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The freeness of ideal subarrangements of Weyl arrangements

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Abstract. A Weyl arrangement is the arrangement defined by the root system of a finite Weyl group. When a set of positive roots is an ideal in the root poset, we call the corresponding arrangement an ideal subarrangement. Our main theorem asserts that any ideal subarrangement is a free arrangement and that its exponents are given by the dual partition of the height distribution, which was conjectured by Sommers-Tymoczko. In particular, when an ideal subarrangement is equal to the entire Weyl arrangement, our main theorem yields the celebrated formula by Shapiro, Steinberg, Kostant, and Macdonald. The proof of the main theorem is classification-free. It heavily depends on the theory of free arrangements and thus greatly differs from the earlier proofs of the formula.

Résumé. Un arrangement de Weyl est défini par l'arrangement d'hyperplans du système de racines d'un groupe de Weyl fini. Quand un ensemble de racines positives est un idéal dans le poset de racines, nous appelons l'arrangement correspondant un sous-arrangement idéal. Notre théorème principal affirme que tout sous-arrangement idéal est un arrangement libre et que ses exposants sont donnés par la partition duale de la distribution des hauteurs, ce qui avait été conjecturé par Sommers-Tymoczko. En particulier, quand le sous-arrangement idéal est égal à l'arrangement de Weyl, notre théorème principal donne la célèbre formule par Shapiro, Steinberg, Kostant et Macdonald. La démonstration du théorème principal n'utilise pas de classification. Elle dépend fortement de la théorie des arrangements libres et diffère ainsi grandement des démonstrations précédentes de la formule.

Keywords: Weyl arrangement, root system, ideal, free arrangements, exponents, height

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1 Introduction

Let Φ be an irreducible root system of rank ℓ and fix a simple system (or basis) $\Delta = \{\alpha_1, \ldots, \alpha_\ell\}$. Define the partial order $\geq$ on the set Φ^+ of positive roots such that $\alpha \geq \beta$ if $\alpha - \beta \in \mathbb{Z}_{\geq 0}\alpha_1 + \cdots + \mathbb{Z}_{\geq 0}\alpha_\ell$ for $\alpha, \beta \in \Phi^+$. A subset I of Φ^+ is called an **ideal** if each positive root β satisfying $\alpha \geq \beta$ for some $\alpha \in I$ belongs to I. The height $\mathrm{ht}(\alpha)$ of a positive root $\alpha = \sum_{i=1}^{\ell} c_i\alpha_i$ is defined to be $\sum_{i=1}^{\ell} c_i$. Define $m := \max\{\mathrm{ht}(\alpha) \mid \alpha \in I\}$. The **height distribution** in I is a sequence of positive integers $(i_1, i_2, \ldots, i_m)$, where $i_j := |\{\alpha \in I \mid \mathrm{ht}(\alpha) = j\}|$. The **dual partition** $\mathcal{DP}(I)$ of the height distribution in I is given by a multiset of ℓ integers:

$$\mathcal{DP}(I) := ((0)^{\ell - i_1}, (1)^{i_1 - i_2}, \ldots, (m-1)^{i_{m-1} - i_m}, (m)^{i_m}),$$

where $(a)^b$ implies that the integer a appears exactly b times.

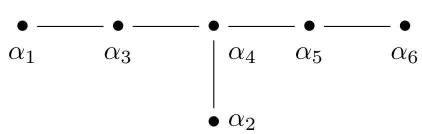
For $\alpha \in \Phi^+$ let H_α denote the hyperplane orthogonal to α. For each ideal $I \subseteq \Phi^+$, define the **ideal subarrangement** $\mathcal{A}(I) := \{H_\alpha \mid \alpha \in I\}$. In particular, when $I = \Phi^+$, $\mathcal{A}(\Phi^+)$ is called the **Weyl arrangement** which is known to be a **free arrangement**. (See §2 and [10] for basic definitions and results concerning free arrangements.) Our main theorem is the following:

Theorem 1.1 *Any ideal subarrangement $\mathcal{A}(I)$ is free with the exponents $\mathcal{DP}(I)$.*

Theorem 1.1 was conjectured by Sommers and Tymoczko in [12] where they defined and studied the ideal exponents, which is essentially the same as our $\mathcal{DP}(I)$. They also verified Theorem 1.1 when Φ is not of type F_4, E_6, E_7 or E_8 by using the addition-deletion theorem ([14]). Our proof is classification-free. If we set $I = \Phi^+$ in Theorem 1.1, we get the following:

Corollary 1.2 (Steinberg [13], Kostant [6], Macdonald [7]) *The exponents of the Weyl arrangement $\mathcal{A}(\Phi^+)$ are given by $\mathcal{DP}(\Phi^+)$.*

Example 1.3 ($I = \Phi^+$) *Consider a root system of type E_6. The Dynkin diagram is*



The positive roots of height one are $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6$. The roots of height two are $\alpha_1 + \alpha_3, \alpha_2 + \alpha_4, \alpha_3 + \alpha_4, \alpha_4 + \alpha_5, \alpha_5 + \alpha_6$. The highest root $\tilde{\alpha} = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6$ is of height 11. It is not hard to find the height of every positive root. The height distribution of Φ^+ turns out to be $6, 5, 5, 5, 4, 3, 3, 2, 1, 1, 1$. On the other hand, the exponents are $1, 4, 5, 7, 8, 11$. Corollary 1.2 asserts that both the height distribution and the exponents appear in the following figure:

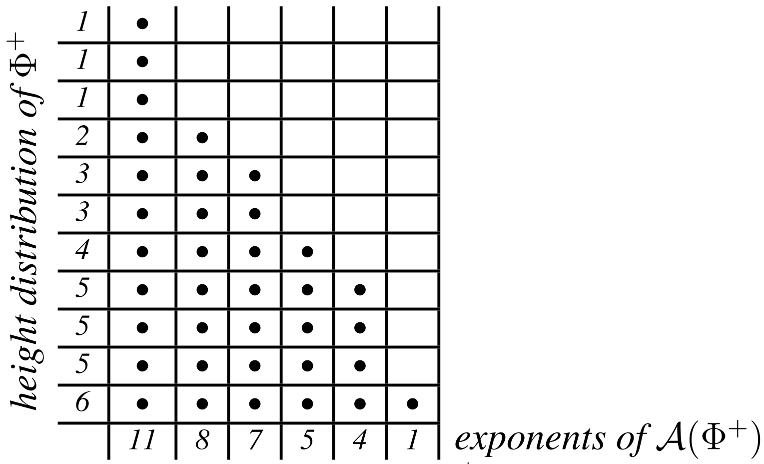


Fig. 1: $I = \Phi^+$

Corollary 1.2, which was referred to as "the remarkable formula of Kostant, Macdonald, Shapiro, and Steinberg" in [2], was first discovered by A. Shapiro (unpublished). Then R. Steinberg found it independently in [13]. It was B. Kostant [6] who first proved it without using the classification by studying the principal three-dimensional subgroup of the corresponding Lie group. I. G. Macdonald gave a proof using generating functions in [7]. An outline of Macdonald's proof is presented in [5, (3.20)]. G. Akyildiz-J. Carrell [1, 2] generalized the remarkable formula in a geometric setting. Theorem 1.1 is another generalization in the language of the theory of free hyperplane arrangements. Consequently our proof, which heavily depends on the theory of free arrangements, greatly differs from the earlier proofs of the formula.

Corollary 1.4 *Let h be the Coxeter number of Φ. For an integer j with $1 \le j \le h-1$, define*

$$\Phi_j^+ := \{\alpha \in \Phi^+ \mid \operatorname{ht}(\alpha) \le j\}.$$

Then Φ_j^+ is an ideal (which we call the j-th ***height ideal****) and the arrangement $\mathcal{A}(\Phi_j^+)$ is free with the exponents $\mathcal{DP}(\Phi_j^+)$.*

Corollary 1.5 *Suppose that $\Phi^+ = \{\beta_1, \beta_2, \dots, \beta_s\}$ with $\operatorname{ht}(\beta_1) \le \operatorname{ht}(\beta_2) \le \cdots \le \operatorname{ht}(\beta_s)$. Choose an integer t with $1 \le t \le s$. Let*

$$I := \{\beta_1, \beta_2, \dots, \beta_t\} \quad (1 \le t \le s).$$

Then I is an ideal and the arrangement $\mathcal{A}(I)$ is free with the exponents $\mathcal{DP}(I)$.

Corollary 1.6 *For any ideal $I \subseteq \Phi^+$, the characteristic polynomial $\chi(\mathcal{A}(I), t)$ splits as*

$$\chi(\mathcal{A}(I), t) = \prod_{i=1}^{\ell} (t - d_i),$$

where the nonnegative integers $d_1, \dots, d_\ell$ coincide with $\mathcal{DP}(I)$.

Corollary 1.7 *For any ideal $I \subseteq \Phi^+$, let $\mathcal{A}(I)_{\mathbb{C}}$ denote the complexified arrangement of $\mathcal{A}(I)$. Then*

$$\mathrm{Poin}(M(\mathcal{A}(I)_{\mathbb{C}}), t) = \prod_{i=1}^{\ell}(1 + d_i t),$$

where $\mathrm{Poin}(M(\mathcal{A}(I)_{\mathbb{C}}), t)$ is the Poincaré polynomial of the complement $M(\mathcal{A}(I)_{\mathbb{C}})$ of $\mathcal{A}(I)_{\mathbb{C}}$ and the nonnegative integers $d_1, \ldots, d_\ell$ coincide with $\mathcal{DP}(I)$.

The organization of this article is as follows. In §2 we review basic definitions and results about free arrangements. Then in §3 we introduce a new tool to prove the freeness of arrangements. It is called the multiple addition theorem (MAT). In §4, we verify all the three conditions in the MAT so that we may apply the MAT to prove Theorem 1.1. In §5, we complete the proof of Theorem 1.1 and its corollaries.

2 Preliminaries

In this section we review some basic concepts and results concerning free arrangements. Our standard reference is [10].

Let V be an ℓ-dimensional vector space over a field k. An **arrangement (of hyperplanes)** is a finite set of linear hyperplanes in V. Let $S := S(V^*)$ be the symmetric algebra of the dual space V^*. The defining polynomial $Q(\mathcal{A})$ of an arrangement $\mathcal{A}$ is

$$Q(\mathcal{A}) := \prod_{H \in \mathcal{A}} \alpha_H \in S,$$

where $\alpha_H \in V^*$ is a defining linear form of $H \in \mathcal{A}$. The derivation module $\mathrm{Der}\, S$ is the collection of all k-linear derivations from S to itself. It is a free S-module of rank ℓ. Define the module of logarithmic derivations by

$$D(\mathcal{A}) := \{\theta \in \mathrm{Der}\, S \mid \theta(\alpha_H) \in \alpha_H S \text{ for any } H \in \mathcal{A}\}.$$

We say that $\mathcal{A}$ is **free** with the **exponents** $(d_1, \ldots, d_\ell)$ if $D(\mathcal{A})$ is a free S-module with a homogeneous basis $\theta_1, \ldots, \theta_\ell$ such that $\deg \theta_i = d_i$ $(i = 1, \ldots, \ell)$. In this case, we use the expression $\exp(\mathcal{A}) = (d_1, \ldots, d_\ell)$. Define the **intersection lattice** by

$$L(\mathcal{A}) := \left\{ \bigcap_{H \in \mathcal{B}} H \mid \mathcal{B} \subseteq \mathcal{A} \right\}, \tag{1}$$

where the partial order is given by reverse inclusion (containment). Agree that $V \in L(\mathcal{A})$ is the minimum. For $X \in L(\mathcal{A})$, define

$$\mathcal{A}_X := \{H \in \mathcal{A} \mid X \subseteq H\} \quad \text{(localization), and} \tag{2}$$
$$\mathcal{A}^X := \{H \cap X \mid H \in \mathcal{A} \setminus \mathcal{A}_X\} \quad \text{(restriction).} \tag{3}$$

The **Möbius function** $\mu : L(\mathcal{A}) \to \mathbb{Z}$ is the function characterized by

$$\mu(V) = 1, \quad \mu(X) = -\sum_{X \subsetneq Y \subseteq V} \mu(Y).$$

Define the **characteristic polynomial** $\chi(\mathcal{A}, t)$ of $\mathcal{A}$ by

$$\chi(\mathcal{A}, t) := \sum_{X \in L(\mathcal{A})} \mu(X) t^{\dim X}.$$

Theorem 2.1 (Factorization theorem, [15, 8, 10]) *If $\mathcal{A}$ is free with $\exp(\mathcal{A}) = (d_1, \ldots, d_\ell)$, then*

$$\chi(\mathcal{A}, t) = \prod_{i=1}^{\ell} (t - d_i).$$

Assume that $\mathcal{A}$ is a free arrangement in the complex space $V = \mathbb{C}^\ell$ with $\exp(\mathcal{A}) = (d_1, \ldots, d_\ell)$. Define the complement of $\mathcal{A}$ by

$$M(\mathcal{A}) := V \setminus \bigcup_{H \in \mathcal{A}} H.$$

Then the Poincaré polynomial of the topological space $M(\mathcal{A})$ splits as

$$\mathrm{Poin}(M(\mathcal{A}), t) = \prod_{i=1}^{\ell} (1 + d_i t).$$

3 Multiple addition theorem

In this section, the root system Φ does not appear. The following is a variant of the addition theorem in [14], which we call the **multiple addition theorem (MAT)**.

Theorem 3.1 (Multiple addition theorem (MAT)) *Let $\mathcal{A}'$ be a free arrangement with exponents multiset $\exp(\mathcal{A}') = (d_1, \ldots, d_\ell)$, where $d_1 \leq \cdots \leq d_\ell$. Let $1 \leq p \leq \ell$ be the multiplicity of the highest exponent $d := d_\ell$. Let $H_1, \ldots, H_q$ be hyperplanes with $H_i \notin \mathcal{A}'$ for $i = 1, \ldots, q$. Define*

$$\mathcal{A}''_j := (\mathcal{A}' \cup \{H_j\})^{H_j} = \{H \cap H_j \mid H \in \mathcal{A}'\}$$

for each $j = 1, \ldots, q$. Assume that the following three conditions are satisfied:

(1) $X := H_1 \cap \cdots \cap H_q$ *is q-codimensional.*

(2) $X \not\subseteq \bigcup_{H \in \mathcal{A}'} H$.

(3) $|\mathcal{A}'| - |\mathcal{A}''_j| = d$ *for each j with $1 \leq j \leq q$.*

Then $\mathcal{A} := \mathcal{A}' \cup \{H_1, \ldots, H_q\}$ is free with $\exp(\mathcal{A}) = (d_1, \ldots, d_{\ell-q}, (d+1)^q)$ and $q \leq p$.

Proof. Assume $1 \leq j \leq q$. Let $\nu_j : \mathcal{A}''_j \to \mathcal{A}'$ be a map satisfying

$$\nu_j(Y) \cap H_j = Y$$

for each $Y \in \mathcal{A}''_j$. Define a polynomial

$$b_j := Q(\mathcal{A}') / \left(\prod_{Y \in \mathcal{A}''_j} \alpha_{\nu_j(Y)} \right),$$

where $\alpha_{\nu_j(Y)}$ is a defining linear form of $\nu_j(Y)$. Then it is known that

$$D(\mathcal{A}')\alpha_{H_j} := \big\{\theta(\alpha_{H_j}) \mid \theta \in D(\mathcal{A}')\big\} \subseteq \big(\alpha_{H_j}, b_j\big)$$

where $\big(\alpha_{H_j}, b_j\big)$ is the ideal of S generated by the two polynomials α_{H_j} and b_j. (See [14] and [10, p. 114] for example.) Let $\theta_1, \ldots, \theta_\ell$ be a basis for $D(\mathcal{A}')$ with $\deg\theta_i = d_i$ for each $i = 1, \ldots, \ell$ and $\deg\theta_1 \leq \cdots \leq \deg\theta_{\ell-p} = d_{\ell-p} < d$. Since

$$\deg b_j = |\mathcal{A}'| - |\mathcal{A}''_j| = d$$

by condition (3), the above inclusion implies that

$$\theta_i \in D(\mathcal{A})$$

for each $i = 1, \ldots, \ell - p$. Define

$$\varphi_i := \theta_{\ell-i+1}$$

for each $i = 1, \ldots, p$. Note that $\varphi_1, \ldots, \varphi_p$ are of degree d. Again, since $\deg b_j = d$ we may express

$$\varphi_i(\alpha_{H_j}) \equiv c_{ij} b_j \bmod (\alpha_{H_j})$$

for some constants c_{ij}. Let C be the $(p \times q)$-matrix $C = (c_{ij})_{i,j}$.

By condition (2), we may choose a point $z \in X \setminus \bigcup_{H\in\mathcal{A}'} H$. Then the evaluation of $D(\mathcal{A}')$ at the point z is the tangent space $T_{V,z}$ of V at z. Thus

$$T_{V,z} = \mathrm{ev}_z(D(\mathcal{A}')) = \mathrm{ev}_z\langle\varphi_1, \ldots, \varphi_p\rangle \oplus \mathrm{ev}_z\langle\theta_1, \ldots, \theta_{\ell-p}\rangle.$$

Let

$$\pi : T_{V,z} \longrightarrow T_{V,z}/T_{X,z}$$

be the natural projection. Note that the definition of the matrix C shows that

$$\operatorname{rank} C = \dim \pi(\mathrm{ev}_z\langle\varphi_1, \ldots, \varphi_p\rangle).$$

Since $\mathrm{ev}_z\langle\theta_1, \ldots, \theta_{\ell-p}\rangle \subseteq T_{X,z}$, one has

$$\operatorname{rank} C = \dim \pi(\mathrm{ev}_z\langle\varphi_1, \ldots, \varphi_p\rangle) = \dim\,(T_{V,z}/T_{X,z}) = q,$$

where the last equality is condition (1). Hence $q \leq p$ and we may assume that

$$C = \begin{pmatrix} E_q \\ O \end{pmatrix}$$

by applying elementary row operations. Therefore

$$\theta_1, \ldots, \theta_{\ell-q}, \alpha_{H_1}\varphi_1, \ldots, \alpha_{H_q}\varphi_q$$

form a basis for $D(\mathcal{A})$. Hence $\mathcal{A}$ is a free arrangement with $\exp(\mathcal{A}) = (d_1, \ldots, d_{\ell-q}, (d+1)^q)$. $\square$

4 Coheights, local-global formula and positive roots of the same height

In this section we will verify the three conditions in the MAT (Theorem 3.1) for all ideals of Φ^+, starting with a result for Φ^+. From now on we will use the notation of §1 and §2. We will often denote the Weyl arrangement $\mathcal{A}(\Phi^+)$ simply by $\mathcal{A}$. Our standard references on root systems are [3] and [5].

Let $\alpha \in \Phi^+$. Define $\mathcal{A}^\alpha$ to be the restriction of the Weyl arrangement $\mathcal{A}$ to H_α. In other words, define

$$\mathcal{A}^\alpha := \mathcal{A}^{H_\alpha} = \{K \cap H_\alpha \mid K \in \mathcal{A} \setminus \{H_\alpha\}\}.$$

Define the **coheight** of α by

$$\operatorname{coht}_\Phi \alpha := h - 1 - \operatorname{ht}(\alpha),$$

where h is the Coxeter number of Φ. For $X \in L(\mathcal{A})$, let $\Phi_X := \Phi \cap X^\perp$. Then Φ_X is a root system of rank $\operatorname{codim} X$. Note that Φ_X may be reducible. When Φ_X is irreducible, define

$$\operatorname{coht}_X \alpha := \operatorname{coht}_{\Phi_X} \alpha.$$

When Φ_X is not irreducible, we interpret

$$\operatorname{coht}_X \alpha := \operatorname{coht}_\Psi \alpha,$$

where Ψ is the irreducible component of Φ_X which contains α.

To verify condition (3) in the MAT for ideal subarrangements, we need the following theorem together with Proposition 4.2:

Theorem 4.1 (Local-global formula for coheights) *For $\alpha \in \Phi^+$, we have*

$$\operatorname{coht}_\Phi \alpha = \sum_{X \in \mathcal{A}^\alpha} \operatorname{coht}_X \alpha.$$

Proof. We proceed by an ascending induction on $\operatorname{coht}_\Phi \alpha$. When α is the highest root, then both sides of the equation are equal to zero. Now suppose $0 < \operatorname{coht}_\Phi \alpha < h-1$. Let $\alpha_1 \in \Delta$ be a simple root such that $\beta := \alpha + \alpha_1 \in \Phi^+$. Let $X_0 := H_\alpha \cap H_\beta$. Then $\{\alpha_1, \alpha, \beta\} \subseteq \Phi_{X_0}$. Set

$$C_\Phi(\alpha) := \sum_{X \in \mathcal{A}^\alpha} \operatorname{coht}_X \alpha.$$

If we verify

$$C_1 := C_\Phi(\alpha) - C_\Phi(\beta) - 1 = 0,$$

then we will obtain

$$C_\Phi(\alpha) = C_\Phi(\beta) + 1 = \operatorname{coht}_\Phi \beta + 1 = \operatorname{coht}_\Phi \alpha$$

by the induction assumption. So it remains to show $C_1 = 0$. Note that $\operatorname{coht}_{X_0} \alpha - \operatorname{coht}_{X_0} \beta = 1$, that $X_0 \in \mathcal{A}^\alpha$, and that $X_0 \in \mathcal{A}^\beta$. Compute

$$\begin{aligned} C_1 &= C_\Phi(\alpha) - C_\Phi(\beta) - 1 = \sum_{X \in \mathcal{A}^\alpha} \operatorname{coht}_X \alpha - \sum_{Y \in \mathcal{A}^\beta} \operatorname{coht}_Y \beta - 1 \\ &= \sum_{X \in \mathcal{A}^\alpha \setminus \{X_0\}} \operatorname{coht}_X \alpha - \sum_{Y \in \mathcal{A}^\beta \setminus \{X_0\}} \operatorname{coht}_Y \beta. \end{aligned} \tag{4}$$

Let $\mathcal{Z} := \mathcal{A}^{X_0} = \{K \cap X_0 \mid K \in \mathcal{A}, X_0 \not\subseteq K\}$. Define

$$C_2 := \sum_{Z \in \mathcal{Z}} \left(\sum_{\substack{X \in \mathcal{A}^\alpha \setminus \{X_0\} \\ X \supset Z}} \operatorname{coht}_X \alpha - \sum_{\substack{Y \in \mathcal{A}^\beta \setminus \{X_0\} \\ Y \supset Z}} \operatorname{coht}_Y \beta \right).$$

We will show that $C_1 = C_2$. To this end, we show that in the expression of C_2, both (A) every term in (4) appears and (B) each of them appears only once.

(A) We prove that every term in (4) appears in C_2. Let $X \in \mathcal{A}^\alpha \setminus \{X_0\}$. Let $Z := X \cap X_0 \subset X$. Then $\operatorname{codim} Z = 3$ because $X \subset H_\alpha$ and $X_0 \subset H_\alpha$. The same proof is valid for $Y \in \mathcal{A}^\beta \setminus \{X_0\}$.

(B) We prove that each of the terms in (A) appears only once in C_2. Let $X \in \mathcal{A}^\alpha \setminus \{X_0\}$ and $Z_1, Z_2 \in \mathcal{Z}$. Assume that $X \supset Z_1$ and $X \supset Z_2$. Then $Z_1 = X \cap X_0 = Z_2$. The same proof is valid for $Y \in \mathcal{A}^\beta \setminus \{X_0\}$.

Thus we obtain $C_1 = C_2$. It is easy to verify the local-global formula of coheights directly when the root system is either A_3, B_3 or C_3. Also the local-global formula for root systems of rank two is tautologically true. Thus we may assume the local-global formula for Φ_Z with $Z \in \mathcal{Z}$ and we compute

$$\begin{aligned} C_1 &= C_2 = \sum_{Z \in \mathcal{Z}} \left(\sum_{\substack{X \in \mathcal{A}^\alpha \setminus \{X_0\} \\ X \supset Z}} \operatorname{coht}_X \alpha - \sum_{\substack{Y \in \mathcal{A}^\beta \setminus \{X_0\} \\ Y \supset Z}} \operatorname{coht}_Y \beta \right) \\ &= \sum_{Z \in \mathcal{Z}} \left(\sum_{\substack{X \in \mathcal{A}^\alpha \\ X \supset Z}} \operatorname{coht}_X \alpha - \sum_{\substack{Y \in \mathcal{A}^\beta \\ Y \supset Z}} \operatorname{coht}_Y \beta - \operatorname{coht}_{X_0} \alpha + \operatorname{coht}_{X_0} \beta \right) \\ &= \sum_{Z \in \mathcal{Z}} (\operatorname{coht}_{\Phi_Z} \alpha - \operatorname{coht}_{\Phi_Z} \beta - 1) = 0. \end{aligned}$$

This completes the proof. □

Proposition 4.2 *Let $I \subseteq \Phi^+$ be an ideal. Fix $\alpha \in I$ with $k+1 := \operatorname{ht}(\alpha) > 1$. Define*

$$\begin{aligned} \mathcal{B}' &:= \{H_\beta \mid \beta \in I,\ \operatorname{ht}(\beta) \le k\}, \\ \mathcal{B} &:= \mathcal{B}' \cup \{H_\alpha\}, \quad \mathcal{B}'' := \mathcal{B}^{H_\alpha} = \{H \cap H_\alpha \mid H \in \mathcal{B}'\}. \end{aligned}$$

Then

$$|\mathcal{B}'| - |\mathcal{B}''| = k.$$

Proof. When $I = \Phi^+$ we denote the triple $(\mathcal{B}, \mathcal{B}', \mathcal{B}'')$ by $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$. Note that $\mathcal{B}''$ is a subset of $\mathcal{A}'' = \mathcal{A}^\alpha$. For $X \in \mathcal{A}''$, we will verify

$$|\mathcal{A}_X| - 2 - \operatorname{coht}_X \alpha = \begin{cases} |\mathcal{B}_X| - 2 & \text{if } X \in \mathcal{B}'', \\ 0 & \text{otherwise}, \end{cases} \tag{5}$$

where $\mathcal{A}_X$ and $\mathcal{B}_X$ are localizations defined in (2). Recall the height distribution of Φ_X^+ is:

$$i_1 = 2, i_2 = \cdots = i_n = 1$$

for $n = |\Phi_X^+| - 1$.

Case 1. If $X \in \mathcal{B}''$, then $|\mathcal{B}_X| \geq 2$. Since $I_X := I \cap \Phi_X^+$ is an ideal of Φ_X^+ and $|I_X| = |\mathcal{B}_X| \geq 2$, I_X contains the simple system of Φ_X. This implies

$$I_X \quad = \quad \{\beta \in \Phi_X^+ \mid \operatorname{coht}_X \beta \geq \operatorname{coht}_X \alpha\} \quad \text{and} \quad |I_X| = |\Phi_X^+| - \operatorname{coht}_X \alpha.$$

Hence we verify (5) in this case because

$$|\mathcal{A}_X| - 2 - \operatorname{coht}_X \alpha = |\Phi_X^+| - \operatorname{coht}_X \alpha - 2 = |I_X| - 2 = |\mathcal{B}_X| - 2.$$

Case 2. If $X \in \mathcal{A}'' \setminus \mathcal{B}''$, then $\mathcal{B}_X = \{H_\alpha\}$ and $I_X = \{\alpha\}$. Since I_X is an ideal of Φ_X^+, α is a simple root of Φ_X. Hence $\operatorname{coht}_X \alpha = |\mathcal{A}_X| - 2$. This verifies (5).

Combining (5) with Theorem 4.1 we compute

$$\begin{aligned}
|\mathcal{B}'| - |\mathcal{B}''| &= \sum_{X \in \mathcal{B}''} (|\mathcal{B}_X| - 2) = \sum_{X \in \mathcal{B}''} (|\mathcal{A}_X| - 2 - \operatorname{coht}_X \alpha) \\
&= \sum_{X \in \mathcal{A}''} (|\mathcal{A}_X| - 2 - \operatorname{coht}_X \alpha) = \sum_{X \in \mathcal{A}''} (|\mathcal{A}_X| - 2) - \sum_{X \in \mathcal{A}^\alpha} \operatorname{coht}_X \alpha \\
&= |\mathcal{A}'| - |\mathcal{A}''| - \operatorname{coht}_\Phi \alpha = h - 2 - (h - 1 - (k+1)) = k,
\end{aligned}$$

where we used the main result of [9] to get the penultimate equality. □

Next we will verify conditions (1) and (2) in the MAT for all ideals. Both conditions concern positive roots of the same height. A subset A of Φ^+ is said to be an antichain if A is a subset of Φ^+ of mutually incomparable elements with respect to the partial order $\geq$ on Φ^+.

Lemma 4.3 (Panyushev[11], Proposition 2.10) *Let Φ be a root system of rank ℓ and Δ be a simple system of Φ. Suppose that ℓ positive roots $\beta_1, \ldots, \beta_\ell$ form an antichain. Then $\Delta = \{\beta_1, \ldots, \beta_\ell\}$. In particular, $\beta_1, \ldots, \beta_\ell$ are linearly independent.*

Proposition 4.4 *Assume that $\beta_1, \ldots, \beta_q$ are distinct positive roots of the same height $k+1$. Define*

$$X := \bigcap_{i=1}^{q} H_{\beta_i}.$$

Then

(1) X is q-codimensional, and

(2)

$$X \not\subseteq \bigcup_{\substack{\alpha \in \Phi^+ \\ \operatorname{ht}(\alpha) \leq k}} H_\alpha.$$

Proof. (1) Since $\beta_1, \dots, \beta_q$ are distinct positive roots of the same height, they form an antichain. Apply Lemma 4.3.

(2) Since $\operatorname{rank} \Phi_X = \operatorname{codim} X = q$, we may apply Lemma 4.3 again to conclude that $\beta_1, \dots, \beta_q$ form the simple system of Φ_X. Assume that $X \subseteq H_\alpha$ with $\operatorname{ht}(\alpha) \leq k$. Then $\alpha \in \Phi_X$. So α can be expressed as a linear combination of $\beta_1, \dots, \beta_q$ with non-negative integer coefficients. Since the heights of $\beta_1, \dots, \beta_q$ are all $k+1$, this is a contradiction. $\square$

5 Proof of Theorem 1.1

In this section we will complete the proof of Theorem 1.1 and its corollaries before the final remark.

Proof of Theorem 1.1. The proof is by an induction on

$$\operatorname{ht}(I) := \max\{\operatorname{ht}(\alpha) \mid \alpha \in I\}.$$

When $\operatorname{ht}(I) = 1$, $\mathcal{A}(I)$ is a Boolean arrangement and there is nothing to prove.

Assume that $k+1 := \operatorname{ht}(I) > 1$. For any integer j with $1 \leq j \leq k+1$, define

$$I_j := \{\alpha \in I \mid \operatorname{ht}(\alpha) \leq j\}.$$

Then I_j is also an ideal. By the induction hypothesis, Theorem 1.1 holds true for $I_1, \dots, I_k$. In particular, $\mathcal{A}(I_k)$ is free with exponents

$$\exp(\mathcal{A}(I_k)) = (d_1, \dots, d_\ell)$$

which coincide with $\mathcal{DP}(I_k)$. If we put $p := |I_k \setminus I_{k-1}|$, then the induction hypothesis shows that

$$d_1 \leq \cdots \leq d_{\ell-p} < d_{\ell-p+1} = \cdots = d_\ell = k.$$

Let $\{\beta_1, \dots, \beta_q\} := I_{k+1} \setminus I_k$. Let $H_i := H_{\beta_i}$ and define $X := H_1 \cap \cdots \cap H_q$. Then Proposition 4.4 shows that $\operatorname{codim} X = q$ and that

$$X \not\subseteq \bigcup_{H \in \mathcal{A}(I_k)} H.$$

Also, Proposition 4.2 shows that $|\mathcal{A}(I_k)| - |(\mathcal{A}(I_k) \cup \{H_j\})^{H_j}| = k$ for any j. Hence all of conditions (1), (2) and (3) in the MAT are satisfied. Now apply the MAT to $\mathcal{A}(I) = \mathcal{A}(I_k) \cup \{H_1, \dots, H_q\}$. $\square$

Corollary 1.4 holds true because the set Φ_j^+ is an ideal which we call the j-th height ideal.

Example 5.1 *($I = \Phi_5^+$) Consider a root system of type E_6. The height distribution of the fifth height ideal Φ_5^+ is $6, 5, 5, 5, 4$. The exponents are $1, 4, 5, 5, 5, 5$ because Corollary 1.4 asserts that both the height distribution and the exponents appear in the following figure:*

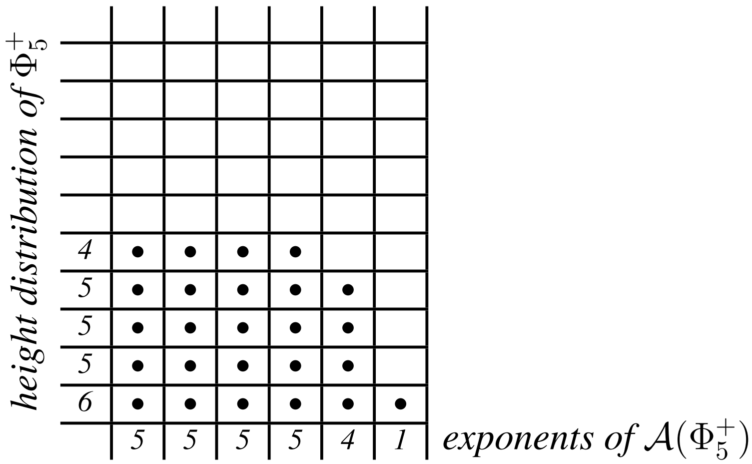


Fig. 2: $I = \Phi_5^+ = \{\alpha \in \Phi^+ \mid \mathrm{ht}(\alpha) \leq 5\}$

Corollary 1.5 holds true because the set I in Corollary 1.5, by definition, is an ideal. Note that I satisfies

$$\Phi_{j-1}^+ \subseteq I \subseteq \Phi_j^+,$$

where $j := \mathrm{ht}(\beta_t) = \mathrm{ht}(I)$. (Our convention is that $\Phi_0^+ = \emptyset$.)

Example 5.2 *Consider a root system of type E_6. Let*

$$\Phi_5^+ \subseteq I \subseteq \Phi_6^+$$

with $|I \setminus \Phi_5^+| = 2$. Then I is an ideal considered in Corollary 1.5. The height distribution of I is $6, 5, 5, 5, 4, 2$. *Thus the exponents of $\mathcal{A}(I)$ are* $1, 4, 5, 5, 6, 6$ *because of Corollary 1.5.*

Applying Theorem 2.1 to the ideal arrangement $\mathcal{A}(I)$, we get Corollaries 1.6 and 1.7.

Remark 5.3 *Note that the product $\mathcal{A}_1 \times \mathcal{A}_2$ of two free arrangements $\mathcal{A}_1$ and $\mathcal{A}_2$ is again free and that* $\exp(\mathcal{A}_1 \times \mathcal{A}_2)$ *is the disjoint union of* $\exp(\mathcal{A}_1)$ *and* $\exp(\mathcal{A}_2)$ *by [10, Proposition 4.28]. Thus it is not hard to see that Theorem 1.1 and its corollaries hold true for all finite root systems including the reducible ones.*

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