Combinatorial Realization of the Hopf Algebra of Sashes

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Abstract. A general lattice theoretic construction of Reading constructs Hopf subalgebras of the Malvenuto-Reutenauer Hopf algebra (MR) of permutations. The products and coproducts of these Hopf subalgebras are defined extrinsically in terms of the embedding in MR. The goal of this paper is to find an intrinsic combinatorial description of a particular one of these Hopf subalgebras. This Hopf algebra has a natural basis given by permutations that we call Pell permutations. The Pell permutations are in bijection with combinatorial objects that we call sashes, that is, tilings of a 1 by n rectangle with three types of tiles: black 1 by 1 squares, white 1 by 1 squares, and white 1 by 2 rectangles. The bijection induces a Hopf algebra structure on sashes. We describe the product and coproduct in terms of sashes, and the natural partial order on sashes. We also describe the dual coproduct and dual product of the dual Hopf algebra of sashes.

Résumé. Une construction générale dans la théorie des treillis dû à Reading construit des sous-algèbres de Hopf de l'algèbre de Hopf de permutations de Malvenuto et Reutenauer (MR). Les produits et coproduits de ces sous-algèbres de Hopf sont définis extrinsèquement en termes du plongement dans MR. Le but de cette communication est de trouver une description combinatoire intrinsèque d'une de ces sous-algèbres de Hopf en particulier. Cette algèbre Hopf a une base naturelle donnée par des permutations que nous appelons permutations Pell. Les permutations Pell sont en bijection avec des objets combinatoires que nous appelons écharpes, c'est-à-dire des pavages d'un rectangle 1-par-n avec trois espèces de tuiles: des carrés noirs 1-par-1, des carrés blancs 1-par-1, et des rectangles blancs 1-par-2. La bijection induit une structure d'algèbre de Hopf sur les écharpes. On décrit le produit et le coproduit en termes d'écharpes, et l'ordre partiel naturel sur les écharpes. On décrit également le coproduit dual et le produit dual de l'algèbre de Hopf dual des écharpes.

Keywords: Algebraic Combinatorics, Hopf Algebra, Permutation, Pattern Avoidance, Enumerative Combinatorics

1 Introduction

The focus of this research is on combinatorial Hopf algebras: Hopf algebras such that the basis elements of the underlying vector space are indexed by a family of combinatorial objects. For each $n \geqslant 0$, let O_n be a finite set of "combinatorial objects". We define a graded vector space over a field $\mathbb{K}$, such that for each grade n the basis vectors of the vector space are indexed by the elements of O_n. That is, the graded

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vector space is: $\mathbb{K}[O_\infty] = \bigoplus_{n\geqslant 0} \mathbb{K}[O_n]$. For simplicity, we refer to a basis element of this vector space by the combinatorial object indexing it. There is a more sophisticated approach for defining combinatorial Hopf algebras. For more information see [1].

Let S_n be the group of permutations of the set of the first n integers $[n] = \{1, 2, \dots, n\}$. Also define $[n, n'] = \{n, n+1, \dots, n'\}$ for $n' \geqslant n$. For $x = x_1x_2\cdots x_n \in S_n$, an *inversion* of x is a pair (x_i, x_j) where $i < j$ and $x_i > x_j$, and the *inversion set of* x is the set of all such inversions. The weak order is the partial order on S_n with $x \leqslant x'$ if and only if the inversion set of x is contained in the inversion set of x'. The weak order is a lattice. The inverse x^{-1} of a permutation $x \in S_n$ is the permutation $x^{-1} = y = y_1 \cdots y_n \in S_n$ such that $y_i = j$ when $x_j = i$.

Let T be a set consisting of integers $t_1 < t_2 < \cdots < t_n$. Given a permutation $x \in S_n$, the notation $(x)_T$ stands for the permutation of T whose one-line notation has t_j in the i^{th} position when $x_i = j$. On the other hand, given a permutation x of T, the *standardization*, $\mathrm{st}(x)$, is the unique permutation $y \in S_n$ such that $(y)_T = x$.

Now let T be a subset of $[n]$. For $x \in S_n$, the permutation $x|_T$ is the permutation of T obtained by removing from the one-line notation for x all entries that are not elements of T.

Example 1.1. Let $x = 31254$, $T_1 = \{2, 3, 6, 8, 9\}$, and $T_2 = \{2, 3, 5\}$. Then, $(x)_{T_1} = 62398$ and thus $\mathrm{st}(62398) = 31254$. Also, $x|_{T_2} = 325$.

The Malvenuto-Reutenauer Hopf algebra MR is a graded Hopf Algebra $(\mathbb{K}[S_\infty], \bullet, \Delta)$. Let $\mathbb{K}[S_\infty] = \bigoplus_{n\geqslant 0} \mathbb{K}[S_n]$ be a graded vector space. Let $x = x_1x_2\cdots x_p \in S_p$ and $y = y_1y_2\cdots y_q \in S_q$. Define $y' = y'_1 \cdots y'_q$ to be $(y)_{[p+1,p+q]}$ so that $y'_i = y_i + p$. A *shifted shuffle* of x and y' is a permutation $z \in S_n$ where $n = p + q$, $z|_{[p]} = x$ and $z|_{[p+1,n]} = y'$. From [3], the product of x and y in MR is the sum of all the shifted shuffles of x and y. Equivalently,

$$x \bullet y = \sum [x \cdot y', y' \cdot x] \tag{1}$$

where $x \cdot y'$ is the concatenation of the permutations x and y', and $\sum[x \cdot y', y' \cdot x]$ denotes the sum of all the elements in the weak order interval $[x \cdot y', y' \cdot x]$. The poset induced on S_n by the weak order is a lattice (also denoted by S_n). The coproduct in MR is:

$$\Delta(x) = \sum_{i=0}^{p} \mathrm{st}(x_1 \cdots x_i) \otimes \mathrm{st}(x_{i+1} \cdots x_p) \tag{2}$$

where $\mathrm{st}(x_1 \cdots x_0)$ and $\mathrm{st}(x_{p+1} \cdots x_p)$ are both interpreted as the empty permutation $\varnothing$.

Define the map $\mathrm{Inv} : S_n \to S_n$ by $\mathrm{Inv}(x) = x^{-1}$ and extend the map linearly to a map $\mathrm{Inv} : \mathbb{K}S_\infty \to \mathbb{K}S_\infty$. MR is known to be self dual [4] and specifically Inv is an isomorphism from $(\mathbb{K}[S_\infty], \bullet, \Delta)$ to the graded dual Hopf algebra $(\mathbb{K}[S_\infty], \Delta^*, m^*)$. Let $x \in S_p$, $y \in S_q$, and $z \in S_n$, where $p + q = n$. Given a subset T of p elements of $[n]$, T^C denotes the complement of T in $[n]$. The dual product is given by:

$$\Delta^*(x \otimes y) = \mathrm{Inv}(x^{-1} \bullet y^{-1}) = \sum_{\substack{T\subseteq[n],\\ |T|=p}} (x)_T \cdot (y)_{T^C}, \tag{3}$$

and the dual coproduct is:

$$m^*(z) = (\mathrm{Inv} \otimes \mathrm{Inv})(\Delta(z^{-1})) = \sum_{i=0}^{n} z|_{[i]} \otimes \mathrm{st}(z|_{[i+1,n]}) \tag{4}$$

where $z|_{[0]}$ and $z|_{[n+1,n]}$ are both interpreted as the empty permutation $\varnothing$.

Now that we have explicitly described both the Hopf algebra of permutations and the dual Hopf algebra of permutations, we will present a family of Hopf subalgebras that are defined by a particular pattern-avoidance condition. We give more detail on the particular pattern-avoidance condition below, but in general the condition requires patterns of length k to have k and 1 adjacent. This family of Hopf algebras is defined by Reading [5].

For some $k \geqslant 2$, let $V \subseteq [2, k-1]$ such that $|V| = j$ and let V^C be the complement of V in $[2, k-1]$. A permutation $x \in S_n$ *avoids* the pattern $V(k1)V^C$ if for every subsequence $x_{i_1}x_{i_2}\cdots x_{i_k}$ of x with $i_{j+2} = i_{j+1} + 1$, the standardization $\mathrm{st}(x_{i_1}x_{i_2}\cdots x_{i_k})$ is not of the form $v(k1)v'$ for any permutation v of the set V and any permutation v' of V^C. In the notation of Babson and Steingrimsson [2] avoiding $V(k1)V^C$ means avoiding all patterns of the form $v_1 - \cdots - v_j - k1 - v'_1 - \cdots - v'_{k-j-2}$, where $v_1 \cdots v_j$ is a permutation of V and $v'_1 \cdots v'_{k-j-2}$ is a permutation of V^C.

Let U be a set of patterns of the form $V(k1)V^C$, where $|V|$ and k can vary. Define Av_n to be the set of permutations in S_n that avoid all of the patterns in U. We define a graded Hopf algebra $(\mathbb{K}[\mathrm{Av}_\infty], \bullet_{\mathrm{Av}}, \Delta_{\mathrm{Av}})$ as a graded Hopf subalgebra of MR. Let $\mathbb{K}[\mathrm{Av}_n]$ be a vector space, over a field $\mathbb{K}$, with basis vectors indexed by the elements of Av_n, and let $\mathbb{K}[\mathrm{Av}_\infty]$ be the graded vector space $\bigoplus_{n \geqslant 0} \mathbb{K}[\mathrm{Av}_n]$. The product and coproduct on $\mathbb{K}[\mathrm{Av}_\infty]$ are described below.

We define a map $\pi_\downarrow : S_n \to \mathrm{Av}_n$ recursively. If $x \in \mathrm{Av}_n$ then define $\pi_\downarrow(x) = x$. If $x \in S_n$, but $x \notin \mathrm{Av}_n$, then x contains an instance of a pattern $V(k1)V^C$ in U. That is, there exists some subsequence $x_{i_1}x_{i_2}\cdots x_{i_k}$ of x, where $i_{j+2} = i_{j+1} + 1$ and $j = |V|$, such that $st(x_{i_1}x_{i_2}\cdots x_{i_k}) = vk1v'$ for some permutations v and v' of V and V^C. Exchange $x_{i_{j+1}}$ and $x_{i_{j+2}}$ in x to create a new permutation x', calculate $\pi_\downarrow(x')$ recursively and set $\pi_\downarrow(x) = \pi_\downarrow(x')$. The recursion must terminate because an inversion of x is destroyed at every step, and because the identity permutation is in Av_n. The map $\pi_\downarrow$ is well-defined as explained in [5, Remark 9.5]. We emphasize that the definition of $\pi_\downarrow$ is dependent on U.

The map $\pi_\downarrow$ defines an equivalence relation with permutations $x, x' \in S_n$ equivalent if and only if $\pi_\downarrow(x) = \pi_\downarrow(x')$. The set Av_n is a set of representatives of these equivalence classes. This equivalence relation is a lattice congruence on the weak order. Therefore the poset induced on Av_n by the weak order is a lattice (also denoted by Av_n) and the map $\pi_\downarrow$ is a lattice homomorphism from the weak order to Av_n. The congruence classes defined by $\pi_\downarrow$ are intervals, and $\pi_\downarrow$ maps an element to the minimal element of its congruence class. Let $\pi^\uparrow$ be the map that takes an element to the maximal element of its congruence class.

The following proposition is a special case of [5, Proposition 2.2]. The congruence on S_n defined by $\pi_\downarrow$ is denoted by Θ. For $x \in S_n$, the congruence class of x mod Θ is denoted by $[x]_\Theta$.

Proposition 1.2. *Given S_n a finite lattice, Θ a congruence on S_n, and $x \in S_n$, the map $y \to [y]_\Theta$ restricts to a one-to-one correspondence between elements of S_n covered by $\pi_\downarrow(x)$ and elements of* Av_n *covered by $[x]_\Theta$.*

Both $\pi_\downarrow$ and $\pi^\uparrow$ are order preserving and $\pi^\uparrow \circ \pi_\downarrow = \pi^\uparrow$ and $\pi_\downarrow \circ \pi^\uparrow = \pi_\downarrow$. A $\pi_\downarrow$*-move* is the result of switching two adjacent entries of a permutation in the manner described above. That is, it changes $\cdots k1 \cdots$ to $\cdots 1k \cdots$ for some pattern in U. A $\pi^\uparrow$*-move* is the result of switching two adjacent entries of a permutation in a way such that a $\pi^\uparrow$-move undoes a $\pi_\downarrow$-move. That is, it changes $\cdots 1k \cdots$ to $\cdots k1 \cdots$.

We define a map $r : \mathbb{K}[S_\infty] \to \mathbb{K}[\mathrm{Av}_\infty]$ that identifies the representative of a congruence class. Given

$x \in S_n$,

$$r(x) = \begin{cases} x & \text{if } x \in \mathrm{Av}_n \\ 0 & \text{otherwise.} \end{cases}$$

Similarly, we define a map $c : \mathbb{K}[\mathrm{Av}_\infty] \to \mathbb{K}[S_\infty]$ that takes an avoider to the sum of its congruence class:

$$c(x) = \sum_{\substack{y \text{ such that} \\ \pi_\downarrow(y)=x}} y.$$

We now describe the product and coproduct in $(\mathbb{K}[\mathrm{Av}_\infty], \bullet_{\mathrm{Av}}, \Delta_{\mathrm{Av}})$. Let $x \in \mathrm{Av}_p$, and let $y \in \mathrm{Av}_q$. Then:

$$m_{\mathrm{Av}}(x \otimes y) = x \bullet_{\mathrm{Av}} y = r(x \bullet y). \tag{5}$$

Just as the product in MR is $\sum[x \cdot y', y' \cdot x]$, we can view this product as:

$$x \bullet_{\mathrm{Av}} y = \sum[x \cdot y', \pi_\downarrow(y' \cdot x)], \tag{6}$$

where $[x \cdot y', \pi_\downarrow(y' \cdot x)]$ is an interval on the lattice Av_n.

The coproduct is:

$$\Delta_{\mathrm{Av}}(z) = (r \otimes r)(\Delta(c(z))). \tag{7}$$

We now describe the Hopf algebra $(\mathbb{K}[\mathrm{Av}_\infty], \Delta^*_{\mathrm{Av}}, \bullet^*_{\mathrm{Av}})$ that is dual to $(\mathbb{K}[\mathrm{Av}_\infty], \bullet_{\mathrm{Av}}, \Delta_{\mathrm{Av}})$. We extend the map $\pi_\downarrow$ linearly, so $\pi_\downarrow$ is a map from $\mathbb{K}[S_\infty]$ to $\mathbb{K}[\mathrm{Av}_\infty]$. The map that is dual to the map c is $c^* : \mathbb{K}[S_\infty] \to \mathbb{K}[\mathrm{Av}_\infty]$, where $c^*(x) = \pi_\downarrow(x)$ for $x \in \mathbb{K}[S_\infty]$. The map that is dual to the map r is $r^* : \mathbb{K}[\mathrm{Av}_\infty] \to \mathbb{K}[S_\infty]$, where $r^*(x) = x$ for $x \in \mathbb{K}[\mathrm{Av}_\infty]$.

Let $z \in \mathrm{Av}_n$, where $n = p + q$. The dual coproduct is given by dualizing Equation (5), so that:

$$m^*_{\mathrm{Av}}(z) = m^*(z). \tag{8}$$

The dual product Δ^*_{Av} is given by dualizing Equation (7):

$$\Delta^*_{\mathrm{Av}}(x \otimes y) = \pi_\downarrow \Delta^*(x \otimes y). \tag{9}$$

Combining Equation (9) with Equation (3), we have:

$$\Delta^*_{\mathrm{Av}}(x \otimes y) = \sum_{\substack{T \subseteq [n] \\ |T|=p}} \pi_\downarrow((x)_T \cdot (y)_{T^C}) \tag{10}$$

Equation (10) leads to the following order theoretic description of the coproduct Δ_{Av}, which was worked out jointly with Nathan Reading.

Given $z \in \mathrm{Av}_n$, a subset $T \subseteq [n]$ is *good with respect to* z if there exists a permutation $z' = z'_1 \cdots z'_n$ with $\pi_\downarrow(z') = z$ such that $T = \{z'_1, \ldots, z'_{|T|}\}$. Suppose T is good with respect to z, let $p = |T|$ and let $q = n - p$. Let $z_{\min}$ be minimal, in the weak order on S_n, among permutations equivalent to z and whose first p entries are the elements of T. Let $z_{\max}$ be maximal, in the weak order, among such permutations. Define I_T to be the sum over the elements in the interval $[\mathrm{st}(z_{\min}|_T), \pi_\downarrow \mathrm{st}(z_{\max}|_T)]$ in Av_p and define J_T to be the sum over the elements in the interval $[\mathrm{st}(z_{\min}|_{T^C}), \pi_\downarrow \mathrm{st}(z_{\max}|_{T^C})]$ in Av_q.

Theorem 1.3. *Let $z \in \mathrm{Av}_n$. Then*

$$\Delta_{\mathrm{Av}}(z) = \sum_{T \text{ is good}} I_T \otimes J_T$$

where $I_T = \sum[\mathrm{st}(z_{\min}|_T), \pi_\downarrow \mathrm{st}(z_{\max}|_T)]$, $J_T = \sum[\mathrm{st}(z_{\min}|_{T^C}), \pi_\downarrow \mathrm{st}(z_{\max}|_{T^C})]$.

To prove Theorem 1.3, we first need several lemmas.

Lemma 1.4. *The elements in the interval $[z_{\min}, z_{\max}]$ are equivalent to z and their first p entries are the elements of T.*

Lemma 1.5. *Suppose $T \subseteq [n]$ with $|T| = p$. Let $q = n - p$. Suppose also that $x_1 \leqslant x_2 \leqslant x_3$ in Av_p, and that $y_1 \leqslant y_2 \leqslant y_3$ in Av_q. If $\pi_\downarrow((x_1)_T \cdot (y_1)_{T^C}) = \pi_\downarrow((x_3)_T \cdot (y_3)_{T^C}) = z$, then $\pi_\downarrow((x_2)_T \cdot (y_2)_{T^C}) = z$.*

Lemma 1.6. *Suppose $x_1, x_2 \in S_p$ and $y_1, y_2 \in S_q$. Suppose $T \subseteq [n]$, where $n = p + q$, and with $|T| = p$. The following identities hold:*

$$(x_1)_T \cdot (y_1)_{T^C} \vee (x_2)_T \cdot (y_2)_{T^C} = (x_1 \vee x_2)_T \cdot (y_1 \vee y_2)_{T^C}$$
$$(x_1)_T \cdot (y_1)_{T^C} \wedge (x_2)_T \cdot (y_2)_{T^C} = (x_1 \wedge x_2)_T \cdot (y_1 \wedge y_2)_{T^C}$$

The proof of Theorem 1.3 begins by defining $\mathrm{terms}(z, T)$ to be the set $\{x \otimes y : \pi_\downarrow((x)_T \cdot (y)_{T^C}) = z\}$, and showing that $\mathrm{terms}(z, T)$ is nonempty if and only if T is good with respect to z. Next, we show that for each good subset T, the set $\mathrm{terms}(z, T)$ is of the form $I_T \otimes J_T$. This proof also establishes the following more detailed statement.

Proposition 1.7. *For some $T \subseteq [n]$, $x \otimes y \in \mathrm{terms}(z, T)$ if and only if $x \otimes y$ is a term of $I_T \otimes J_T$ in $\Delta_{\mathrm{Av}}(z)$.*

Proof: Since $x \otimes y \in \mathrm{terms}(z, T)$ means that $\pi_\downarrow\big((x)_T \cdot (y)_{T^C}\big) = z$, we see from Equation(10) and Theorem 1.3 that for a fixed set T, $x \otimes y$ is a term of the summand indexed by T in $\Delta_{\mathrm{Av}}(z)$ if and only if z is the summand indexed by T in $\Delta^*_{\mathrm{Av}}(x \otimes y)$. □

2 Pell Permutations and Sashes

Given a permutation $x = x_1 x_2 \cdots x_n \in S_n$, for each $i \in [n-1]$, there is a nonzero integer j such that $x_i = x_{i+1} + j$. If $j > 0$, then there is an *descent* of size j in the i^{th} position of x. A Pell permutation is a permutation of $[n]$ with no descents of size larger than 2, and such that for each descent $x_i = x_{i+1} + 2$, the element $x_{i+1} + 1$ is to the right of x_{i+1}. We write P_n for the set of Pell permutations in S_n.

Let us consider how many Pell permutations of length n there are. Given $x \in P_{n-1}$, we can place n at the end of x or before $n-1$. We can also place n before $n-2$, but only if $n-1$ is the last entry of x. Therefore $|P_n| = 2|P_{n-1}| + |P_{n-2}|$. This recursion, with the initial conditions $|P_0| = 0$ and $|P_1| = 1$, defines the Pell numbers as defined by [6, Sequence A000129].

Lemma 2.1. $P_n = \mathrm{Av}_n$ *for* $U = \{2(31), (41)23\}$.

Proof: Suppose $x \in P_n$. Since x does not have any descents larger than 2, it avoids $(41)23$. For each descent $x_i = x_{i+1} + 2$ in x, the element $x_{i+1} + 1$ is to the right of x_{i+1}. Thus x also avoids $2(31)$. Now suppose $x \in \mathrm{Av}_n$. Suppose x has a descent $x_i = x_{i+1} + j$. Because x avoids $2(31)$, the entries

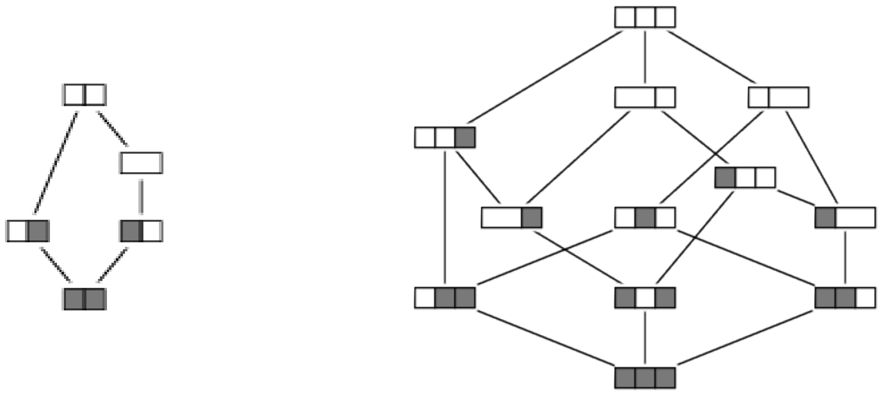


Fig. 1: The elements of Σ_3 (left) and Σ_4 (right).

$x_{i+1}+1, ..., x_{i+1}+j-1$ are to the right of the x_{i+1}. Thus, since x avoids $(41)23$ we see that $j \leqslant 2$ and conclude that $x \in P_n$. $\square$

The poset induced on P_n by the weak order is a lattice (also denoted by P_n). As a consequence of Lemma 2.1, there is a Hopf algebra $(\mathbb{K}[\mathrm{Av}_\infty], \bullet_{\mathrm{Av}}, \Delta_{\mathrm{Av}})$ of Pell permutations. For the rest of this paper we fix $U = \{2(31), (41)23\}$.

There is a combinatorial object in bijection with Pell permutations that will allow us to have a more natural understanding of the Hopf algebra of Pell permutations.

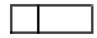
A *sash* of length n is a tiling of a $1 \times n$ rectangle by black 1×1 squares, white 1×1 squares, and/or white 1×2 rectangles. The set of sashes of length n is called Σ_n. There are no sashes of length -1 so $\Sigma_{-1} = \varnothing$, and there is one sash of length 0, a 1 by 0 rectangle denoted $\Vert$, so $|\Sigma_0| = 1$. There are two sashes of length 1: $\blacksquare$ and $\square$. The five sashes of length 2 and the twelve sashes of length 3 are shown in Figure 1. The poset structure of these sashes will be explained later in this section.

A sash of length n starts with either a black square, a white square, or a rectangle. Thus $|\Sigma_n| = 2|\Sigma_{n-1}| + |\Sigma_{n-2}|$. Since $|\Sigma_{-1}| = 0$ and $|\Sigma_0| = 1$, there is a bijection between Pell permutations of length n and sashes of length $n-1$. We now describe a bijection that we use to induce a Hopf Algebra structure on sashes.

Definition 2.2. We define a map σ from S_n to Σ_{n-1}. Let $x \in S_n$. We build a sash $\sigma(x)$ from left to right as we consider the entries in x from 1 to $n-1$. For each value $i \in [n-1]$, if $i+1$ is to the right of i, place a black square on the sash, and if $i+1$ is to the left of i, place a white square on the sash. There is one exception: If $i+1$ is to the right of i, and $i+2$ is to the left of i (and of $i+1$), then place a rectangle in the i^{th} and $(i+1)^{st}$ positions of the sash. We also define $\sigma(1) = \Vert$ and $\sigma(\varnothing) = \varnothing$.

From the definition of the map σ we see that σ sometimes involves replacing an adjacent black square and white square by a rectangle. Later, we will sometimes break a rectangle into a black square and a white square.

Example 2.3. Here is the procedure for computing $\sigma(421365)$.

2 is to the left of 1 → 
3 is to the right of 2 → 
4 is to the left of 3 and also to the left of 2 → 
5 is to the right of 4 → 
6 is to the left of 5 but to the right of 4 → 

Let T be a set of n integers and let x be a permutation of T. We define $\sigma(x) = \sigma(\mathrm{st}(x))$.

Example 2.4. $\sigma(742598) = \sigma(\mathrm{st}(742598)) = \sigma(421365) =$ 

Definition 2.5. We define a map $\eta : \Sigma_{n-1} \to P_n$. To calculate $\eta(A)$ for a sash $A \in \Sigma_{n-1}$, we place the numbers 1 through n one at a time. Place the number 1 to begin and let i run from 1 to $n-1$. If A has either a black square or the left half of a rectangle in the i^{th} position, place $i+1$ at the right end of the permutation. If A has either a white square or the right half of a rectangle in the i^{th} position, place $i+1$ immediately to the left of i or $i-1$ respectively. We also define $\eta(\|) = 1$ and $\eta(\varnothing) = \varnothing$.

It is immediate that this construction yields a Pell permutation because the output has no descents of size larger than 2, and for each descent of size 2, the value in between the values of the descent is to the right of the descent.

Example 2.6. Here are the steps to calculate $\eta(A)$ for $A =$ .

→ 1
 → 21
 → 213
 → 4213
 → 42135
 → 421365

Theorem 2.7. *The restriction of σ to the Pell permutations is a bijection $\sigma : P_n \to \Sigma_{n-1}$ whose inverse is given by $\eta : \Sigma_{n-1} \to P_n$.*

Theorem 2.7 is proven by showing that $\sigma(\eta(A)) = A$ and using the fact $|P_n| = |\Sigma_{n-1}|$.

Proposition 2.8. *$x, y \in S_n$ are equivalent if and only if $\sigma(x) = \sigma(y)$.*

We prove the forward direction of Proposition 2.8 by considering the case where y is obtained from x by a single $\pi_{\downarrow}$-move. The reverse direction is shown by contradiction.

The partial order on Σ_{n-1} is such that the map $\sigma : P_n \to \Sigma_{n-1}$ is an order isomorphism from the lattice of Pell permutations to Σ_{n-1}. We refer to this lattice as Σ_{n-1}.

From Proposition 1.2, the cover relations in Σ_{n-1} are exactly the relations $\sigma(y) \lessdot \sigma(x)$ where $x \in P_n$ and y is covered by x in S_n.

Proposition 2.9. *The cover relations on sashes are*

1. *A**B* $\lessdot$ *A**B for any sash A and for a sash B whose leftmost tile is not a white square*
2. *A**B* $\lessdot$ *A**B for any sash A and any sash B*
3. *A**B* $\lessdot$ *A**B for any sash A and any sash B*

Example 2.10. See Figure 1 for the poset on Σ_3 and Σ_4.

3 The Hopf Algebra (and Dual Hopf Algebra) of Sashes

The bijection σ allows us to carry the Hopf algebra structure on Pell permutations to a Hopf algebra structure $(\mathbb{K}[\Sigma_\infty], \bullet_S, \Delta_S)$ on sashes and a dual Hopf algebra $(\mathbb{K}[\Sigma_\infty], \Delta_S^*, m_S^*)$ on sashes, where $\mathbb{K}[\Sigma_\infty]$ is a vector space, over a field $\mathbb{K}$, whose basis elements are indexed by sashes. In order to do this, we extend σ and η to linear maps. For each grade n of the vector space, the basis elements are represented by the sashes of length $n-1$. Recall that the sash of length -1 is represented by $\varnothing$, and the sash of length 0 is represented by $\Vert$. Let A, B, and C be sashes. Using σ, we define a product, coproduct, dual product, and dual coproduct of sashes:

$$m_S(A,B) = A \bullet_S B = \sigma\big(\eta(A) \bullet_{\mathrm{Av}} \eta(B)\big) \tag{11}$$

$$\Delta_S(C) = (\sigma \otimes \sigma)\big(\Delta_{\mathrm{Av}}(\eta(C))\big) \tag{12}$$

$$\Delta_S^*(A \otimes B) = \sigma\Big(\Delta_{\mathrm{Av}}^*\big(\eta(A) \otimes \eta(B)\big)\Big) \tag{13}$$

$$m_S^*(C) = (\sigma \otimes \sigma)\big(m_{\mathrm{Av}}^*(\eta(C))\big) \tag{14}$$

These operation definitions are somewhat unsatisfying because they require computing the operation in MR. That is, calculating a product or coproduct in this way requires mapping sashes to permutations, performing the operations in MR, throwing out the non-avoiders in the result, and then mapping the remaining permutations back to sashes. In the rest of this chapter we show how to compute these operations directly in terms of sashes.

3.1 Product

Proposition 3.1. *The empty sash $\varnothing$ is the identity for the product $\bullet_S$. For sashes $A \neq \varnothing$ and $B \neq \varnothing$, the product $A \bullet_S B$ equals:*

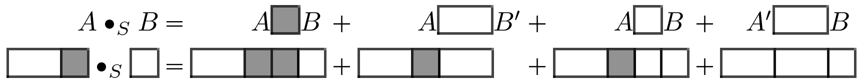
$$\begin{cases} \sum\Big[A\blacksquare B, A'\boxed{}B\Big] & \textit{if } A = A'\blacksquare \\ \sum\Big[A\blacksquare B, A\square B\Big] & \textit{if } A \neq A'\blacksquare \end{cases}$$

where $\sum[D,E]$ is the sum of all the sashes in the interval $[D,E]$ on the lattice of sashes.

The case where $A = \Vert$ is an instance of $A \neq A'\blacksquare$, and similarly for $B = \Vert$. In informal terms, the product of two sashes is the sum of the sashes created by joining the two sashes with a black square and a white square, and if by so doing an adjacent black square to the left of a white square is created, then the product has additional terms with rectangles in the places of the adjacent black square and white square.

The proof is obtained by applying the map σ to the product of Pell permutations which is the sum over the interval $[x \cdot y', \pi_\downarrow(y' \cdot x)]$ in the lattice of Pell permutations, where $x \in P_p$, $y \in P_q$, and $y' = (y)_{[p+1,n]}$.

Example 3.2. Let $A =$ and let $B =$. Notice that $A = A'$, where $A' =$, and $B =$ B', where $B' = \|$.



3.2 Dual Coproduct

From Equation (8) and Equation (4), it follows that:

$$m_S^*(C) = \sum_{i=0}^{n} \sigma\big(\eta(C)|_{[i]}\big) \otimes \sigma\big(\eta(C)|_{[i+1,n]}\big) \tag{15}$$

Proposition 3.3. *The dual coproduct on a sash $C \in \Sigma_n$ is given by:*

$$m_S^*(C) = \sum_{i=-1}^{n} C_i \otimes C^{n-i-1}$$

Where $C_i \in \Sigma_i$ is a sash identical to the first i positions of C (unless C has in position i, in which case C_i ends with), and $C^{n-i-1} \in \Sigma_{n-i-1}$ is a sash identical to the last $n-i-1$ positions of C (unless C has in position $i+2$, in which case C^{n-i-1} begins with), and we define $C_0 = C^0 = \|$ and $C_{-1} = C^{-1} = \varnothing$.

The proof is given by showing that C is a term of $A \bullet_S B$ if and only if $A \otimes B$ is a term of $m_S^*(C)$.

3.3 Dual Product

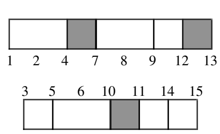
From Equation (9), it follows that:

$$\Delta_S^*(A \otimes B) = \sum_{\substack{T \subseteq [n] \\ |T| = p}} \sigma\Big(\big(\eta(A)\big)_T \cdot \big(\eta(B)\big)_{T^C}\Big) \tag{16}$$

We now prepare to describe the dual product Δ_S^* directly on sashes.

Definition 3.4. Given a set $T \subseteq [n]$ such that $|T| = p$ and $n = p + q$, and given sashes $D \in \Sigma_{p-1}$ and $E \in \Sigma_{q-1}$, define a sash $\gamma_T(D \otimes E)$ by the following steps. First, write D above E. Then, label D with T, by placing the elements of T in increasing order between each position of D, including the beginning and end. Label E similarly using the elements of T^C.

Example 3.5. Let $T = \{1, 2, 4, 7, 8, 9, 12, 13\}$, $D =$, and $E =$



Next, draw arrows from i to $i+1$ for all $i \in [n-1]$. Lastly, follow the path of the arrows placing elements in a new sash based on the following criteria:

Place a rectangle in the i^{th} and $(i+1)^{st}$ positions of the new sash if either of the following conditions are met:

1. if the i^{th} arrow is from D to E, the $(i+1)^{st}$ arrow is from E to D, and there is a or in D in between i and $i+2$

2. if the i^{th} arrow is from E to E, the $(i+1)^{st}$ arrow is from E to D, and there is a or in E in between i and $i+1$

If the above criteria are not met, then the following rules apply:

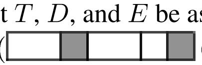
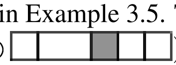
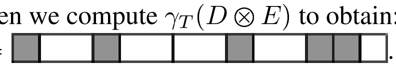
1. if the i^{th} arrow is from D to D (or from E to E), place whatever is in between i and $i+1$ in D (or in E) in the i^{th} position.

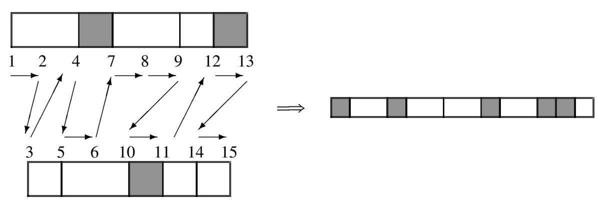
2. if the i^{th} arrow is from D to E, place a black square in the i^{th} position.

3. if the i^{th} arrow is from E to D, place a white square in the i^{th} position.

Note that it may be necessary to replace the left half of a rectangle by a black square or to replace the right half of a rectangle by a white square (as in the first step of the example below).

Example 3.6. Let T, D, and E be as in Example 3.5. Then we compute $\gamma_T(D \otimes E)$ to obtain:

$\gamma_{\{1,2,4,7,8,9,12,13\}}($  $\otimes$  $) =$ .



Theorem 3.7. *The dual product of sashes $D \in \Sigma_{p-1}$ and $E \in \Sigma_{q-1}$, for $p+q=n$, is given by:*

$$\Delta_S^*(D \otimes E) = \sum_{\substack{T \subseteq [n], \\ |T|=p}} \gamma_T(D \otimes E)$$

3.4 Coproduct

We now describe the coproduct in the Hopf algebra of sashes and we begin with some definitions.

Definition 3.8. For $C \in \Sigma_{n-1}$, a *dotting of* C is C with a dot in any subset of the $n-1$ positions of C. An *allowable dotting of* C is a dotting of C that meets all of the following conditions

1. has at least one dot

2. the first dot can be in any position, and dotted positions alternate between a black square (or the left half of a rectangle) and a white square (or the right half of a rectangle)

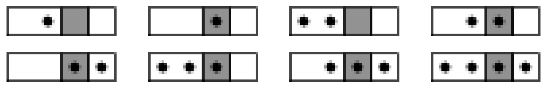


Fig. 2: The allowable dottings of a sash

3. has no instances of or

Figure 2 shows the allowable dottings of the sash .

Consider an allowable dotting $d = c_1 \bullet_1 c_2 \bullet_2 \cdots c_j \bullet_j c_{j+1}$ of a sash C, where each c_i is a sub sash of C without any dots, and $\bullet_i$ is a single dotted position. If any $\bullet_i$ is on the right half of a rectangle, then the the left half of the rectangle in the last position of c_i is replaced by a black square. If $\bullet_i$ and $\bullet_{i+1}$ are in adjacent positions, then $c_{i+1} = \|$. (If any $\bullet_i$ is on the left half of a rectangle, then $\bullet_{i+1}$ is on the right half of the same rectangle, so $c_{i+1} = \|$.)

We use C and d to define two objects A and B that are similar to sashes, but have an additional type of square , which we call a mystery square. If $\bullet_1$ is on a black square or the left half of a rectangle, then let A be the concatenation of the *odd* c_i with a mystery square in between each c_i (where i is odd), and let B be the concatenation of the *even* c_i with a mystery square in between each c_i (where i is even). For example, if $\bullet_1$ is on a black square and j is even, then $A = c_1$ c_3 $\cdots$ c_{j+1} and $B = c_2$ c_4 $\cdots$ c_j. If $\bullet_1$ is on a white square or the right half of a rectangle, then let A be the concatenation of the *even* c_i with a mystery square in between each c_i, and let B be the concatenation of the *odd* c_i with a mystery square in between each c_i.

We use the objects A and B to define four sashes $\underline{A}$, $\overline{A}$, $\underline{B}$, and $\overline{B}$.

To compute $\underline{A}$, consider each mystery square in A. If the mystery square follows c_i and the i^{th} and $(i+1)^{st}$ dots of d are on the same rectangle, , then replace the mystery square after c_i with a white square. Otherwise replace the mystery square with a black square.

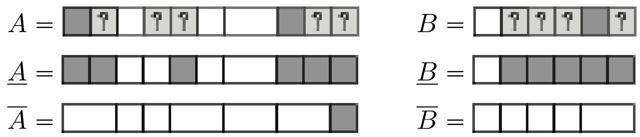
To compute $\overline{A}$, consider each mystery square in A from left to right. If the mystery square follows c_i and the i^{th} and $(i+1)^{st}$ dots of d are on an adjacent black square and white square, , then we check to see whether or not the mystery square is followed by a white square. If the mystery square is followed by a white square (*i.e.* if c_{i+2} starts with a white square), then replace the mystery square and the white square with a rectangle. Otherwise replace the mystery square with a black square. If the mystery square follows c_i and the i^{th} and $(i+1)^{st}$ dots of d are *not* on an adjacent black square and white square, then we check to see whether or not the mystery square is preceded by a black square. If either c_i ends in a black square or $c_i = \|$ and the previous mystery square has been changed to a black square, then replace the mystery square after c_i and the black square before it with a rectangle. Otherwise replace the mystery square after c_i with a white square.

To compute $\underline{B}$, replace all mystery squares of B with black squares.

To compute $\overline{B}$, replace all mystery squares of B with white squares, unless the mystery square is preceded by a black square, in which case replace both the black square and the mystery square with a rectangle.

Example 3.9. If $d =$, then $c_1 =$, $c_2 = c_3 =$, $c_4 = c_5 = c_6 = \|$, $c_7 =$, $c_8 =$, $c_9 = c_{10} = c_{11} = \|$, and $\bullet_1$ is on a black square. Thus,

$A = c_1 \boxed{\cdot} c_3 \boxed{\cdot} c_5 \boxed{\cdot} c_7 \boxed{\cdot} c_9 \boxed{\cdot} c_{11}$ and $B = c_2 \boxed{\cdot} c_4 \boxed{\cdot} c_6 \boxed{\cdot} c_8 \boxed{\cdot} c_{10}$. Using the rules above to compute A, B, and the four sashes $\underline{A}$, $\overline{A}$, $\underline{B}$, and $\overline{B}$, we have:



Given an allowable dotting d of a sash C we define $I_d = \sum [\underline{A}, \overline{A}]$ and $J_d = \sum [\underline{B}, \overline{B}]$ for $\underline{A}$, $\overline{A}$, $\underline{B}$, and $\overline{B}$ computed as above. Thus the notation $I_d \otimes J_d$ denotes $\sum_{\substack{D \in [\underline{A}, \overline{A}] \\ E \in [\underline{B}, \overline{B}]}} D \otimes E$.

Theorem 3.10. *Given $C \in \Sigma_{n-1}$:*

$$\Delta_S(C) = \varnothing \otimes C + C \otimes \varnothing + \sum_{\substack{\text{allowable} \\ \text{dottings} \\ d \text{ of } C}} I_d \otimes J_d$$

References

[1] Marcelo Aguiar, Nantel Bergeron, and Frank Sottile. Combinatorial Hopf algebras and generalized Dehn-Sommerville relations. *Compos. Math.*, 142(1):1–30, 2006.

[2] Eric Babson and Einar Steingrímsson. Generalized permutation patterns and a classification of the Mahonian statistics. *Sém. Lothar. Combin.*, 44:Art. B44b, 18 pp. (electronic), 2000.

[3] Jean-Louis Loday and María O. Ronco. Hopf algebra of the planar binary trees. *Adv. Math.*, 139(2):293–309, 1998.

[4] Claudia Malvenuto. Produits et coproduits des fonctions quasi-symétriques et de l'algèbre des descentes. *(LACIM)*, 16, 1994.

[5] Nathan Reading. Lattice congruences, fans and Hopf algebras. *J. Combin. Theory Ser. A*, 110(2):237–273, 2005.

[6] N. J. A. Sloane. The on-line encyclopedia of integer sequences. `http://oeis.org`. 2013.