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Finiteness Conditions for Graph Algebras over Tropical Semirings

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Abstract. Connection matrices for graph parameters with values in a field have been introduced by M. Freedman, L. Lovász and A. Schrijver (2007). Graph parameters with connection matrices of finite rank can be computed in polynomial time on graph classes of bounded tree-width. We introduce join matrices, a generalization of connection matrices, and allow graph parameters to take values in the tropical rings (max-plus algebras) over the real numbers. We show that rank-finiteness of join matrices implies that these graph parameters can be computed in polynomial time on graph classes of bounded clique-width. In the case of graph parameters with values in arbitrary commutative semirings, this remains true for graph classes of bounded linear clique-width. B. Godlin, T. Kotek and J.A. Makowsky (2008) showed that definability of a graph parameter in Monadic Second Order Logic implies rank finiteness. We also show that there are uncountably many integer valued graph parameters with connection matrices or join matrices of fixed finite rank. This shows that rank finiteness is a much weaker assumption than any definability assumption.

Résumé. Les matrices de connection pour des fonctions sur les graphes á valeurs dans un corps ont étés introduites par M. Freedman, L. Lovász and A. Schrijver (2007). Une fonctions sur les graphes ayant des matrices de connection de rang fini peut être calculée en temps polynomial sur sur toute famille de graphes de largeur arborescente ("tree-width") bornée. Nous introduisons des matrices de joimture ("join matrices") qui généralisent les matrices de connection, et nous permettons aux fonctions sur les graphes de prendre leurs valeurs dans des semianneaux tropicaux réels. Nous montrons qu'une fonctions sur les graphes ayant des matrices de jointure de rang fini peut être calculée en temps polynomial sur des graphes de largeur de clique ("clique-width") bornée. Dans le cas des semi-anneaux commutatifs, cela reste vrai pour les graphes de largeur de clique linéaire bornée. B. Godlin, T. Kotek and J.A. Makowsky (2008) ont montré que certaines hypothšes de definissabilité en Logique du Second Ordre Monadique concernant des opérations sur les graphes entraine la finitude des rangs. Nous exhibons un ensemble non dénombrable d'opérations ayant une matrice de connection et des matrices de joimture de rang fini. Cela démontre que l'hypothèse de rang fini est beaucoup plus faible que l'hypothèse de definissabilité.

Keywords: Graph parameters, Hankel matrices, graph algebras, connection matrics, tree-width, clique-width

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1 Introduction and Summary

Connection matrices of graph parameters with values in a field $\mathcal{K}$ have been introduced by M. Freedman, L. Lovász and A. Schrijver (2007). Graph parameters with connection matrices of finite rank exhibit many nice properties. In particular, as was shown by L. Lovász, (L.Lovász, 2012, Theorem 6.48), they can be computed in polynomial time on graph classes of bounded tree-width. This is a logic-free version of the celebrated theorem by B. Courcelle, cf. (Downey and Fellows, 1999, Chapter 6.5) and (Flum and Grohe, 2006, Chapter 11.4-5). The theorem is proved using the formalism of graph algebras as developed in L.Lovász (2012).

In this paper we introduce join matrices, a generalization of connection matrices, which will allow us to replace the condition on tree-width to weaker conditions involving clique-width. Courcelle's theorem was extended to this case in Courcelle et al. (1998); Oum (2005). Furthermore we study graph parameters which take values in the tropical semirings $\mathcal{T}_{max}$ and $\mathcal{T}_{min}$ (max-plus algebras) over the real numbers, as opposed to values in a field. We shall call them tropical graph parameters in contrast to real graph parameters.

There are several notions of rank for matrices over commutative semirings. All of them coincide in the case of a field, and some of them coincide in the tropical case, Butkovič (2010); Guterman (2009); Cuninghame-Green and Butkovič (2004). We shall work with two specific notions: row-rank in the tropical case, and a finiteness condition introduced by G. Jacob Jacob (1975), which we call J-finiteness, in the case of arbitrary commutative semirings.

A typical example of a tropical graph parameter with finite row-rank of its connection matrix is $\omega(G)$, the maximal size of a clique in a graph G. If viewed as a real graph parameter, its connection matrix has infinite rank.

Main results

We adapt the formalism of graph algebras to tropical semirings with an inner product derived from the join matrices. Superficially this adaption may seem straightforward. However, there are several complications to be overcome: (i) the definition of the join matrix, (ii) the choice of the finiteness condition on the join matrices, and (iii) the choice of the definition of the quotient algebra.

(Theorem 6.2) We show that row-rank finiteness of join matrices implies that tropical graph parameters can be computed in polynomial time on graph classes of bounded clique-width.

(Theorem 6.3) A similar result holds in arbitrary commutative semirings when we replace row-rank finiteness with J-finiteness and bounded clique-width with bounded linear clique-width.

It was shown by B. Godlin, T. Kotek and J.A. Makowsky (2008) that definability of the graph parameter in Monadic Second Order Logic implies rank finiteness.

(Theorems 4.4,4.6) We show that there are uncountably many integer valued graph parameters with connection matrices or join matrices of fixed finite rank. This shows that (row)-rank finiteness is a much weaker assumption than any definability assumption.

It is well known that graph classes of bounded tree-width are also of bounded clique-width, therefore we restrict our presentation to the case of bounded clique-width. All results stated in this paper for tropical or arbitrary commutative semirings hold for fields as well.

Outline of the paper

In Section 2 we give the background on k-graphs, k-colored graphs, tree-width and clique width. In Section 3 we introduce join-matrices, and more generally, Hankel matrices and their ranks. In Section 4 we show that there are uncountably many graph parameters with Hankel matrices of fixed finite rank. In Section 5 we construct the graph algebras for join-matrices of finite row-rank. Finally, in Section 6 we show our main theorems for graph classes of bounded (linear) clique-width. In Section 7 we discuss our achievements and remaining open problems.

2 Prerequisites

2.1 k-graphs and k-colored graphs

Let $k \in \mathbb{N}$. A k-graph is a graph $G = (V(G), E(G))$ together with a partial map $\ell : [k] \to V(G)$. ℓ is called a *labeling* and the images of ℓ are called labels.

A k-colored graph is a graph $G = (V(G), E(G))$ together with a map $C : [k] \to 2^{V(G)}$. C is called a *coloring* and the images of C are called colors.

ℓ and C are often required to be injective, but this is not necessary. If ℓ is partial not all labels in $[k]$ are assigned values in $V(G)$. This corresponds to C having as values the empty set in $V(G)$. The labeling ℓ can be viewed as a special case of the coloring C, where $C(i)$ is a singleton for all $i \in [k]$.

We denote the class of graphs by $\mathcal{G}$, the class of k-graphs by $\mathcal{G}_k$, and the class of k-colored graphs by $\mathcal{CG}_k$.

2.2 Gluing and joining

We consider binary operations $\Box$ on k-graphs, resp. k-colored graphs. Specific examples are the following versions of gluing and joining, but if not further specified, $\Box$ can be any isomorphism preserving binary operation.

Two k-graphs (G_1, ℓ_1) and (G_2, ℓ_2) can be *glued together* producing a k-graph $(G, \ell) = (G_1, \ell_1) \sqcup_k (G_2, \ell_2)$ by taking the disjoint union of G_1 and G_2 and ℓ_1 and ℓ_2 and identifying elements with the same label.

For two k-colored graphs (G_1, C_1) and (G_2, C_2) we have similar operations. Let $i, j \in [k]$ be given. We define their (i, j)-*join* by

$$\overline{\eta}_{i,j}((G_1, C_1), (G_2, C_2)) = (G, C)$$

by taking disjoint unions for

(i) $V(G) = V(G_1) \sqcup V(G_2)$ and

(ii) for all $i \in [k]$ $C(i) = C_1(i) \sqcup C_2(i)$, and

(iii) $E(G) = E(G_1) \sqcup E(G_2) \cup \{(u, v) \in V(G) : u \in C(i), v \in C(j)\}$, which connects in the disjoint union all vertices in $C(i)$ with all vertices in $C(j)$.

$\overline{\eta}_{i,j}$ is a binary version of the operation $\eta_{i,j}$ used in the definition of the clique-width of a graph, cf. Courcelle et al. (1998).

Proposition 2.1 *The operations $\sqcup_k$ and $\overline{\eta}_{i,j}$ are commutative and associative.*

2.3 Inductive definition of tree-width and clique-width

As we do not need much of the theory of graphs of bounded tree-width and clique-width, the following suffices for our purpose. The interested reader may consult Hlinený et al. (2008). In Makowsky (2004) the following equivalent definitions of the class of (labeled or colored) graphs of tree-width at most k ($\mathrm{TW}(k)$), path-width at most k ($\mathrm{PW}(k)$), clique-width at most k ($\mathrm{CW}(k)$), and linear clique-width at most k ($\mathrm{LCW}(k)$) were given:

Tree-width

(i) Every k-graph of size at most $k+1$ is in $\mathrm{TW}(k)$ and $\mathrm{PW}(k)$.

(ii) $\mathrm{TW}(k)$ is closed under disjoint union $\sqcup$ and gluing $\sqcup_k$.

(iii) $\mathrm{PW}(k)$ is closed under disjoint union $\sqcup$ and small gluing $\sqcup_k$ where one operand is k-graph of size at most $k+1$.

(iv) Let $\pi : [k] \to [k]$ be a partial relabeling function. If $(G, \ell) \in \mathrm{TW}(k)$ then also $(G, \ell') \in \mathrm{TW}(k)$ where $\ell'(i) = \ell(\pi(i))$. The same holds for $\mathrm{PW}(k)$.

Clique-width

(i) Every single-vertex k-colored graph is in $\mathrm{CW}(k)$ and $\mathrm{LCW}(k)$.

(ii) $\mathrm{CW}(k)$ is closed under disjoint union $\sqcup$ and (i,j)-joins for $i,j \leq k$ and $i \neq j$.

(iii) $\mathrm{LCW}(k)$ is closed under disjoint union $\sqcup$ and small (i,j)-joins for $i,j \leq k$ and $i \neq j$, where one operand is a single-vertex k-colored graph.

(iv) Let $\rho : 2^{[k]} \to 2^{[k]}$ be a recoloring function. If $(G, C) \in \mathrm{CW}(k)$ then also $(G, C') \in \mathrm{CW}(k)$ where $C'(I) = C(\rho(I))$. The same holds for $\mathrm{LCW}(k)$.

A graph G is of clique-width at most 2^k iff there is a coloring C such that $(G, C) \in \mathrm{CW}(k)$. The discrepancy between 2^k and k comes from the fact that we allow overlapping colorings. Note that in the original definition a unary operation $\eta_{i,j}$ is used instead of the binary (i,j)-join $\overline{\eta}_{i,j}$. However, the two are interdefinable with the help of disjoint union. For a detailed discussion of various width parameters, cf. Hlinený et al. (2008).

A *parse tree for G* is a witness for the inductive definition describing how G was constructed. Parse trees for $G \in \mathrm{TW}(k)$ and $G \in \mathrm{PW}(k)$ can be found in polynomial time, Bodlaender and Kloks (1991). For $G \in \mathrm{CW}(k)$ the situation seems slightly worse. It was shown in Oum (2005):

Proposition 2.2 (S. Oum) *Let G be a graph of clique-width at most k. Then we can find a parse tree for $G \in \mathrm{CW}(3k)$ in polynomial time.*

3 Graph parameters with values in a semiring and their Hankel matrices

An $\mathcal{S}$-valued graph parameter f is a function $f : \mathcal{G} \to \mathcal{S}$ which is invariant under graph isomorphisms. If we consider $f : \mathcal{G}_k \to \mathcal{S}$ or $f : \mathcal{CG}_k \to \mathcal{S}$ then we require that f is also invariant under labelings and colorings.

Let $X_i : i \in \mathbb{N}$ be an enumeration of all colored graphs in $\mathcal{CG}_k$. For a binary operation $\square$ on labeled or colored graphs, and a graph parameter f, we define the *Hankel matrix* $\mathrm{H}(f, \square)$ with

$$\mathrm{H}(f, \square)_{i,j} = f(X_i \square X_j).$$

If the operation $\Box$ is $\sqcup_k$, the Hankel matrix $\mathrm{H}(f, \Box)$ is the connection matrix $M(f, k)$ of L.Lovász (2012).

Given a Hankel matrix $\mathrm{H}(f, \Box)$ we associate with it the semimodule $\mathrm{MH}(f, \Box)$ generated by its rows. If there exist finitely many elements $g_1, \ldots, g_m \in \mathrm{MH}(f, \Box)$ which generated $\mathrm{MH}(f, \Box)$, we say that $\mathrm{MH}(f, \Box)$ is *finitely generated*.

3.1 Notions of rank for matrices over semirings

Semimodules over semirings are analogs of vector spaces over fields. However, in contrast to vector spaces, there are several ways of defining the notion of independence for semimodules. For our purposes we adopt the definition 3.4 used in (Guterman, 2009, Section 3) and in Cuninghame-Green and Butkovič (2004), but see also Develin et al. (2005); Akian et al. (2009). A set of elements P from a semimodule U over a semiring $\mathcal{S}$ is *linearly independent* if there is no element in P that can be expressed as a linear combination of other elements in P.

Using this notion of linear independence, we define the notions of basis and dimension as in Guterman (2009); Cuninghame-Green and Butkovič (2004): a *basis* of a semimodule U over a semiring $\mathcal{S}$ is a set P of linearly independent elements from U which generate it, and the *dimension* of a semimodule U is the cardinality of its smallest basis.

Given a Hankel matrix $\mathrm{H}(f, \Box)$ with its associated semimodule $\mathrm{MH}(f, \Box)$, we define the *row-rank* $r(\mathrm{H}(f, \Box))$ of the matrix as the dimension of $\mathrm{MH}(f, \Box)$. In addition, we say that $\mathrm{H}(f, \Box)$ has *maximal row-rank* $mr(\mathrm{H}(f, \Box)) = k$ if $\mathrm{H}(f, \Box)$ has k linearly independent rows and any $k+1$ rows are linearly dependent. These definitions are the definitions used in Guterman (2009), applied to infinite matrices.

As stated in Guterman (2009); Cuninghame-Green and Butkovič (2004), in the case of tropical semirings, we have $r(\mathrm{H}(f, \Box)) = rm(\mathrm{H}(f, \Box))$.

Lemma 3.1 *If a Hankel matrix* $\mathrm{H}(f, \Box)$ *over a tropical semiring has row-rank* $r(\mathrm{H}(f, \Box)) = m$*, then there are* m *rows in* $\mathrm{H}(f, \Box)$ *which form a basis of* $\mathrm{MH}(f, \Box)$.

Remark 3.1 *If the matrix* $\mathrm{H}(f, \Box)$ *is over a general semiring* $\mathcal{S}$*, a smallest basis of* $\mathrm{MH}(f, \Box)$ *does not necessarily reside in* $\mathrm{H}(f, \Box)$.

Proof: $r(\mathrm{H}(f, \Box)) = m$, so by definition the dimension of $\mathrm{MH}(f, \Box)$ is m. Suppose the set $\mathcal{B} = \{g_1, \ldots, g_m\}$ is a smallest basis for $\mathrm{MH}(f, \Box)$. Each g_p is in $\mathrm{MH}(f, \Box)$, therefore there is a finite linear combination of rows from $\mathrm{H}(f, \Box)$ such that $g_p = \bigoplus_{i_p=1}^{\ell_p} \alpha_{i_p} r_{i_p}$. Consider the set of all the rows that appear in any of these linear combinations: $\mathcal{R} = \bigcup_{p=1}^{m} \left(\bigcup_{i_p=1}^{\ell_p} \alpha_{i_p} r_{i_p} \right)$. Since $\mathrm{H}(f, \Box)$ is over a tropical semiring, it holds that $mr(\mathrm{H}(f, \Box)) = r(\mathrm{H}(f, \Box)) = m$. Therefore, any set of $m+1$ rows from $\mathrm{H}(f, \Box)$ is linearly dependent. Consider the result of the following process:

(i) Set $i = |\mathcal{R}|$, and $B_i = \mathcal{R}$, note that B_i is of size i and generates $\mathcal{B}$. Repeat until $i = m$:

(ii) Let $r' \in B_i$ be a row that can be expressed using other rows in B_i. Such an element must exist, as $|B_i| > m$. Set $B' = B_i - r'$, set $i = i - 1$ and $B_i = B'$.

Note that B_i is still of size (now smaller) i and it still generates $\mathcal{B}$.

When $i = m$ is reached, we have B_m of size m which generates $\mathcal{B}$. This set must be independent: if it were not, we could perform more iterations of the above process and obtain a linearly independent set of size $< m$ which generates $\mathcal{B}$. But the existence of such a set contradicts $\mathcal{B}$ being a smallest basis, Therefore, B is linearly independent and generates $\mathcal{B}$. Since $\mathcal{B}$ generate $\mathrm{MH}(f, \Box)$, so does B, making B a basis for $\mathrm{MH}(f, \Box)$ which resides in $\mathrm{H}(f, \Box)$. $\Box$

After establishing the fact that there lies a basis B of $\mathrm{MH}(f, \Box)$ in $\mathrm{H}(f, \Box)$, we can find it in finite time, due to (Cuninghame-Green and Butkovič, 2004, Theorems 2.4 and 2.5).

4 Graph parameters with join matrices of finite (row-)rank

4.1 Graph parameters definable in Monadic Second Order Logic

It follows from Godlin et al. (2008); Kotek and Makowsky (2012, 2013) that for graph parameters definable in Monadic Second Order Logic (MSOL) or MSOL with modular counting quantifiers (CMSOL), the connection matrices and join matrices all have finite rank over fields, and finite row-rank over tropical semirings.

Let $H = (V(H), E(H))$ be a weighted graph with weight functions on vertices and edges $\alpha : V(H) \to \mathbb{R}$ and $\beta : E(H) \to \mathbb{R}$. The *tropical partition function* $Z_{H,\alpha,\beta}$ on graphs G is defined by

$$Z_{H,\alpha,\beta}(G) = \bigoplus_{h:G\to H} \left(\bigotimes_{v\in V(G)} \alpha(h(v)) \bigotimes_{(u,v)\in E(G)} \beta(h(u), h(v)) \right)$$

In the tropical ring this can be written as:

$$Z_{H,\alpha,\beta}(G) = \max_{h:G\to H} \left(\sum_{v\in V(G)} \alpha(h(v)) \sum_{(u,v)\in E(G)} \beta(h(u), h(v)) \right)$$

where h ranges over all homomorphisms $h : G \to H$.

It is easy to verify that $Z_{H,\alpha,\beta}$ is MSOL-definable. Hence we have:

Proposition 4.1 $\mathrm{H}(Z_{H,\alpha,\beta}, \overline{\eta}_{i,j})$ *has finite row-rank.*

The independence number $\alpha(G)$, which is the cardinality of the largest independent set, is a special case of a tropical partition function.

There are many graph parameters which have infinite connection rank, but finite row-rank if interpreted over tropical semirings. Examples for this phenomenon are the clique number $\omega(G)$ and the independence number $\alpha(G)$. Many other examples may be found in Arnborg et al. (1991).

4.2 Uncountably many graph parameters with finite (row-)rank

Here we show that both over fields and tropical semirings, most of the graph parameters with finite (row-)rank of connection or join matrices are not definable in the above mentioned logics.

We first need an observation. A graph is *k-connected*, if there is no set of k vertices, such that their removal results in a graph which is not connected. Obviously we have:

Lemma 4.2 *Let G_1 and G_2 be two k-graphs and $G = G_1 \sqcup_k G_2$. Then G is not $k+1$-connected.*

For a subset $A \subseteq \mathbb{N}$ we define graph parameters

$$f_A(G) = \begin{cases} |V(G)| & G \text{ is } k_0+1\text{-connected and } |V(G)| \in A \\ 0 & \text{else} \end{cases}$$

Lemma 4.3 *Let $\mathcal{S}$ be a commutative semiring which contains $\mathbb{N}$. Let $k_0 \in \mathbb{N}$ and $A \subseteq \mathbb{N}$ with $1 \in A$. Then for every $k \leq k_0$ the semimodule of the rows of $\mathrm{H}(f_A, \sqcup_k)$ is generated by the two rows*

$$(1, 0, \ldots) \text{ and } (\ldots, f_A(\emptyset, G), \ldots)$$

If $\mathcal{S}$ is a field, $\mathrm{H}(f_A, \sqcup_k)$ has rank at most 2.

Proof: By Lemma 4.2, if the graph $G_1 \sqcup_k G_2$ is k_0+1-connected, then either G_1 is k_0+1-connected and G_2 is the empty graph, or vice versa. So the non-zero entries in $\mathrm{H}(f_A, \sqcup_k)$ are in the first row and the first column. As $1 \in A$, we have a row $(1, 0, \ldots)$ which generates all the rows but the first one. □

Theorem 4.4 *Let $k_0 \in \mathbb{N}$ and $\mathcal{S}$ a field. There are continuum many graph parameters f with values in $\mathcal{S}$ with $r(f, \sqcup_k) \leq 2$ for each $k \leq k_0$.*
The same holds for tropical semirings and row-rank.

Proof: There are continuum many subsets $A \subseteq \mathbb{N}$ and for two different sets $A, B \subseteq \mathbb{N}$ the parameters f_A and f_B are different. □

Let $(G_1, C_1), (G_2, C_2)$ be two 2-colored graphs.

Lemma 4.5 *Let $(G, C) = \overline{\eta}_{1,2}((G_1, C_1), (G_2, C_2))$ and let $C_1(2) = C_2(1) = \emptyset$.*
(i) G is the disjoint union of G_1 and G_2 iff $C_1(1)$ or $C_2(2)$ are empty.
(ii) If both $C_1(1)$ and $C_2(2)$ are not empty, there is a vertex in $C_1(1)$ which has a higher degree in G than it had in G_1.

Let $r \in \mathbb{N}$ and $A \subseteq \mathbb{N}$. We define graph parameters with values in $\mathbb{N}$:

$$g^r_A(G) = \begin{cases} |V(G)| & G \text{ is } r\text{-regular and connected and } |V(G)| \in A \\ 0 & \text{else} \end{cases}$$

Theorem 4.6 *Let $\mathcal{S}$ be a field of characteristic 0. There are continuum many graph parameters g^r_A with values in $\mathcal{S}$ such that $r(f, \overline{\eta}_{1,2}) \leq 2$.*
Similarly for commutative semirings.

Proof: Use Lemma 4.5. □

5 Graph algebras

This section presents our adaptation of the formalism of graph algebras, to tropical semirings with an inner product derived from the join matrices of tropical graph parameters.

5.1 Quantum graphs

A formal linear combination of a finite number of k-colored graphs F_i with coefficients from $\mathcal{T}_{max}$ ($\mathcal{T}_{min}$) is called a *quantum graph*. The set of k-colored[(i)] quantum graphs is denoted $\mathcal{Q}_k$.

Let X, Y be quantum graphs: $X = \bigoplus_{i=1}^{m} a_i F_i$, and $Y = \bigoplus_{i=1}^{n} b_i F_i$. Note that some of the coefficients may be $-\infty$ (∞).

$\mathcal{Q}_k$ is a semimodule with the operations:

- $x \oplus y = (\bigoplus_{i=1}^{m} a_i F_i) \oplus (\bigoplus_{i=1}^{n} b_i F_i) = \bigoplus_{i=1}^{\max\{m,n\}} (a_i \oplus b_i) F_i$, and
- $\alpha \otimes x = \bigoplus_{i=1}^{n} (\alpha \otimes a_i) F_i$

We extend any binary operation $\Box$ to quantum graphs by

$$\Box(X, Y) = \bigoplus_{i,j=1}^{m,n} (a_i \otimes b_j) \Box(F_i, F_j)$$

We extend any graph parameter f to quantum graphs linearly

$$f(X) = \bigoplus_{i=1}^{m} a_i f(F_i)$$

From now on we assume that $\Box$ is a *commutative* graph operation. Given a Hankel matrix $\mathrm{H}(f, \Box)$, we turn $\mathcal{Q}_k$ into a commutative algebra by defining an inner product on X, Y:

$$\langle X, Y \rangle_{f,\Box} = f(\Box(X, Y)) = \bigoplus_{i,j=1}^{m,n} ((a_i \otimes b_j) \otimes f(\Box(F_i, F_j)))$$

5.2 Equivalence relations over $\mathcal{Q}_k$

Given a Hankel matrix $\mathrm{H}(f, \Box)$, we define an equivalence relation in the following way:

$$Ker_f^{\Box} = \{(X, Y) \in \mathcal{Q}_k \times \mathcal{Q}_k | \forall Z \in \mathcal{Q}_k : f(\Box(X, Z)) = f(\Box(Y, Z))\}$$

Note that this definition is reminiscent to the equivalence relation used in the Myhill-Nerode Theorem characterizing regular languages, cf. Hopcroft and Ullman (1980).

We denote the set of equivalence classes of this relation by $\mathcal{Q}_k / Ker_f^{\Box}$. $\mathcal{Q}_k / Ker_f^{\Box}$ is a semimodule with the operations:

$$[X]_f^{\Box} \oplus [Y]_f^{\Box} = [X \oplus Y]_f^{\Box}$$

and

$$\alpha [X]_f^{\Box} = [\alpha X]_f^{\Box}$$

We turn $\mathcal{Q}_k / Ker_f^{\Box}$ into a quotient algebra by extending the binary operation $\Box$ to these equivalence classes. We define

$$\Box([X]_f^{\Box}, [Y]_f^{\Box}) = [\Box(X, Y)]_f^{\Box}$$

It can be easily verified that the following properties hold for $X' \in [X]_f^{\Box}$ and $Y' \in [Y]_f^{\Box}$:

(i) In L.Lovász (2012) this notations is used only for k-graphs and real coefficients. As k-graphs are a special case of k-colored graphs our notations also includes his.

Proposition 5.1 *Let $\Box$ be a commutative and associative operation on graphs.*

(i) $X' \oplus Y' \in [X \oplus Y]_f^{\Box} = [X]_f^{\Box} \oplus [Y]_f^{\Box}$

(ii) $\alpha X' \in [\alpha X]_f^{\Box} = \alpha [X]_f^{\Box}$

(iii) $\Box(X', Y') \in [\Box(X, Y)]_f^{\Box}$

5.3 Finiteness condition on Hankel matrices

Given the Hankel matrix $\mathrm{H}(f, \Box)$ we denote by $\mathrm{MH}(f, \Box)$ the semimodule generated by the rows of $\mathrm{H}(f, \Box)$.

Lemma 5.2 *Assume the semimodule* $\mathrm{MH}(f, \Box)$ *is generated by the rows* $Gen = \{r_1, \ldots, r_m\}$ *of* $\mathrm{H}(f, \Box)$, *where each row corresponds to a graph* G_q. *Then* $\mathcal{Q}_k / Ker_f^{\Box}$ *is generated by* $\mathcal{B}_k = \{[G_1]_f^{\Box}, \ldots, [G_m]_f^{\Box}\}$.

Proof: Let $X \in \mathcal{Q}_k$, where $X = \bigoplus_{i=1}^{n} a_i F_i$. Each F_i a linear combination of the generators $G_1, \ldots, G_m$, $F_i = \bigoplus_{j=1}^{m} \alpha_{i,j} G_j$. By Proposition 5.1(i)-(ii) we have

$$X \in \bigoplus_{i=1}^{n} a_i [F_i]_f^{\Box} = \bigoplus_{i=1}^{n} a_i \bigoplus_{j=1}^{m} \alpha_{i,j} [G_j]_f^{\Box}$$

$\Box$

6 Graphs of clique-width at most k

Let $k \in \mathbb{N}$ be fixed. From now on $\Box = \overline{\eta}_{1,2}$ on k-colored graphs, and we write simply η^k instead of $\overline{\eta}_{1,2}$. We omit k when it is clear from the context. The Hankel matrix $\mathrm{H}(f, \eta)$ is called the join matrix.

We note that because the rows and columns correspond to all the graphs with all the possible k-colorings, all the join matrices $\mathrm{H}(f, \overline{\eta}_{i,j})$ are submatrices of $\mathrm{H}(f, \eta)$, after a suitable recoloring.

6.1 Representing G in the graph algebra

Given a graph G of clique-width at most k, together with its parse tree of the inductive definition from Section 2.3, we want to find $[X_G]_f^{\eta} \in \mathcal{Q}_k / Ker_f^{\eta}$ s.t. $f(X_G) = f(G)$. Furthermore, $[X_G]_f^{\eta}$ will be a linear combination of generators of $\mathcal{Q}_p / Ker_f^{\eta}$ and will be computable in polynomial time.

The same result for tree-width follows from the result on clique-width, but it can also be directly obtained using the inductive definition of tree-width from Section 2.3.

Lemma 6.1 *Let G of clique-width at most k be given together with its parse tree T, and let $\mathcal{B} = \{[F_1]_f^{\eta}, \ldots, [F_m]_f^{\eta}\}$ be a basis of $\mathcal{Q}_k / Ker_f$. Then there exists $[X_G]_f^{\eta} \in \mathcal{Q}_k / Ker_f^{\eta}$ s.t. $f(X_G) = f(G)$, and $[X_G]_f^{\eta}$ can be represented as a linear combination of $\{[F_1]_f^{\eta}, \ldots, [F_m]_f^{\eta}\}$.*

Proof:

Let $S_1, \ldots, S_\ell \in \mathcal{Q}_k$ be the single-vertex k-colored graphs (we later refer to them as small graphs), and let $[S_i]_f^{\eta} = \bigoplus_j s_{ij} [F_j]_f^{\eta}$ be their representations in the basis $\mathcal{B}$.

Let $\eta([F_i]^\eta_f, [F_j]^\eta_f)$ be the representations of the results of the η operation on elements from the basis $\mathcal{B}$.
Let G be a graph of clique-width at most k, and let T be its parse tree. We proceed by induction on T.
If $G = S_i$ then set $X_G = S_i$. The graph S_i is a single-vertex graph, and we have a representation for X_G.
Assume that for G_1, G_2, there exist $[X_{G_1}]^\eta_f, [X_{G_2}]^\eta_f$ and that there are representations $[X_{G_1}]^\eta_f = \bigoplus_{i=1}^n a_i[F_i]^\eta_f$, $[X_{G_2}]^\eta_f = \bigoplus_{i=1}^m b_i[F_i]^\eta_f$ for them.
If $G = \eta(G_1, G_2)$, then by Proposition 5.1(iii) we have $\eta(G_1, G_2) \in [\eta(G_1, G_2)]^\eta_f = \eta([X_{G_1}]^\eta_f, [X_{G_2}]^\eta_f)$.
We have representations for the operations $\eta([F_i]^\eta_f, [F_j]^\eta_f)$ on the basis elements, so we replace the expressions $\eta([F_i]^\eta_f, [F_j]^\eta_f)$ in $\eta([X_{G_1}]^\eta_f, [X_{G_2}]^\eta_f)$ by these representations and obtain a representation of $X_G = \eta(G_1, G_2) \in [\eta(G_1, G_2)]^\eta_f$.
If $G = \rho_{i,j}(G_1)$, we replace the basis elements $[F_i]^\eta_f$ in the representation of X_{G_1} by the representations of $[\rho_{i,j}(F_i)]^\eta_f$ and obtain a representation of X_G. □

6.2 Computing $f(G)$

Theorem 6.2 *Let f be a tropical graph parameter. Let the row-rank $r(\mathrm{H}(f, \eta))$ be finite, and let G be a graph of clique width at most k. Then $f(G)$ can be computed in polynomial time.*

Proof: We first use Proposition 2.2 to find a parse tree for $G \in \mathrm{CW}(3k)$. Next, we use dynamic programming to build a representation of the given graph G in the basis $\mathcal{B}$ in order to obtain $f(G)$. The algorithm requires a finite amount of preprocessing:
Find basis elements $\mathcal{B}$. By Lemma 3.1 and Theorems 2.4 and 2.5 in Cuninghame-Green and Butkovič (2004) this can be done in finite time.
Compute representations of all the small graphs by basis elements
Compute representations of the product η for all basis elements and all small graphs
Compute the value of f on all the small graphs and basis elements
The algorithm works with the provided parse tree T from the bottom up, following the inductive definition given in the proof of Lemma 6.1. When the top of the tree is reached, we have a representation of $[X_G]^\eta_f$ using only basis elements $[F_i]^\eta_f$. We then use the precomputed values of f in order to compute the value of $f(X_G) = f(G)$. □

6.3 Commutative semirings

In the case of $\mathcal{S}$ being an arbitrary commutative semiring we use following finiteness condition first introduced in Jacob (1975):

A Hankel matrix $\mathrm{H}(f, \Box)$ of an $\mathcal{S}$-valued graph parameter f is *J-finite* if $MH(f, \Box)$ is finitely generated. This does not necessarily imply that $\mathrm{H}(f, \Box)$ has a finite row-rank. However, in automata theory it suffices to prove the following: Let f be a $\mathcal{S}$-valued function on words in Σ^* (for a finite alphabet Σ). Then f is recognizable by a multiplicity automaton iff $\mathrm{H}(f, \circ)$ is J-finite, Berstel and Reutenauer (1984). Using virtually the same proof we can show:

Theorem 6.3 *Let $\mathcal{S}$ be an arbitrary commutative semiring. Let f be an $\mathcal{S}$-valued graph parameter and $k \in \mathbb{N}$ be fixed.*

(i) If $\mathrm{H}(f, \sqcup_i)$ is J-finite for all $i \leq k$, then f can be computed in polynomial time on graphs of path-width at most k.

(ii) If $\mathrm{H}(f, \eta^k)$ *is J-finite, then* f *can be computed in polynomial time on graphs of linear clique-width at most* k*.*

7 Conclusions

L. Lovász showed a "logic-free" version of Courcelle's famous theorem, cf. (Downey and Fellows, 1999, Chapter 6.5) and (Flum and Grohe, 2006, Chapter 11.4-5).

Theorem 7.1 (Theorem 6.48 of L.Lovász (2012)) *Let* f *be a real-valued graph parameter and* $k \geq 0$*. If* $r(f, \sqcup_k)$ *is finite, then* f *can be computed in polynomial time for graphs of tree-width at most* k*.*

The proof in L.Lovász (2012) is rather sketchy in its part relating to tree-decompositions. In particular, the role of relabelings, admittedly not very critical, is not spelled out at all.

In this paper we extended Theorem 7.1 to Theorems 6.2 and 6.3 in two ways.

(i) We showed how to prove the theorem for bounded clique-width instead of bounded tree-width.

(ii) We showed how to prove the theorem for tropical graph parameters, and more generally for graph parameters in an arbitrary commutative semirings.

In order to do this we introduced Hankel matrices for binary graph operations, in particular for a binary version of the basic operations used in the definition of clique-width.

The main differences between our proofs and the proof in L.Lovász (2012) are:

(i) the definition of the join matrix,

(ii) the choice of the finiteness condition of the join matrices, and

(iii) the choice of the definition of the equivalence relation used for the quotient algebra.

We also had to spell out the role of parse trees for clique-width in the dynamic programming part of the polynomial time algorithm.

Our approach also works for

• *other notions of width for graphs*, such as rank-width and modular width, and other inductively defined graph classes, cf. Makowsky (2004); Hlinený et al. (2008).

• *other notions of connection matrices*, cf. Schrijver (2012a,b).

In the full paper we shall discuss these extensions in detail.

Tropical graph parameters occur naturally in optimization theory. Graph parameters with values in polynomial rings are called graph polynomials, and are widely studied in diverse fields as statistical mechanics, computational biology and mathematics of finance. It remains open to identify the most suitable finiteness condition on Hankel matrices in the case where the graph parameter has its values in a arbitrary ring or semiring.

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