, The representation of a permutation as the product of a minimum number of transpositions and its connection with the theory of graphs, *Publ. Math. Institute Hung. Acad. Sci.* **4** (1959), pp. 63–70.

[3] F.G. Frobenius, Uber Gruppencharacktere (1896), in Gesammelte Abhandlungen III, Springer-Verlag, 1968.

[4] A.M. Garsia and M. Haiman, A random (q, t)-hook walk and a sum of Pieri coefficients, *J. Combin. Theory, Ser. A* **82** (1998), 74–111.

[5] G. Gasper and M. Rahman, Basic Hypergeometric Series, second edition, Cambridge University Press, Cambridge, 2004.

[6] J.A. Green, The characters of the finite general linear groups, *Trans. Amer. Math. Soc.* **80** (1955), 402–447.

[7] M. Hildebrand, Generating random elements in $SL_n(\mathbb{F}_q)$ by random transvections, *J. Algebraic Combin.* **1** (1992), 133–150.

[8] A. Hurwitz, Ueber Riemann'sche Flächen mit gegebenen Verzweigungspunkten, *Math. Ann.* **39** (1891), 1–60.

[9] D.M. Jackson, Counting cycles in permutations by group characters, with an application to a topological problem, *Trans. Amer. Math. Soc.* **299** (1987), 785–801.

[10] D.M. Jackson, Some combinatorial problems associated with products of conjugacy classes of the symmetric group, *J. Combin. Theory Ser. A* **49** (1988), 363–369.

[11] S.V. Kerov, q-analogue of the hook walk algorithm and random Young tableaux, *Funktsional. Anal. i Prilozhen.* **26** (1992), no. 3, 35–45; translation in *Funct. Anal. Appl.* **26** (1992), no. 3, 179–187.

[12] S. Lang, Algebra, revised third edition, *Graduate Texts in Mathematics* **211**, Springer-Verlag, New York, 2002.

[13] J.B. Lewis, V. Reiner, and D. Stanton, Reflection factorizations of Singer cycles, arXiv preprint `arXiv:1308.1468`. To appear, *J. Algebraic Combin.*

[14] I.G. Macdonald, Symmetric functions and Hall polynomials, second edition, *Oxford Mathematical Monographs*. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1995.

[15] A.J. Silberger and E.-W. Zink, The characters of the generalized Steinberg representations of finite general linear groups on the regular elliptic set, *Trans. Amer. Math. Soc.* **352** (2000), 3339–3356.

[16] R.P. Stanley, Enumerative combinatorics, vol. 2, *Cambridge Studies in Advanced Mathematics* **62**, Cambridge University Press, Cambridge, 1999.

[17] V. Reiner, D. Stanton, and P. Webb, Springer's regular elements over arbitrary fields, *Math. Proc. Cambridge Philos. Soc.* **141** (2006), 209–229.

[18] V. Reiner, D. Stanton, and D. White, The cyclic sieving phenomenon, *J. Combin. Theory Ser. A* **108** (2004), 17–50.

[19] R. Steinberg, The representations of $GL(3, q), GL(4, q), PGL(3, q)$, and $PGL(4, q)$, *Canadian J. Math.* **3**, (1951), 225–235.

[20] A.V. Zelevinsky, Representations of finite classical groups: a Hopf algebra approach, *Lecture Notes in Mathematics* **869**. Springer-Verlag, Berlin-New York, 1981.