Chromatic Polynomials and Rings in Species

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Abstract. We present a generalization of the chromatic polynomial, and chromatic symmetric function, arising in the study of combinatorial species. These invariants are defined for modules over lattice rings in species. The primary examples are graphs and set partitions. For these new invariants, we present analogues of results regarding stable partitions, the bond lattice, the deletion-contraction recurrence, and the subset expansion formula. We also present two detailed examples, one related to enumerating subgraphs by their blocks, and a second example related to enumerating subgraphs of a directed graph by their strongly connected components.

Resumé. Nous présentons une généralisation du polynôme chromatique et de la fonction symétrique chromatique, qui apparaissent dans l'étude des espèces de structures. Ces invariants sont définis pour modules sur anneaux réticulés aux espéces de structures. Les exemples principaux sont les graphes et les partitions d'entiers. Pour ces invariants nouveaux, nous présentons d'analogues de rsultats concernants les partitions stables, le treillis de liaisons, la rélation de contraction-suppression, et la formule d'expansion en termes de sous-ensembles. Nous présentons aussi deux exemples détaillś, l'un lié à l'énumération des sous-graphes par ses blocs, et l'autre lié à l'énumération des sous-graphes d'un graphe dirigé par ses composantes fortement connexes.

Keywords: Graph Polynomials, Chromatic Symmetric Functions, Combinatorial Species, Combinatorial Hopf Algebras, Galois Connections

One well-known invariant of a graph g is its chromatic polynomial $\chi(g; q)$, whose evaluation at a nonnegative integer k counts the number of proper k-colorings of a graph [BL46]. Moreover, there are many generalizations, to the Tutte polynomial [Tut54] and the chromatic symmetric function [Sta95]. Here are some classical results:

Theorem 1 *Let g be a graph with edge set E.*

1. $\chi(g; q) = \sum_{\pi} q(q - 1) \cdots (q - |\pi| + 1)$, *where we sum over all stable partitions of g.*

2. *([Tut54])* $\chi(g; q) = \chi(g - e; q) - \chi(g/e; q)$ *for any edge e of g, where g/e is contraction.*

3. *([Whi32])* $\chi(g; q) = \sum_{\pi \in L_g} \mu(\hat{0}, \pi) q^{|\pi|}$, *where L_g is the bond lattice of g.*

4. *([Whi32])* $\chi(g; q) = \sum_{A \subseteq E} (-1)^{|A|} q^{c(A)}$, *where $c(A)$ is the number of components of $(V(g), A)$.*

All terminology needed to understand these results appear in this paper. Several of these identities have been extended in various ways [Tut54, Ber73, Sta95, GS01, Hum11]. The goal of this present paper is to present a generalization of chromatic polynomials and symmetric functions, and extend the above results, by working with lattice rings in species. While the details are abstract, our results are similar to the Möbius Inversion formula: general algebraic results which give combinatorial results as corollaries. In particular, given any combinatorial setting where our framework applies, one immediately gets an analogue of the Tutte polynomial, and a variation of Theorem 1 for this invariant.

Our philosophy is to replace graphs with any notion of combinatorial objects which have operations analogous to disjoint union and taking induced subgraphs. Such a construction is an abelian Hopf monoid in species [AM10]. Given an abelian Hopf monoid in species, there is a notion of coloring, and the first three results of Theorem 1 generalize to this setting.

Our motivation comes from the work of Aguiar, Bergeron, and Sottile [ABS06], who showed that quasisymmetric functions form the terminal object in the category of combinatorial Hopf algebras. For example, graphs form a graded Hopf algebra, and the resulting map to quasisymmetric functions sends a graph g to its chromatic symmetric function. Thus, one obtains analogues of chromatic symmetric functions. However, the results of Theorem 1 have not been extended to arbitrary combinatorial Hopf algebras. Moreover, known results can have very involved proofs. For instance, in [HM12], a formula for the *antipode* of the Hopf algebra of graphs is proven, using a very complicated argument. However, with a deletion-contraction recurrence, one could reprove their result following Stanley's proof that $(-1)^{|g|}\chi(g;-1)$ is the number of acyclic orientations of g [Sta73]. So our goal is to create new invariants, *and* prove new results about these invariants.

The foundation for our work is Joyal's category of species [Joy81], where we can define the bond lattice. Moreover, the notion of bond lattice comes from the study of Galois connections, and a natural action of set partitions that is analogous to the action of integers on any abelian group. Naturally, in order to say that set partitions act on abelian Hopf monoids, we need to define the notion of ring in species. Rings in species are defined in ongoing joint work with Aguiar and Mahajan [AMW], as an attempt to generalize and understand Hopf rings [RW74] over other categories. We mention some of the axioms for the definition in this paper, as well as the result that every abelian Hopf monoid in species is a module over set partitions.

Our goal is to study 'graded lattice rings' and their modules. Given a module over a graded lattice ring, we can define a generalization of the chromatic polynomial, where the definition involves Möbius inversion, similar to Whitney's formula for the Tutte polynomial of a matroid. To justify that our setting is non-trivial, we give two examples of new enumerative invariants in Sections 5 and 6.

The paper is organized as follows. First, in Section 1, we review the definition of rings in species, and some fundamental results. Then in Section 2, we define our characteristic polynomials and chromatic symmetric functions, and verify Theorem 1, part 1. In Section 3, we prove some basic properties, and a generalization of Theorem 1, part 2, regarding deletion and contraction. In Section 4, we study coclosure operators, Galois connections, and generalizations of Theorem 1, part 3 and part 4. The last two section involves studying new examples related to blocks of graphs, and digraphs.

We acknowledge that we have had many fruitful conversations with Aguiar and Mahajan regarding this present work, which is related to an ongoing project [AMW]. Results obtained in joint work which we have chosen to mention in this abstract will be given appropriate attribution. Finally, we assume familiarity with terminology from the theory of posets, as presented in [Sta97, Rot64].

1 Rings in Species

We begin with some review of known and new material, regarding rings of species. Most of this comes from ongoing joint work with Aguiar and Mahajan [AMW], or previous material [AM10, Joy81].

Definition 2 *A species (with restrictions) is a contravariant functor* $\mathrm{P} : \mathsf{set}^{\times} \to \mathsf{Set}$ *from the category of finite sets with injections to the category of all sets. It is* connected *if* $|\mathrm{P}_{\emptyset}| = 1$.

In this paper, all species are assumed to be connected. In particular, for every finite set I we have a set P_I of I-structures with label set I. Moreover, any injection $\sigma : I \to J$ lifts to a map $\mathrm{P}_{\sigma} : \mathrm{P}_J \to \mathrm{P}_I$. For a subset $S \subset I$, we call this the *restriction* map $\rho_{I,S} : \mathrm{P}_I \to \mathrm{P}_S$. For $x \in \mathrm{P}_I$, we let $x|_S$ denote its image under $\rho_{I,S}$. Species (with restrictions) were originally introduced by Joyal [Joy81], and have found applications in the theory of enumeration [BLL98].

Example 3 *For these examples, fix a finite set I, and a subset $S \subset I$.*

- *Let G_I denote the collection of all graphs with vertex set I. This is a species with restrictions: for a graph g with vertex set I, $g|_S$ is the subgraph induced by S.*

- *Let Π_I denote the collection of all set partitions on I. Given a set partition π, $\pi|_S$ is given by replacing each part B of π by $B \cap S$, and then removing empty parts. For instance, let $\pi = \{\{1,2,3\},\{4,5\},\{6,7\}\} = 123|45|67$. Then $\pi|_{\{1,2,5\}} = 12|5$.*

Hopf monoids are a generalization of Hopf algebras [Mon93], defined with respect to any braided monoidal category [AM10]. However, for the abstract, we work with species.

Definition 4 *A species (with restrictions) P is a* Hopf monoid *if there is a collection of maps $\uplus_{S,T} : \mathrm{P}_S \times \mathrm{P}_T \mapsto \mathrm{P}_{S \sqcup T}$ for any disjoint sets S and T, which is natural in S and T, and is associative. Given $x \in \mathrm{P}_S, y \in \mathrm{P}_T$, we let $x \uplus y$ denote their image under the map $\uplus_{S,T}$. Then associativity means that, for three disjoint sets R, S, T, and $x \in \mathrm{P}_R, y \in \mathrm{P}_S, z \in \mathrm{P}_T$, we have $(x \uplus y) \uplus z = x \uplus (y \uplus z)$. The Hopf monoid is abelian if $x \uplus y = y \uplus x$, where the equality involves identifying $\mathrm{P}_S \times \mathrm{P}_T \simeq \mathrm{P}_T \times \mathrm{P}_S$.*

Viewing P_I as a collection of combinatorial objects, $\uplus_{S,T}$ is a rule for how to combine combinatorial objects with disjoint labels, where the order does not matter, and satisfying axioms which allow us to define $\uplus_{S_1,\ldots,S_k}$ for any collection of finite sets. There are other axioms involving units and counits, and there is a notion of negation (called the *antipode*). We omit these details for length considerations.

Definition 5 *Let R be an abelian Hopf monoid, such that each R_I is also a commutative monoid with multiplication $\cdot_I$ and identity 1_I. Then R is a ring of species if multiplication distributes over addition: for disjoint sets S, T, $x \in \mathrm{R}_{S \cup T}, y \in \mathrm{R}_S, z \in \mathrm{R}_T$, we have $x \cdot (y \uplus z) = x|_S \cdot y \uplus x|_T \cdot z$.*

Ring in species is a special case of a more general notion of a ring in a distributive category [AMW]. The main feature of a ring in species is that multiplication is defined for objects with the *same* label set, and addition is defined for objects with *disjoint* label sets.

Example 6 *Now we mention some examples of rings in species.*

- *Consider G, the species of graphs. Addition is given by sending a pair of graphs (g, h) to their disjoint union $g \uplus h$, while multiplication is given by sending pairs of graphs to their intersection $g \cap h$, the graph which contains only edges that are in both g and h.*

- *The species* Π *is also a ring in species, where addition is given by disjoint union. That is, given set partitions* π, σ *on disjoint label sets,* $\pi \uplus \sigma$ *is the set partition consisting of all the parts of* π *and* σ. *For instance,* $12|3 \uplus 456 = 12|3|456$. *If* π *and* σ *are partitions of the same set, then their product is* $\pi \wedge \sigma$, *their most common refinement. For instance,* $13|245 \wedge 1|23|45 = 1|2|3|45$.

Definition 7 *Let* R *be a ring in species, and let* P *be an abelian Hopf monoid. Then* P *is a module over* R *if, for each* I R_I *acts on* P_I. *For each* $r \in R_I, p \in P_I$, *let* $r \lhd p \in P_I$ *denote the action of* r *on* p.

1. *For* $r, s \in R_I$, *and* $p \in P_I$, *we have* $r \lhd (s \lhd p) = (r \cdot s) \lhd p$. *Moreover,* $1_I \lhd p = p$.

2. *For a disjoint union* $I = S \cup T$, *and* $r \in R_I$, $p \in P_S$, $q \in P_T$, *we have* $r \lhd (p \uplus q) = r_S \lhd p \uplus r|_T \lhd q$.

Moreover, if P *is also a ring,* R *is commutative, and* $r \lhd (s \cdot p) = (r \lhd s) \cdot p = s \cdot (r \lhd p)$, *for all* $r \in R_I$, $s, p \in P_I$, *then* R *is an* algebra *over* P.

In the work of Aguiar and Mahajan [AM13], they study a 'characteristic' action of set partitions on every Hopf monoid. Using rings in species, we can restate their results in a succint way:

Proposition 8 ([AMW]) *Let* P *be an abelian Hopf monoid in species. Then* P *is a module over* Π. *If* P *is a ring, then* P *is an algebra over* Π.

The action is given as follows: let $p \in P_I$, $\pi \in \Pi_I$. Then $\pi \lhd p = \uplus_{B \in \pi} p|_B$, where we are taking the sum over all parts of the partition π. This simple proposition is what allows us to prove results regarding bond lattices, stable partitions, and other simple properties of chromatic polynomials. This proposition is also the primary motivation for working with species. If we worked with graded Hopf algebras instead, then it turns out that Π is replaced by the Hopf ring of symmetric functions. Unfortunately, the resulting action is not as well-behaved, as it involves Kronecker products, which do not turn symmetric functions into a graded lattice. Using this action, we generalize the notion of 'connected components' of a graph to any element x in any abelian Hopf monoid P.

Definition 9 *Given* P *and* $x \in P_I$, *let* π_x *be the most refined partition for which* $\pi \lhd x = x$. *We call the parts of* π_x *the* components *of* x.

For generalizations of Tutte polynomials, we need a ring in species to have extra structure.

Definition 10 *A* lattice ring *in speces is a ring* R *such that:*

1. *The multiplication on* R_I *is idempotent and commutative.*

2. R_I *is a finite set.*

3. *Addition is an order-preserving operation* $\uplus_{S,T} : R_S \times R_T \to R_I$, *where we are using cartesian product of posets.*

4. *Moreover, if each* R_I *is graded, then we call* R *a* graded *lattice ring if for all* $x \in R_S$, $y \in R_T$, *we have* $\rho_I(x \uplus y) = \rho_S(x) + \rho_T(y)$, *where* ρ_A *is the rank function of* R_A.

The species G and Π are examples of graded lattice rings. Also, the first two axioms force R_I to be a lattice for all I, hence the name *(graded) lattice ring*.

2 Characteristic invariants for Abelian Hopf Monoids

In this section we define the polynomial and symmetric invariants which are the focus of our study. In each case, we focus on the example of graphs. Throughout, we let $\mathbb{Z}[t]$ be the polynomial ring in the variable t, and let Λ be the ring of symmetric functions in the commuting indeterminates $\mathbf{x} = x_1, x_2, \ldots$, with coefficients in a field $\mathbb{K}$.

Given a graded poset P with bottom element $\hat{0}$ and top element $\hat{1}$, let $\chi(P; t) = \sum_{p \in P} \mu(\hat{0}, p) t^{\rho(\hat{1}) - \rho(p)}$ denote the characteristic polynomial of P, where ρ is the rank function of P.

Definition 11 *Let* M *be a module over a graded lattice ring* R*, and let* $\mathbb{Z}[x](\mathrm{M}_I)$ *denote the free* $\mathbb{Z}[x]$-*module with generating set* $\{m : m \in \mathrm{M}_I\}$*. Given a finite set I, and $m \in \mathrm{M}_I$, we define the* characteristic polynomial *of m to be*

$$\chi_{\mathrm{R,M}}(m; x) = \sum_{r \in \mathrm{R}_I} \chi([r, \hat{1}]; x)(r \lhd m)$$

which is an element of $\mathbb{Z}[x](\mathrm{M}_I)$*.*

We write

$$\chi_{\mathrm{R,M}}(m; x) = \sum_{m' \in \mathrm{M}_I} \chi_{\mathrm{R,M}}(m, m'; x) m',$$

and call these coefficients the characteristic polynomials for the pair (m, m'). There are three reasons we work in $\mathbb{Z}[x](\mathrm{M}_I)$. First, there is no general analogue of the edgeless graph for an arbitrary module. Second, in the next section we derive some combinatorial results about invariants for pairs (m, m') that we would not have noticed if we only focused on specific values of m'. Finally, by choosing to work in a free module, we obtain different results for any specialization where we replace elements of M_I by polynomials or symmetric functions.

Let $\mathrm{R} = \Pi$, and $\mathrm{M} = \mathrm{G}$, and let g be a graph with vertex set I. Finally, let d_I denote the edgeless graph on I. Then $\chi_{\Pi,\mathrm{G}}(g, d; x) = \sum_{\pi : \pi \lhd g = d} \chi([\pi, \hat{1}]; x) = \sum_{\pi : \pi \lhd g = d} x(x-1) \cdots (x - |\pi| + 1)$. Set partitions π for which $\pi \lhd g = d$ are called stable partitions. Thus, by Theorem 1, part 1, $\chi_{\Pi,\mathrm{G}}(g, d; x) = \chi(g; x)$. Similarly, if we replace the graphs h by $(t+1)^{e(h)}$, where e is the number of edges of h, then $\chi_{\Pi,\mathrm{G}}(g, x)$ specializes to the *bad* chromatic polynomial, which enumerates all colorings, weighting each coloring by $(t+1)^m$, where m is the number of monochromatic edges in the coloring.

Now allow M to be any abelian Hopf monoid in species, and let $m \in \mathrm{M}_I$, and fix a set S. We generalize the chromatic symmetric function to this setting. Given any function $f : I \to S$, and $m \in \mathrm{M}$, we define $m|_f = \biguplus_{i \in S} m|_{f^{-1}(i)}$. For the case of graphs, given a graph g, and a coloring f, $g|_f$ is the subgraph of g with the same vertex set, and only the edges of g whose endpoints get the same color. Given $\mathbf{x} = x_1, \ldots$, a sequence of commuting indeterminates, and a coloring f, we weight f by $\mathbf{x}^f = \prod_{i \in I} x_{f(i)}$. Then

$$\hat{\chi}_{\mathrm{M}}(m; \mathbf{x}) = \sum_{f : I \to \mathbb{P}} \mathbf{x}^f m|_f$$

is the chromatic symmetric function of m, which lies in the free Λ-module on the set M_I.

Proposition 12 *We have the following results.*

- $\chi_{\Pi,\mathrm{M}}(m; x) = \sum\limits_{\pi \in \Pi_I} x(x-1) \cdots (x - |\pi| + 1)(\pi \lhd m).$

- *For $x \in \mathbb{N}$, $\chi_{\Pi,\mathrm{M}}(m;x) = \displaystyle\sum_{f:I \to \{1,\ldots,x\}} m|_f$.*

- *$\hat{\chi}_{\mathrm{M}}(m;\boldsymbol{x}) = \displaystyle\sum_{\pi \in \Pi_I} m_{\lambda(\pi)}(\pi \lhd m)$, where $m_{\lambda(\pi)}$ are the augmented monomial symmetric functions, and $\lambda(\pi)$ consists of the sizes of the blocks of π, written in decreasing order.*

In particular, for $\mathrm{R} = \Pi$, the characteristic polynomial has an interpretation in terms of coloring, and it generalizes to a symmetric function invariant. This leads to the following open question:

Question 1 *What sufficient conditions on a graded lattice ring R are needed so that $\chi_{\mathrm{R},\mathrm{M}}(m;x) \in \mathbb{N}$ for all $x \in \mathbb{N}$, $m \in \mathrm{M}_I$, and for all modules M over R?*

Now we generalize the Tutte polynomial, following the definition given by Sokal [Sok05], which is equivalent to the classical Tutte polynomial.

Definition 13 *Let R be an algebra over a graded lattice ring L in species, and assume that R is also a lattice ring. Given $r \in \mathrm{R}_I$, let ℓ_r be the minimum element of L_I such that $\ell_r \lhd r = r$. We call this the L-closure of r. Letting ρ denote the rank function of L_I, the Tutte polynomial is defined by*

$$T_{\mathrm{L},\mathrm{R}}(r;x) = \sum_{s \leq r} x^{\rho(\hat{1}) - \rho(\ell_s)} s.$$

We have not shown that the L-closure is a well-defined operation: this is the content of Lemma 18. As we shall see in Theorem 22, the Tutte polynomial and characteristic polynomials are related by Möbius inversion. Let $\mathrm{L} = \Pi$ and $\mathrm{R} = \mathrm{G}$. Then $T_{\Pi,\mathrm{G}}(g;x) = \sum_{h \subset g} x^{c(h)} h$, where we are summing over all spanning subgraphs of G. Replacing h with $t^{e(h)}$, we obtain Sokal's definition of the Tutte polynomial.

3 Simple Identities and Deletion-Contraction

In this section, we describe a few simple identities for characteristic invariants, including a generalization of the deletion-contraction recurrence. This generalizes Theorem 1, part 2. All of these identities follow from elementary manipulations, and generalizations of properties of rings and their modules.

Proposition 14 ([AMW]) *Let P be an abelian Hopf monoid. Fix a finite set I, and $p, p' \in \mathrm{P}_I$. Then*

$$\hat{\chi}_{\mathrm{P}}(p;\boldsymbol{x} + \boldsymbol{y}) = \sum_{S \subset I} \hat{\chi}_{\mathrm{P}}(p|_S;\boldsymbol{x})\hat{\chi}_{\mathrm{P}}(p|_{I \setminus S};\boldsymbol{y})$$

where $\boldsymbol{x} + \boldsymbol{y}$ means that we set $x_i = x_i + y_i$ for each i, where the x_i and y_i are commuting indeterminates.

Also, by working with pairs (m, m'), we have found a new identity for chromatic polynomials.

Proposition 15 *Let P be an abelian Hopf monoid. Fix a finite set I, $p, p' \in \mathrm{P}_I$, and let $q, t \in \mathbb{Z}$. Then*

$$\chi_{\Pi,\mathrm{P}}(p, p'; qt) = \sum_{p'' \in \mathrm{P}_I} \chi_{\Pi,\mathrm{P}}(p, p''; q)\chi_{\Pi,\mathrm{P}}(p'', p'; t)$$

For graphs, this identity is equivalent to $\chi(g; qt) = \sum_{h \subset g} \chi(g/h; q)\chi(h; t)$, where we sum over all spanning subgraphs h of g. It is an easy exercise to find a combinatorial proof.

Proposition 16 *Let* R *be a ring in species, and let* M *be a module over* R. *Fix a finite set* I, *and* $r \in R_I, m, m' \in M_I$. *Then*

$$\chi_{R,M}(rm, m'; x) = \sum_{m'' \in M_I : rm'' = m'} \chi_{R,M}(m, m''; x)$$

Moreover, this identity remains true if M *is an algebra over* R, *and we choose* $r \in M_I$ *instead.*

Consider the case $R = \Pi$ and $M = G$, the ring of graphs. Let g, h be graphs, and let $r = g - e$, for some edge $e \in E(g) \setminus E(h)$. Applying the proposition in this case, we obtain the deletion contraction recurrence. A similar result holds when working with chromatic symmetric functions, obtained with Aguiar and Mahajan. We also obtained the following corollary.

Corollary 17 ([AMW]) *Let* G *be the ring of graphs, and let* I *be a finite set,* $g, h \in G_I$. *Then*

1. *If* $E(g) \subset E(h)$, *then* $\hat{\chi}_G(g, h; \boldsymbol{x}) = p_{\lambda(h)}$, *where* $\lambda(h)$ *is the sizes of the components of* h, *written in decreasing order.*

2. *Otherwise, given* $e \in E(g) \setminus E(h)$, *we have*

$$\hat{\chi}_G(g, h; \boldsymbol{x}) = \hat{\chi}_G(g - e, h; \boldsymbol{x}) - \hat{\chi}_G(g, h + e; \boldsymbol{x}).$$

4   Coclosure Operators and Galois Connections

Given a module M over a graded lattice ring R, and an element $m \in M_I$, we can define a coclosure operator on R_I. Moreover, when M is also a lattice ring, we can obtain a Galois connection. We use the coclosure operator to define the bond lattice. In this section, we detail these two results, and demonstrate how to compute the characteristic invariants from the bond lattices, thus generalizing Theorem 1, parts 3 and 4. We define coclosure operator in Proposition 19, and define Galois connection in Proposition 23. The bond lattice $L(g)$ of g is the collection of all set partitions π of $V(g)$ such that, for each block B of π, $g|_B$ is connected. The partial order for $L(g)$ is refinement. The bond lattice is defined by a coclosure operator on Π_I, which is the basis of our generalization.

Lemma 18 *Let* R *be a graded lattice ring,* M *an* R-*module,* $m \in M_I$. *If* $r \lhd m = s \lhd m$ *for* $r, s \in R_I$, *then* $(r \wedge s) \lhd m = r \lhd m$.

For $m \in M_I$, let r_m denote its R-closure, which is well-defined. For fixed m, and $s \in R_I$, define $s^m = r_{s \lhd m}$. Then s^m is the minimum $r \in R_I$ such that $r \lhd m = s \lhd m$.

Proposition 19 *Let* $m \in M_I$, *where* M *is a module over a graded lattice ring* R *in species. Then the map* $(-)^m : R_I \to R_I$ *which sends* s *to* s^m *is a coclosure operator, meaning that, for all* $s, t \in R_I$, $s^m \leq t$ *if and only if* $s^m \leq t^m$.

A ring element r is m-closed if $r^m = r$. The collection of closed elements forms the m-closure lattice R_m, with the same join operation from R_I. For G, and $R = \Pi$, then Π_g is the bond lattice of g.

Theorem 20 *Let* R *be a graded lattice ring with rank function* ρ, M *be an* R-*module,* I *be a finite set, and* $m \in M_I$. *Furthermore, let* L_m *denote the lattice of* m-*closed elements in* R_I, *and let* $r \in L_m$. *Then*

$$\chi_{R,M}(m, r \lhd m; x) = \sum_{s \geq r} \mu_{L_m}(r, s) x^{\rho(1) - \rho(s)}.$$

Theorem 21 *Let* M *be an abelian Hopf monoid,* $m \in M_I$. *Furthermore, let* L_m *denote the lattice of* m-*closed elements in* Π_I, *and let* $\sigma \in L_m$. *Then, in terms of power sum symmetric functions,*

$$\hat{\chi}_M(m, \sigma \lhd m; \boldsymbol{x}) = \sum_{\pi \geq \sigma} \mu_{L_m}(\sigma, \pi) p_{\lambda(\pi)}.$$

Now we extend Theorem 1, part 4, which is the subset expansion formula for chromatic polynomials. Our extension comes from two facts: first, the Tutte polynomial and the *bad* chromatic polynomial of a graph are equivalent, up to substitution. Second, the subset expansion is a summation of Möbius function values over the lattice of graphs, ordered by inclusion of edge sets. Consider the free $\mathbb{Z}[x]$-module $\mathbb{Z}[x](R_I)$, where R_I is a lattice ring in species, and let $\mu_R : \mathbb{Z}[x](R_I) \rightarrow \mathbb{Z}[x](R_I)$ be the linear transformation defined by $\mu_R(r) = \sum_{s \leq r} \mu_{R_I}(s, r)s$. This linear transformation is invertible, with inverse $\zeta_R : \mathbb{Z}[x](R_I) \rightarrow \mathbb{Z}[x](R_I)$ given by $\zeta_R(r) = \sum_{s \leq r} s$. The Tutte polynomial and characteristic polynomial are equivalent, up to Möbius inversion.

Theorem 22 *Let* S *be a lattice ring in species, and an algebra over the graded lattice ring* R. *Let* I *be a finite set,* $s \in S_I$.

1. $\chi_{R,S}(r; x) = \mu_S \circ T_{R,S}(r; x)$

2. $T_{R,S}(r; x) = \zeta_R \circ \chi_{R,S}(r; x)$

In particular, the characteristic polynomial for a pair (r, r') *are given by*

$$\chi_{R,S}(r, r'; x) = \sum_{r'' \in S_I : r' \leq r'' \leq r} \mu_{S_I}(r', r'') x^{\rho(\hat{1}) - \rho(r'')}.$$

Similarly, the chromatic symmetric function for a pair (r, r') *is given by*

$$\hat{\chi}_S(r, r'; \boldsymbol{x}) = \sum_{r'' \in S_I : r' \leq r'' \leq r} \mu_{S_I}(r', r'') p_{\lambda(r'')}.$$

The fact that Möbius functions may vary for different elements in a lattice ring is the primary reason why we have chosen our invariants to lie in free modules $\mathbb{Z}[x](M_I)$, rather than being elements of $\mathbb{Z}[x]$. When $M = G$ and $R = \Pi$, then the Möbius function is $\mu_{R_I}(h, g) = (-1)^{e(g) - e(h)}$, so one recovers the familiar result that the bad coloring polynomial and the Tutte polynomial are equivalent, up to substitution.

The proof of Theorem 22 relies on basic facts about Möbius inversion and Galois connections, as in [Rot64]. When M is a lattice ring in species, and an algebra over R, then the coclosure operator comes from a natural Galois connection. Let $\varphi : M_I \rightarrow R_I$ be the order-preserving map given by $m \mapsto r_m$. For a fixed $m \in M_I$, let $\psi : R_I \rightarrow [\hat{0}, m]$ be the order-preserving map given by $r \mapsto r \lhd m$.

Proposition 23 *Let* M *be a lattice ring in species, and an algebra over* R, *a graded lattice ring in species. Then for* $m \in M_I$, *the pair* $(\varphi|_{[\hat{0},m]}, \psi)$ *forms a Galois connection. In other words, for all* $n \in [\hat{0}, m]$ *and* $s \in R_I$, *we have* $r_n \leq s$ *if and only if* $n \leq s \lhd m$.

5   Block Tutte Polynomial of a Graph

In this section, we apply our results to the study of new polynomial invariants of graphs, based on studying the blocks of a graph. A (non-trivial) *block* of a graph g is a maximal subgraph h on at least two vertices with no cut vertex. Then h is either a bridge of g, or a maximal 2-connected subgraph. We identify blocks with their vertex sets for the rest of this section. Background regarding blocks can be found in [Die10].

The structure of the blocks of a graph g is known, although our approach to the structure will use hypergraphs. Recall that a hypergraph is a pair $h = (V, E)$, where $E \subset 2^V$. Given a graph g, let h be the hypergraph on the same vertex set with hyperedges S, one for each block S of g. Then h is a linear hyperforest, the *block forest* of g. A hypergraph h is linear if any two hyperedges of h intersect in at most one vertex. A linear hyperforest is a linear hypergraph which does not have an elementary hypercycle, which is any sequence $(v_0, e_0, v_1, e_2, \ldots, e_k, v_0)$ such that $\{v_i, v_{i+1}\} \subset e_i$, $v_0 \in e_k$, and $v_0, \ldots, v_k$, $e_0, \ldots, e_k$ are all distinct. We also assume that linear hyperforests have no hyperedges of size one.
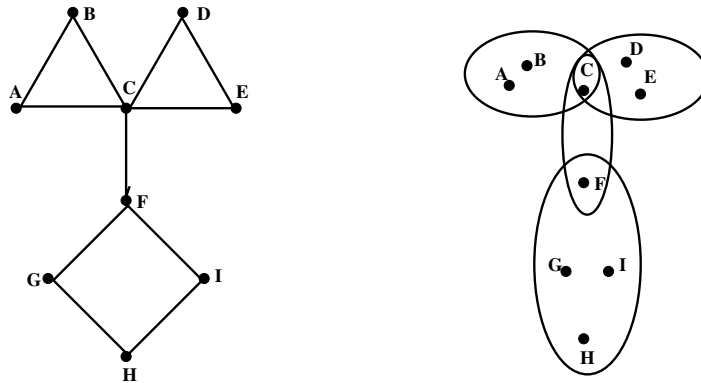

Fig. 1: A graph and its corresponding block hyperforest

Linear hyperforests form a graded lattice ring F, in species, which act on G. The species F is an abelian Hopf monoid, where addition is given by disjoint union. Given two linear hyperforests f and f', we define their product as follows: $f \cdot f'$ has edge set $\{e \cap e' : e \in f, e' \in f'\} \setminus \{\emptyset\}$. As such, Π is a subring of F. Thus F_I is a graded lattice, and is isomorphic to the lattice $\Sigma_{n,2}^2$ appearing in [BBL$^+$99]. The rank of a hyperforest f is $2|I| - 2c(f) - e(f)$, where $c(f)$ is the number of components, and $e(f)$ is the number of hyperedges. Finally, the rank of $\hat{1}$ is $2|I| - 3$.

Now let G be the lattice ring of graphs. Then F acts on G as follows: given $f \in F_I$, and $g \in G_I$, we let $f \lhd g$ be the graph with edge set $\{e \in g : e \subseteq e' \in f \text{ for some } e'\}$. That is, we remove any edge from g that is not contained in some hyperedge of f. Then G is an algebra over F. Consider a fixed graph g. Then F_g consists of all linear hyperforests f on $V(G)$ such that, for each hyperedge S of f, $g|_S$ has no cut vertex. Moreover, f_g, the F-closure of G, is the block hyperforest of g, as in Figure 5. Thus, for $g \in G$, we have

$$T_{F,G}(g;x) = \sum_{h \subset g} x^{2c(h)+b(h)-3} h$$

where $b(h)$ is the number of non-trivial blocks of h. From our general theory, we obtain a natural invariant

of graphs, which enumerates spanning subgraphs, with weights based on the number of connected components and the number of blocks. Moreover, from our general framework we obtain analogues of Theorem 1. An open question is to determine nice combinatorial interpretations for $\chi_{\mathrm{F,G}}(g; x)$ when $x \in \mathbb{N}$.

6 Strong Tutte Polynomial of a Directed Graph

In this section, we apply our results to the study of new polynomial invariants of directed graphs, or digraphs, based on studying the strongly connected components. A strongly connected component of a digraph $\vec{g}$ is a maximal subgraph $\vec{h}$, such that, for every pair of distinct vertices u and v, there is a directed path from u to v. There is a combinatorial structure encoding the strongly connected components of $\vec{g}$. A DAP (directed acyclic partition) is a pair $\vec{\pi} = (\pi, \vec{g})$ where π is a set partition, and $\vec{g}$ is an acyclic digraph with vertex set given by the blocks of π. Given $\vec{g}$, let π denote the partition of $V(\vec{g})$ into its strongly connected components. Then direct an edge between two parts of π if there is an edge between the corresponding components. We call this the associated DAP. An example appears in Figure 6. It turns out that DAPS form a graded lattice ring $\vec{\Pi}$ in species, which act on $\vec{G}$, the lattice ring of directed graphs. The species $\vec{G}$ forms a ring in a way similar to graphs: restriction is given by induced subdigraphs, addition is given by disjoint union, and multiplication is given by intersection.
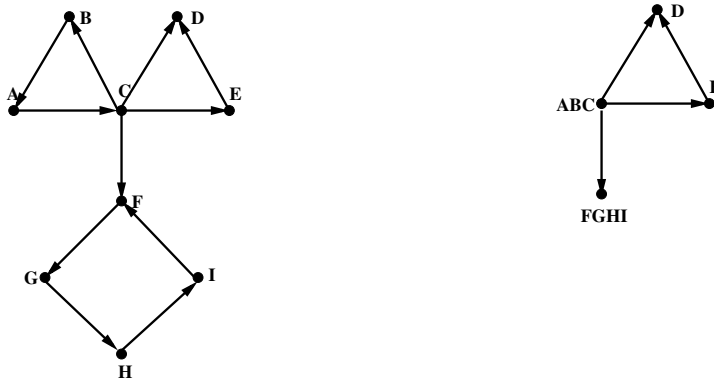

Fig. 2: A digraph and its corresponding DAP

For $\vec{\pi} \in \vec{\Pi}_I$, and $S \subset I$, the restriction $\vec{\pi}|_S$ is given by restricting each part of π to S, and sending directed edges (B, B') to $(B \cap S, B' \cap S)$ when both sets are non-empty. Addition is given by disjoint union. For the multiplication structure, it is easier to define the cover relations of the resulting partitial order on $\vec{\Pi}_I$: given a DAP π, there are two types of cover relations:

- Suppose $\vec{\pi}$ has blocks B, B' where (B, B') is an arc. Then replacing B and B' with their union, and contracting the arc (B, B') gives a new DAP $\vec{\sigma}$ which covers $\vec{\pi}$.

- Suppose $\vec{\pi}$ has blocks B, B' that have no arcs between them. If adding the arc B, B' does not create a cycle, then we may add it, to obtain a new DAP $\vec{\sigma}$, which covers $\vec{\pi}$.

This does turns $\vec{\Pi}_I$ into a graded lattice, with rank function $\rho(\vec{\pi}) = 2n - |\pi| - c(\pi)$, where $c(\pi)$ are the number of components of the digraph on the blocks of π. The rank of $\hat{1}$ is $2|I| - 2$. Then $\vec{\Pi}$ acts on $\vec{G}$

as follows: given $\vec{\pi} \in \vec{\Pi}_I$, and $\vec{g} \in \vec{G}_I$, we let $\vec{\pi} \lhd \vec{g}$ be the directed graph obtained from $\vec{g}$ by removing any directed edge u, v, if u and v are in distinct parts B, B' of $\vec{\pi}$, unless (B, B') is a directed edge of $\vec{\pi}$. Moreover, for fixed $\vec{g}$, $\vec{\Pi}_{\vec{g}}$ consists of all daps $\vec{\pi}$ such that, for each block B, $\vec{g}|_B$ is strongly connected, and such that there are no directed edges from parts B, B' whenever there is no arc of $\vec{g}$ with source in B and target in B'. Moreover, $\vec{\pi}_{\vec{g}}$, the closure of $\vec{g}$, forms an analogue of the notion of block forest. It is obtained by partitioning $\vec{g}$ into its strongly connected components, and then taking the quotient digraph, (so two parts are connected by an arc if and only if there is an arc between the corresponding vertex sets in $\vec{g}$). This quotient is sometimes called the condensation of $\vec{g}$. Note that since we are working with species, the vertices of the condensation are labeled by the sets we quotiented out by, which is how we arrived at the notion of a DAP. Thus, for $\vec{g} \in \vec{G}_I$, we have

$$T_{\vec{\Pi}, \vec{G}}(\vec{g}; x) = \sum_{\vec{h} \subset \vec{g}} x^{c(\vec{h}) + s(\vec{h}) - 2} \vec{h}$$

where $s(\vec{h})$ is the number of strongly connected components of $\vec{h}$. From our general theory, we obtain a very natural invariant of digraphs, which enumerates spanning subdigraphs, with weights based on the number of components and strongly connected components. Due to all the results in Theorem 1 generalizing to this situation, this makes T a natural analogue of the Tutte polynomial for digraphs. Of course, this is not the only possible analogue [CG95], although it does have the advantage that we get a notion of 'bond lattice' for our new invariant.

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