How to count genus one partitions

Robert Cori[1]* and Gábor Hetyei[2]†

[1] *Labri, Université Bordeaux 1, France*
[2] *Department of Mathematics and Statistics, UNC-Charlotte, Charlotte NC, USA*

Abstract. We prove the conjecture by M. Yip stating that counting genus one partitions by the number of their elements and parts yields, up to a shift of indices, the same array of numbers as counting genus one rooted hypermonopoles. Our proof involves representing each genus one permutation by a four-colored noncrossing partition. This representation may be selected in a unique way for permutations containing no trivial cycles. The conclusion follows from a general generating function formula that holds for any class of permutations that is closed under the removal and reinsertion of trivial cycles. Our method also provides another way to count rooted hypermonopoles of genus one, and puts the spotlight on a class of genus one permutations that is invariant under an obvious extension of the Kreweras duality map to genus one permutations.

Résumé. Nous démontrons la conjecture de M. Yip affirmant que compter les partitions de genre un par le nombre de leurs éléments et leurs parties donne, à une décalage d'indices près, la même gamme de nombres que celle qui résulte en comptant les hypermonopoles de genre un. Notre preuve utilise une représentaton de chaque permutation de genre un par une partition non-croisé quatricolorée. Cette représentation peut être choisi d'une manière unique pour les permutations qui ne contiennent pas des cycles triviaux. La conclusion suit d'une formule des fonctions génératrices générale qui vaut pour toute classe de permutations qui est fermé sous l'enlèvement et la reinsertion des cycles triviaux. Notre méthode offre une autre manière de compter les hypermonopoles enracinés de genre un, et dirige l'attention sur une extension évident de la dualité de Kreweras sur les permutations de genre un.

Keywords: set partitions, noncrossing partitions, genus of a hypermap

Introduction

Noncrossing partitions, first defined in G. Kreweras' seminal paper [13], have a vast literature. Using combinatorial results on the representation of graph embeddings by permutations (see [1, Section 4.1] or [21]) leads to define the genus of a permutation. This definition uses the number of cycles of it and that of its product with the circular permutation $(1, 2, \ldots, n)$. It is not difficult to prove that a permutation has genus 0 if and only if its cycles defines a partition which is noncrossing.

It is then natural to define the genus of partitions by considering permutations whose cycles may be written as lists of elements in increasing order and to consider partitions of genus 1 trying to prove properties for them similar to those of noncrossing ones. This was done by M. Yip in her Master thesis where

*Email: cori@labri.fr. Partially supported by the ANR project (BLAN-0204.07 Magnum to Robert Cori)
†Email: ghetyei@uncc.edu. Partially supported by a grant from the Simons Foundation (#245153 to Gábor Hetyei)

she collected a great deal of numerical evidence, and made the following conjecture [21, Conjecture 3.15]: the number of genus 1 partitions on n elements and k parts is the same as the number of genus one permutations of $n-1$ elements having $k-1$ cycles. We will outline a proof of this conjecture and provide further insight into the structure of genus 1 partitions and permutations. This will allow us to give a simple description of genus 1 permutations and partitions which is illustrated in Figure 2 below.

Counting partitions of a given genus, greater than 1, seems surprisingly hard, considering the fact that, for permutations, a general machinery was built by S. Cautis and D. M. Jackson [1] and some explicit formulas were given by A. Goupil and G. Schaeffer [8].

We will proceed as follows. After reviewing some basic terminology and facts in Section 1, in Section 2 we outline a way to represent every permutation of genus 1 by a four-colored noncrossing partition. In Section 3 we will see that, if the permutation of genus 1 is *reduced* in the sense that it contains no *trivial cycle* (consisting of consecutive elements in the circular order) then we may select a unique four-colored noncrossing partition representation of our permutation which we call the canonical representation. This unicity enables us to count reduced permutations and partitions of genus 1 in Section 4. We only need to account for the possibility of having trivial cycles. In Section 5 we show how to do this. We then combine the formula stated in Theorem 5.3 with the generating function formulas stated in Section 4 and obtain the generating function formulas counting genus 1 permutations and partitions with given number of permuted elements and cycles. Since the resulting formulas stated in Theorems 6.1 and 6.3 differ only by a factor of xy, the validity of M. Yip's conjecture is at this point verified. In Section 7 we show how to extract the coefficients from our generating functions to find the number of partitions of genus 1. The generalized formula stated in Section 7 links the problem of counting genus 1 permutations and partitions to the problem of counting type B noncrossing partitions, convex polyominos and Jacobi configurations (at least numerically). The explanation of these connections, together with ideas of possible simplifications and further questions, are collected in the concluding Section 8.

This extended abstract omits most proofs to meet the 12 page limit. Full details may be found in the full preprint [4].

1 On the genus of permutations and partitions

A hypermap is a pair of permutations (σ, α) on a set $\{1, 2, \ldots, n\}$, generating a transitive permutation group. Its *genus* $g(\sigma, \alpha)$, a nonnegative integer (see [12]) is defined by

$$n + 2 - 2g(\sigma, \alpha) = z(\sigma) + z(\alpha) + z(\alpha^{-1}\sigma) \quad (1)$$

where $z(\alpha)$ is the number of cycles of α. The relation between the combinatorial and topological notions of genus, as presented in in [7] and [11], will be explained in more detail in our presentation. The genus $g(\alpha)$ of a permutation α is defined to be $g(\alpha, \zeta_n)$ where ζ_n is permutation whose only cycle is $(1, 2, \ldots, n)$. Thus we have $n + 1 - 2g(\alpha) = z(\alpha) + z(\alpha^{-1}\zeta_n)$. To a partition $P = (P_i)_{i=1,k}$ of $\{1, 2, \ldots n\}$ we associate the permutation α_P with k cycles, each one corresponding to one of the P_i written with the elements in increasing order. We define the genus of the partition P to be $g(\alpha_P)$. A permutation α is of genus 0, if and only if $\alpha = \alpha_P$ for some *noncrossing partition* P (see [5, Theorem 1]) . Two cycles in a permutation α are *crossing* if there exist two elements a, a' in one of them and b, b' in the other such that $a < b < a' < b'$. An element i of $1, 2, \ldots, n$ is a *back point* of the permutation α if $\alpha(i) < i$ and $\alpha(i)$ is not the smallest element in its cycle (i. e. there exists k such that $\alpha^k(i) < \alpha(i)$). A *twisted cycle* is a cycle $(b_1, b_2, \ldots, b_p)$ containing a back point.

The genus of a permutation may be determined by counting back points as the following variant of [2, Lemma 5] shows.

Lemma 1.1 *For any permutation $\alpha \in \mathrm{Sym}\,(n)$, the sum of the number of back points of the permutation α and the number of those of $\alpha^{-1}\zeta_n$ is equal to $2g(\alpha)$.*

A permutation is associated to a partition if and only if it contains no back point, moreover the partition and the associated permutation are of genus 0 if and only if there are no crossing cycles.

2 Genus one permutations and four-colored noncrossing partitions

We define a four-coloring of a noncrossing partition of the set $\{1, 2, \dots, n\}$ as a partitioning of the n points on the circle into four arcs denoted A, B, C, D in clockwise order where A is the arc containing the point 1 and in which C is only arc allowed to contain no point. We will denote by $\gamma = (A, B, C, D)$ such a 4-coloring. The color sets are uniquely given by setting four integers $i < j \leq k < \ell$ such that

$$\begin{array}{ll} A = \{\ell+1, \dots, n, 1, \dots, i\}, & B = \{i+1, \dots, j\}, \\ C = \{j+1, \dots, k\}, & D = \{k+1, \dots, \ell\}, \end{array} \tag{2}$$

We call (i, j, k, ℓ) a *sequence of coloring points* for the partition P. To any four-colored noncrossing partition (P, γ) we associate a permutation $\alpha = \Phi(P, \gamma)$ in which cycles are obtained from the parts of P by renumbering the points as follows. We leave the numbering of the points in A unchanged and we continue labeling consecutively the elements of D, C, and finally B, in this order. Within each color set, points are labeled in clockwise order. After introducing

$$a = i,\ b = i + \ell - k,\ c = i + \ell - j, \quad \text{and} \quad d = \ell, \tag{3}$$

we obtain that the color sets, in terms of the relabeled elements, are given by

$$\begin{array}{rcl} A & = & \left\{ \begin{array}{ll} \{1, 2, \dots a, d+1, \dots, n\} & \text{if } d \neq n, \\ \{1, 2, \dots, a\} & \text{otherwise;} \end{array} \right. \\ B & = & \{c+1, c+2, \dots, d\}; \quad D = \{a+1, a+2, \dots, b\}; \\ C & = & \left\{ \begin{array}{ll} \{b+1, b+2, \dots, c\} & \text{if } c \neq b, \\ \emptyset & \text{otherwise.} \end{array} \right. \end{array} \tag{4}$$

The linear map taking (i, j, k, ℓ) into (a, b, c, d) is its own inverse:

$$i = a,\ j = a + d - c,\ k = a + d - b \quad \text{and} \quad \ell = d. \tag{5}$$

Once the points are renumbered, each cycle of α is obtained from a part $P_q = \{x_1, x_2, \dots x_p\}$ of P by writing the numbering of the corresponding points $x_1, x_2, \dots x_p$, where the x_i's are in clockwise order.

For the example shown in Figure 1 we obtain the following permutation of genus 1:

$$\alpha \ = \ \Phi(P, \gamma) = (1, 4, 3, 8)(2, 7)(5)(6)$$

We will say that a point p has color X for $X = A, B, C, D$ if $p \in X$, a part P_q will be unicolored, bicolored, three-colored or four-colored depending on the number of different colors its points have.

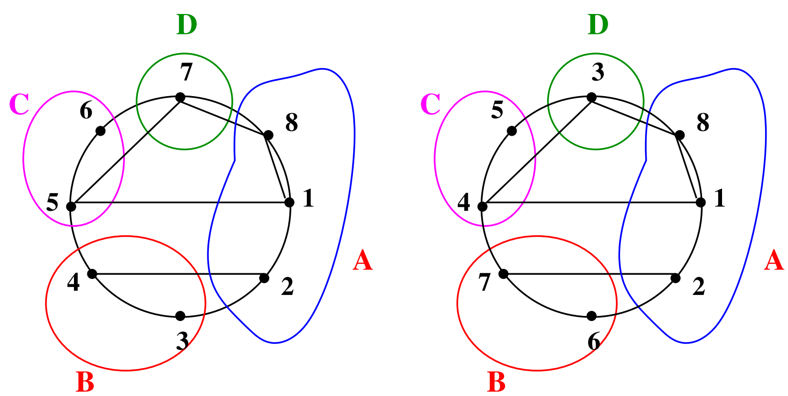

Fig. 1: A four-coloring of P and the induced renumbering of points

Theorem 2.1 *If (P, γ) is a four-colored noncrossing partition then $\Phi(P, \gamma)$ is a permutation of genus 0 or 1. It is of genus 1 if and only if at least one of these two conditions is satisfied:*

1. *There exists a part P_q which is three or four-colored.*
2. *There exists two parts P_q, P_r which are two colored and share a common color, more precisely there are three different colors X, Y, Z such that*

$$P_q \cap X \neq \emptyset,\ \ P_q \cap Y \neq \emptyset,\ \ P_q \subseteq X \cup Y \quad \text{and} \quad P_r \cap X \neq \emptyset,\ \ P_r \cap Z \neq \emptyset,\ \ P_r \subseteq X \cup Z.$$

In the proof of the converse of Theorem 2.1 we used the following notion:

Definition 2.2 *Let α be a permutation of genus 1. We say that the sequence of integers (a, b, c, d) is a* sequence of separating points *for α if the permutation $\theta = \zeta_n(a, c)(b, d)$ is such that the genus of the hypermap (θ, α) is zero and*

$$a < b \leq c < d. \tag{6}$$

It is not hard to see that whenever a permutation α of genus 1 is represented as $\alpha = \Phi(P, \gamma)$ by a four-colored noncrossing partition (P, γ) then the sequence of coloring points (i, j, k, ℓ) gives rise to the sequence of separating points (a, b, c, d) given by (3). The converse is also true.

Proposition 2.3 *Let α be a permutation of genus 1 on n elements that has a sequence of separating points (a, b, c, d). Then there is a noncrossing partition P and a four-coloring $\gamma = (A, B, C, D)$ representing α as $\alpha = \Phi(P, \gamma)$ whose sequence of coloring points (i, j, k, ℓ) is obtained from (a, b, c, d) via (5).*

Definition 2.4 *We call the representation described in Proposition 2.3* the four-colored noncrossing partition representation induced by the sequence of separating points (a, b, c, d).

Now we are ready to state the converse of Theorem 2.1.

Theorem 2.5 *For any permutation α of genus 1, there exists a noncrossing partition P and a four-coloring γ such that $\alpha = \Phi(P, \gamma)$.*

Corollary 2.6 *A permutation α of genus 1 is a partition if and only if it may be represented by a four-colored noncrossing partition (Q, γ) that has no three or four-colored part and has at least two two-colored parts.*

Remark 2.7 *Every genus 1 partition $\alpha \in \mathrm{Sym}\,(n)$ has a three-colored non-crossing partition representation, that is, a four-colored representation with $C = \emptyset$.*

Theorem 2.5 has an interesting topological consequence. Consider a torus represented as a square whose parallel sides are identified. Draw a "large circle" as seen in Figure 2. This circle represents a closed curve, apparently consisting of four disjoint arcs that are contiguous due to the identification rules. If we put n numbered points on this curve in cyclic order, the visual order of these numbered points in the picture will correspond to the visual order of the points in a four-colored representation, after the relabeling. The four color classes correspond to the four arcs. The four-colored representation of the permutation $(1438)(27)(5)(6)$, given in Figure 2, naturally induces a representation of the same permutation on the torus as a union of "polygons" with noncrossing edges. We obtain that every genus

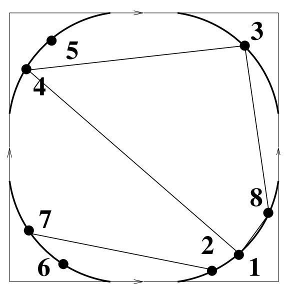

Fig. 2: A representation of a genus 1 permutation on the torus

one permutation may be drawn on a torus in such a way that the permuted elements are on a simple closed curve in the correct cyclic order and the cycles are represented by noncrossing polygons bounded by geodesic lines.

Definition 2.8 *A cycle of α is simply twisted if it contains exactly one back point and it is doubly twisted if it has two back points.*

Remark 2.9 *In a four-colored noncrossing partition representation of a permutation of genus 1, three colored parts correspond to simply twisted cycles and four-colored parts correspond to doubly twisted cycles.*

3 Reduced permutations and partitions

Definition 3.1 *A* trivial *cycle in a permutation is a cycle consisting of consecutive points on the circle,* i. e. *a cycle $C_i = (i, i+1, \ldots, i+p)$ where sums are taken modulo n. A permutation is* reduced *if it contains no trivial cycle.*

It is easy to show that a permutation of genus 1 is reduced if and only if each of its cycles either crosses another one or it is twisted.

Definition 3.2 *Let α be a reduced permutation of genus 1. The* canonical sequence of separating points (a, b, c, d) *of α is defined as follows:*

1. *a is the smallest integer such that $\alpha(a) \neq a + 1$;*
2. *b is the smallest integer satisfying $b > a$ and such that either $\alpha(b) > \alpha(a)$ or $\alpha(b) \leq a$ holds;*
3. *$c = \alpha(a) - 1$;*
4. *$d = n$ if $\alpha(b) = 1$ and $d = \alpha(b) - 1$ otherwise.*

We call the four-colored noncrossing partition representation induced by the canonical sequence of separating points the canonical representation *of α.*

Proposition 3.3 *Every reduced permutation of genus 1 of has a unique canonical sequence (a, b, c, d) of separating points, that induces a four-colored noncrossing partition representation.*

Proposition 3.4 *Let $\alpha = \Phi(\beta, \gamma)$ be the representation of the reduced permutation α of genus 1 induced by its canonical sequence of separating points (a, b, c, d). This representation has the following properties:*

1. *$a < b \leq c < d$ and $\alpha(a) \equiv c + 1, \alpha(b) \equiv d + 1 \mod n$.*
2. *If x and $\alpha(x)$ are in the same subset A, B, C, or D then $\alpha(x) \equiv x + 1 (\mathrm{mod}\, n)$.*
3. *There is no cycle of α containing elements in both A and D except the one containing b and $d + 1$.*
4. *There is no cycle of α containing elements in both B and D except if this cycle is twisted and contains $b \in D, d + 1 \in A$ and an element $x \in B$.*

Proposition 3.5 *Let α be a reduced permutation of genus 1, represented as $\alpha = \Phi(\beta, \gamma)$ by a four-colored noncrossing partition. If this representation satisfies the properties stated in Proposition 3.4 then it is the representation induced by the canonical sequence of separating points.*

Corollary 3.6 *The canonical four-colored noncrossing partition representation of a reduced permutation α of genus 1 may be equivalently defined by requiring that the sequence of separating points inducing it must satisfy the four conditions of Proposition 3.4.*

4 Counting reduced partitions and permutations

4.1 Counting reduced partitions of genus 1

Using the results of the preceding section, one may show the following.

Lemma 4.1 *A reduced partition of genus 1 having k parts is determined by a subset of $2k$ integers in $\{1, 2, \ldots, n\}$ and a sequence of four non-negative integers whose sum is $k - 2$.*

Theorem 4.2 *The number $r_0(n, k)$ of reduced partitions of genus 1, of the set $\{1, \ldots, n\}$, having k blocks is*

$$r_0(n, k) = \binom{n}{2k}\binom{k+1}{3}.$$

Moreover, the ordinary generating function of these partitions is given by

$$R_0(x, y) = \sum_{n,k \geq 0} r_0(n, k) x^n y^k \; = \; \frac{y^2 x^4 (1-x)^3}{((1-x)^2 - yx^2)^4}. \tag{7}$$

Corollary 4.3 *Let $r_0(n)$ be the number of all reduced genus 1 partitions on $\{1,\ldots,n\}$. Then the generating function $R_0(x)=\sum_{n\geq 4} r_0(n)x^n$ is given by $R_0(x)=x^4(1-x)^3/(1-2x)^4$.*

The sequence $r_0(4), r_0(5), \ldots$ appears as sequence A049612 in [14]. They also form the third row of the array given as sequence A049600. This array is essentially the same as the array of *asymmetric Delannoy numbers* in [9].

4.2 Counting reduced permutations of genus 1

Theorem 4.4 *The number of reduced permutations of genus 1 of $\mathrm{Sym}\,(n)$ with k cycles is equal to:*

$$r_*(n,k)=\binom{n+2}{2k+2}\binom{k+1}{3}+\binom{n+1}{2k+2}\binom{k+1}{2}.$$

More precisely, for $j=0,1,2$, the number $r_j(n,k)$ of reduced permutations of genus 1 of $\mathrm{Sym}\,(n)$ with j back points and k cycles is given by the following formulas:

$$r_0(n,k)=\binom{n}{2k}\binom{k+1}{3},\quad r_2(n,k)=\binom{n}{2k+2}\binom{k+2}{3}\quad \textit{and}$$

$$r_1(n,k)=\binom{n}{2k+1}\left(\binom{k+2}{3}+\binom{k+1}{3}\right).$$

Proposition 4.5 *The ordinary generating function for the reduced permutations of genus 1, counting the number of points and cycles, is given by:*

$$R_*(x,y)=\frac{yx^3(1-x)^2(1-x+xy)}{((1-x)^2-yx^2)^4}.$$

More precisely, for $j=0,1,2$, the ordinary generating function for the reduced permutations of genus 1 with j back points, counting the number of points and cycles, is given by:

$$R_0(x,y)=\frac{y^2x^4(1-x)^3}{((1-x)^2-yx^2)^4},\quad R_2(x,y)=\frac{yx^4(1-x)^3}{((1-x)^2-yx^2)^4}\quad \textit{and}$$

$$R_1(x,y)=\frac{yx^3(1-x)^2((1-x)^2+yx^2)}{((1-x)^2-yx^2)^4}.$$

5 Reducing permutations and reinserting trivial cycles

Definition 5.1 A trivial reduction *π' of a permutation π of $\{1,2,\ldots,n\}$ is a permutation obtained from π by removing a trivial cycle $(i,i+1,\ldots,j)$ and decreasing all $k\in\{j+1,j+1,\ldots,n\}$ by $j-\min(0,i-1)$ in the cycle decomposition of π.*

Clearly π' is a permutation of $\{1,\ldots,n'\}$ for $n'=n-|\{i,i+1,\ldots,j\}|$ and has the same genus as π. Conversely, we will say that π is a *trivial extension* (or an extension) of π'. A permutation is *reduced* exactly when it has no trivial reduction. For technical reasons we postulate that *the empty permutation is a reduced permutation of the empty set*.

Proposition 5.2 *For any permutation π of positive genus there is a unique reduced permutation π' that may be obtained by performing a sequence of reductions on π. If π has genus zero then this reduced permutation is the empty permutation on the empty set.*

As a consequence of Proposition 5.2, if a class of permutations is closed under reductions and extensions then we are able to describe this class reasonably well by describing the reduced permutations in the class. The main result of this section states that knowing the number of reduced permutations also allows to *count* all permutation in such a class. Our main result we will use the generating function

$$D(x,y) = \frac{1-x-xy-\sqrt{(x+xy-1)^2-4x^2y}}{2\cdot x}+1 \tag{8}$$

of noncrossing partitions. As it is well-known [14, sequence A001263], $[x^n y^k]D(x,y)$ is the number of noncrossing partitions of the set $\{1,\dots,n\}$ having k parts. Note that we deviate from the usual conventions by defining the constant term to be 1, i.e., we consider that there is one noncrossing partition on the empty set and it has zero blocks.

Theorem 5.3 *Consider a class $\mathcal{C}$ of permutations that is closed under trivial reductions and extensions. Let $p(n,k)$ and $r(n,k)$ respectively be the number of all, respectively all reduced permutations of $\{1,\dots,n\}$ in the class having k cycles. Then the generating functions $P(x,y)=\sum_{n,k}p(n,k)x^n y^k$ and $R(x,y)=\sum_{n,k}r(n,k)x^n y^k$ satisfy the equation*

$$P(x,y) = R(x\cdot D(x,y),y)\cdot\left(1+x\cdot\frac{\frac{\partial}{\partial x}D(x,y)}{D(x,y)}\right).$$

Here $D(x,y)$ is the generating function of noncrossing partitions given in (8).

We conclude this section with rewriting the factor $1+x\cdot\frac{\partial}{\partial x}D(x,y)/D(x,y)$, appearing in Theorem 5.3, in an equivalent form.

Proposition 5.4

$$1+x\cdot\frac{\frac{\partial}{\partial x}D(x,y)}{D(x,y)} = \frac{1-xD(x,y)}{\sqrt{(x+xy-1)^2-4x^2y}}.$$

Corollary 5.5 *The formula stated in Theorem 5.3 is equivalent to stating*

$$P(x,y) = R(x\cdot D(x,y),y)\cdot\frac{1-xD(x,y)}{\sqrt{(x+xy-1)^2-4x^2y}}.$$

6 Counting all partitions and permutations of genus one

Theorem 6.1 *Let the number $p_0(n,k)$ of all partitions of $\{1,\dots,n\}$ of genus one having k parts. Then the generating function*

$$P_0(x,y) = \sum_{n\geq 4}\sum_{k\geq 2} p_0(n,k)x^n y^k$$

is given by the equation

$$P_0(x,y) = \frac{x^4y^2}{(1-2(1+y)x+x^2(1-y)^2)^{5/2}}.$$

We will see in Section 7 that Theorem 6.1 is equivalent to an explicit formula (10) for the numbers $p_0(n,k)$, originally conjectured by M. Yip [21, Conjecture 3.15]. Theorem 6.1 follows from Theorem 5.3, by multiplying the formulas given in Propositions 5.4 and 6.2.

Proposition 6.2 *The generating function $R_0(x,y)$ of reduced partitions of genus one satisfies the equality*

$$R_0(x \cdot D(x,y),y) = \frac{x^4y^2}{(1-xD(x,y))((x+xy-1)^2-4x^2y)^2}.$$

We may follow an analogous procedure to count all permutations of genus 1.

Theorem 6.3 *Let $p_*(n,k)$ be the number of all permutations in $\mathrm{Sym}\,(n)$ of genus one having k cycles. Then the generating function $P_*(x,y)=\sum_{n,k} p_*(n,k)x^ny^k$ is given by the equation*

$$P_*(x,y) = \frac{x^3y}{(1-2(1+y)x+x^2(1-y)^2)^{5/2}}.$$

More precisely, for $j=0,1,2$, let $p_j(n,k)$ be the number of all permutations in $\mathrm{Sym}\,(n)$ of genus one having k cycles and j back points. Then the generating functions $P_j(x,y)=\sum_{n,k} p_j(n,k)x^ny^k$ are given by the formulas

$$P_0(x,y) = \frac{x^4y^2}{(1-2(1+y)x+x^2(1-y)^2)^{5/2}},$$

$$P_2(x,y) = \frac{x^4y}{(1-2(1+y)x+x^2(1-y)^2)^{5/2}} \quad \textit{and}$$

$$P_1(x,y) = \frac{x^3y(1-xy-x)}{(1-2(1+y)x+x^2(1-y)^2)^{5/2}}.$$

7 Extracting the coefficients from our generating functions

In this section we will show how to extract the coefficients from our generating functions to obtain explicit formulas for the numbers of genus 1 partitions and permutations. Our main tool is a generalization of the following equation.

$$\frac{x^4y^2}{(1-2(1+y)x+x^2(1-y)^2)^{5/2}} = \sum_{n\geq 4} \frac{1}{6}\binom{n}{2} x^n \sum_{k=2}^{n-2} \binom{n-2}{k}\binom{n-2}{k-2} y^k. \tag{9}$$

According to this equation, M. Yip's conjecture [21, Conjecture 3.15], stating

$$p_0(n,k) = \frac{1}{6}\binom{n}{2}\binom{n-2}{k}\binom{n-2}{k-2}. \tag{10}$$

is equivalent to our Theorem 6.1 and thus true. Since, by Theorem 6.3, the generating function of genus one permutations only differs by a factor of xy, we also obtain a new way to count these objects, thus providing a new proof of the result first stated by A. Goupil and G. Schaeffer [8].

After dividing both sides by x^4y^2 and shifting n and k down by two, we obtain the following equivalent form of equation (9).

$$\frac{1}{(1-2(1+y)x+x^2(1-y)^2)^{5/2}} = \sum_{n\geq 2} \frac{1}{6}\binom{n+2}{2} x^{n-2} \sum_{k=0}^{n-2} \binom{n}{k+2}\binom{n}{k} y^k. \tag{11}$$

This equation is the special case (when $m = 2$) of Equation (12) below, that holds for all $m \in \mathbb{N}$.

$$\frac{1}{(1-2(1+y)x+x^2(1-y)^2)^{(2m+1)/2}} = \sum_{n\geq m}\sum_{k\geq 0} \frac{\binom{n+m}{m}\binom{n}{k}\binom{n}{m+k}}{\binom{2m}{m}} x^{n-m} y^k. \tag{12}$$

Equation (12) may be obtained from [6, Equation (2)], after substituting $\alpha = (2m+1)/2$ and replacing each appearance of y with xy in that formula (on the right hand side, one also needs to replace the summation indices i, and j respectively, with $n-m-k$ and k, respectively). As pointed out by Strehl [17, p. 180] (see also [6, p. 64]), [6, Equation (2)] is a consequence of classical results in the theory of special functions.

Remark 7.1 *Equation (12) may also be derived directly from classical results as follows. Take the mth derivative with respect to u of the generating function $\sum_{n\geq 0} L_n(u)t^n$ of the Legendre polynomials (given in [3, Ch. V, (2.34)]), multiply both sides by $2^m/(t^m m!)$, use [19, (4.21.2)] to express $L_n(u)$, substitute $u = (1+y)/(1-y)$ and $t = x(1-y)$, and use the Chu-Vandermonde identity.*

We conclude this section with providing explicit formulas for the number of all permutations of genus 1, with a given numbers of points, cycles, and back points.

Theorem 7.2 *The number of all permutations of genus 1 of $\mathrm{Sym}\,(n)$ with k cycles is equal to:*

$$p_*(n,k) = \frac{1}{6}\binom{n+1}{2}\binom{n-1}{k+1}\binom{n-1}{k-1}$$

More precisely, for $j = 0, 1, 2$, the number $p_j(n,k)$ of permutations of genus 1 of $\mathrm{Sym}\,(n)$ with j back points and k cycles is given by the following formulas:

$$p_0(n,k) = \frac{1}{6}\binom{n}{2}\binom{n-2}{k}\binom{n-2}{k-2}, \quad p_2(n,k) = \frac{1}{6}\binom{n}{2}\binom{n-2}{k+1}\binom{n-2}{k-1} \quad \textit{and}$$

$$p_1(n,k) = \frac{1}{3}\binom{n}{2}\binom{n-2}{k}\binom{n-2}{k-1}.$$

8 Concluding remarks

Lemma 1.1 establishes a relationship between α and $\alpha^{-1}\zeta_n$. In the case when $g(\alpha) = 0$, the permutation $\alpha^{-1}\zeta_n$ is known as the *Kreweras dual* of the noncrossing partition represented by α, used by G. Kreweras [13] to show that the lattice of noncrossing partitions is self-dual. M. Yip has shown that the poset of genus 1 partitions is rank-symmetric [21, Proposition 4.5], but not self dual [21, Proposition 4.6] for $n \geq 6$. Lemma 1.1 suggests that maybe true duality could be found between genus 1 partitions and permutations with 2 back points, after defining the proper partial order on the set of all genus 1 permutations.

In this setting, permutations with exactly one back point would form a self-dual subset. Their number $p_1(n,k)$, given in Theorem 7.2, may be rewritten as

$$p_1(n,k) = \binom{n}{3} N(n-2,k-1),$$

where $N(n-2,k-1)$ is a Narayana number. It is a tantalizing thought that this simple formula could have a very simple proof. If this is the case, then the formulas for $p_0(n,k)$ and $p_1(n,k)$ could be easily derived, using Lemma 1.1 and Yip's rank-symmetry result [21, Proposition 4.5] to establish $p_2(n,k) = p_0(n,k+1)$, and then the formula for $p_*(n,k)$ already stated by A. Goupil and Shaeffer [8] to complete a setting in which the formula for $p_0(n,k)$ may be shown by induction on k. A "numerically equivalent" conjecture (albeit for sets of partitions) was stated by M. Yip [21, Conjecture 4.10].

Equation (12) naturally inspires the question: what other combinatorial objects are counted by the coefficients of $x^n y^k$ in the Taylor series of $(1-2(1+y)x+x^2(1-y)^2)^{-(2m+1)/2}$, when m is some other nonnegative integer. For $m=0$, these coefficients count the *type B noncrossing partitions* of rank k of an n-element set. In [16], R. Simion constructed a simplicial polytope in each dimension whose h vector entries are the squares of the binomial coefficients. The number of j-element faces of the n-dimensional polytope is $f_{j-1} = \binom{n+j}{j}$. Another class of simplicial polytopes with the same face numbers was defined in [10] as the class of all simplicial polytopes arising by taking any pulling triangulation of the boundary complex of the "Legendrotope". For higher values of m, taking the mth derivative of $F(u)$ (see Remark 7.1) corresponds to summing over the links of all $(m-1)$ dimensional faces. It is not evident from this interpretation why we should get integer entries, even after dividing by $\binom{2m}{m}$, and it seems an interesting question to see whether for the type B associahedron or for some very regular triangulation of the Legendrotope, symmetry reasons would explain the integrality. For $m=1$, I. Gessel has shown [6] that the coefficients count convex polyominoes. Finally, for general m, the coefficients have a combinatorial interpretation in the work of V. Strehl [18] on Jacobi configurations. Even though V. Strehl uses exponential generating functions, the use of the same coefficients becomes apparent by comparing his summation formula on page 303 with [6, Equation (2)]. It seems worth exploring whether deeper connections exist between the above listed models.

Acknowledgements

The second author wishes to express his heartfelt thanks to Labri, Université Bordeaux I, for hosting him as a visiting researcher in the of Summer 2011 and 2013. This work was partially supported by a grant from the Simons Foundation (#245153 to Gábor Hetyei) and by the ANR project (BLAN-0204.07 Magnum to Robert Cori). We wish to thank to Ira Gessel for pointing us to the right sources to shorten our calculations and to two anonymous referees for their helpful comments and encouragement.

References

[1] S. Cautis and D. M. Jackson, On Tutte's chromatic invariants, *Trans. Amer. Math. Soc.* **362** (2010), 509–535.

[2] G. Chapuy, The structure of unicellular maps, and a connection between maps of positive genus and planar labelled trees, *Probab. Theory Relat. Fields* **147** (2010), 415–447.

[3] T. S. Chihara, “An Introduction to Orthogonal Polynomials,” Gordon and Breach Science Publishers, New York-London-Paris, 1978.

[4] R. Cori and G. Hetyei, Counting genus one partitions and permutations, preprint 2013, arXiv:1306.4628 [math.CO].

[5] R. Cori “Un code pour les Graphes Planaires et ses applications” *Asterisque* **27** (1975).

[6] I. M. Gessel, On the number of convex polyominoes, *Ann. Sci. Math. Québec* **24** (2000), 63–66.

[7] I. P. Goulden and D. M. Jackson, Maps in locally orientable surfaces, the double coset algebra, and zonal polynomials, *Canad. J. Math.* **48** (1996), 569–584.

[8] A. Goupil and G. Schaeffer, Factoring N-cycles and Counting Maps of Given Genus, *European J. Combin.* **19** (1998), 819–834.

[9] G. Hetyei, Central Delannoy numbers and balanced Cohen-Macaulay complexes, *Ann. Comb.* **10** (2006), 443–462.

[10] G. Hetyei, Delannony orthants of Legendre polytopes, *Discrete Comput. Geom.* **42** (2009), 705–721.

[11] D. M. Jackson and T. I. Visentin, Character theory and rooted maps in an orientable surface of given genus: face-colored maps, *Trans. Amer. Math. Soc.* **322** (1990), 365–376.

[12] A. Jacques, Sur le genre d’une paire de substitutions, *C. R. Acad. Sci. Paris* **267** (1968), 625–627.

[13] G. Kreweras, Sur les partitions non croisées d’un cycle, *Discrete Math.* **1** (1972), 333–350.

[14] OEIS Foundation Inc. (2011), “The On-Line Encyclopedia of Integer Sequences,” published electronically at `http://oeis.org`.

[15] J. Rué, I. Saub, D. M. Thilikos, Asymptotic enumeration of non-crossing partitions on surfaces, *Discrete Math.* **313** (2013), 635–649.

[16] R. Simion, A type-B associahedron, *Adv. in Appl. Math.* **30** (2003), 2–25.

[17] V. Strehl, Zykel Enumeration bei lokalstrukturierten Funktionen, Habilitationsschrift, Institut für Mathematische Machinen und Datenverarbeitung der Universität Erlangen Nürnberg, 1990.

[18] V. Strehl, Combinatorics of Jacobi configurations. I. Complete oriented matchings, in: Combinatoire énumérative (Montreal, Que., 1985/Quebec, Que., 1985), 294–307, Lecture Notes in Math., 1234, Springer, Berlin, 1986.

[19] G. Szegő, “Orthogonal Polynomials”, AMS Colloquium Publications Vol. 23, Providence, Rhode Island 2000.

[20] T. R. S. Walsh and A. B. Lehman, Counting rooted maps by genus I, *J. Combinatorial Theory Ser. B* **13** (1972), 192–218.

[21] M. Yip, Genus one partitions, Master’s thesis 2006, Univeristy of Waterloo. Available online at `http://hdl.handle.net/10012/2933`