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Rationally smooth Schubert varieties, inversion hyperplane arrangements, and Peterson translation

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Abstract. We show that an element w of a finite Weyl group W is rationally smooth if and only if the hyperplane arrangement $\mathcal{I}(w)$ associated to the inversion set of w is inductively free, and the product $(d_1+1)\cdots(d_l+1)$ of the coexponents $d_1,\ldots,d_l$ is equal to the size of the Bruhat interval $[e,w]$. We also use Peterson translation of coconvex sets to give a Shapiro-Steinberg-Kostant rule for the exponents of w.

Résumé. Nous montrons qu'un élément w d'un groupe de Weyl fini est rationnellement lisse si et seulement si l'arrangement des hyperplans associé à l'ensemble d'inversion de w est libre, et le produit $(d_1+1)\cdots(d_l+1)$ des coexposants $d_1,\ldots,d_l$ est égal à la cardinalité de l'intervalle $[e,w]$ pour l'ordre de Bruhat. Nous donnons une règle de Shapiro-Steinberg-Kostant pour calculer les exposants de w en utilisant traduction de Peterson sur des sous-ensembles coconvexes.

Keywords: rational smoothness, Schubert varieties, inversion sets, inversion arrangements, hyperplane arrangements, Peterson translation

1 Introduction

Let R be a crystallographic root system in a Euclidean space V, and let R^+ be the subset of positive roots. If we identity V with V^* using the inner product, then the vectors of R^+ cut out a hyperplane arrangement in V. It is well-known that the characteristic polynomial $\chi(t)$ of this arrangement is equal to a product $\prod_{i=1}^{l}(t-m_i)$, where l is the rank of R. The integers $m_1,\ldots,m_l$ that appear in this factorization are called the exponents of R, and arise in many other contexts. In particular, if W is the Weyl group of R, and ℓ is the length function on W, then the Poincare polynomial $P(q)=\sum_{w\in W} q^{\ell(w)}=\prod_{i=1}^{l}[m_i+1]_q$, where $[m]_q$ is the q-integer $(1+q+\ldots+q^{m-1})$. If X is the generalized flag variety of R, then $P(q^2)=\sum_i q^i \dim H^i(X)$, so the exponents can be used to calculate the Betti numbers of X. The exponents can also be calculated directly from R via the Shapiro-Steinberg-Kostant rule: the multiplicity of m as an exponent of R is the number of positive roots of height m minus the number of positive roots of height $m+1$.

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In this paper (an extended abstract of [Slo13b] and [Slo13a]), we show that the picture above extends to any rationally smooth Schubert variety of the flag variety X. Furthermore, by combining the condition that the inversion arrangement be free with a condition introduced by Hultman, Linusson, Shareshian, and Sjöstrand, we can characterize the Schubert varieties which are rationally smooth.

The Schubert varieties $X(w)$ of X are indexed by the elements w of the Weyl group W. Let $\leq$ denote the Bruhat order on W, and let $[e, w]$ denote the interval in Bruhat order between the identity e and the element w. The Poincare polynomial of w is the polynomial

$$P_w(q) = \sum_{x \in [e,w]} q^{\ell(x)}.$$

Note that $P_w(q)$ has degree $\ell(w)$. As with the flag variety, $P_w(q^2) = \sum_i q^i \dim H^i(X(w))$. A theorem of Carrell and Peterson states that $X(w)$ is rationally smooth if and only if $P_w(q)$ is palindromic, meaning that $q^{\ell(w)} P_w(q^{-1}) = P_w(q)$ [Car94]. We say that $w \in W$ is rationally smooth if this latter condition is satisfied. By combining deep results of Gasharov, Billey, Billey-Postnikov, and Akyildiz-Carrell, we get the following result:

Theorem 1.1 ([Gas98], [Bil98], [BP05], [AC12])**.** *Let W be a finite Weyl group. An element $w \in W$ is rationally smooth if and only if*

$$P_w(q) = \prod_{i=1}^{l} [m_i + 1]_q$$

for some collection of non-negative integers $m_1, \ldots, m_l$.

Theorem 1.1 allows us to make the following definition:

Definition 1.2. *Let W be a finite Weyl group. If $w \in W$ is rationally smooth, then the exponents of w are the integers $m_1, \ldots, m_l$ appearing in Theorem 1.1.*

Given an element $w \in W$, the inversion set of w is the set

$$I(w) = \{\alpha \in R^+ : w^{-1}\alpha \in R^-\},$$

where R^- is the set of negative roots of R. The inversion hyperplane arrangement $\mathcal{I}(w)$ of w is the hyperplane arrangement in V cut out by the elements of $I(w)$. If w_0 is the longest element of W, then $X(w_0) = X$, $I(w_0) = R^+$, and $\mathcal{I}(w_0)$ is the arrangement cut out by R^+ mentioned above.

Given an arrangement $\mathcal{A} = \mathcal{A}(T)$ in V cut out by a set of vectors T, let $V_{\mathbb{C}} = V \otimes \mathbb{C}$, $R = S^* V_{\mathbb{C}}$, and Q be the polynomial $\prod_{\alpha \in T} \alpha$ in R cutting out $\mathcal{A}$. Let $\operatorname{Der}(\mathcal{A})$ be the set of derivations of R which preserve the ideal generated by Q. The set $\operatorname{Der}(\mathcal{A})$ is an R-module, called the module of derivations of $\mathcal{A}$. The arrangement $\mathcal{A}$ is said to be free if $\operatorname{Der}(\mathcal{A})$ is a free R-module. In this case, $\operatorname{Der}(\mathcal{A})$ has a homogeneous basis, and the polynomial degrees $d_1, \ldots, d_l$ of the elements of this basis are called the coexponents of $\mathcal{A}$. When $\mathcal{A}$ is free, a theorem of Terao [Ter81] states that the characteristic polynomial $\chi(\mathcal{A}; t)$ of $\mathcal{A}$ factors as

$$\chi(\mathcal{A}; t) = \prod_i (t - d_i).$$

It is well-known that the arrangement $\mathcal{I}(w_0)$ cut out by R^+ is free, with coexponents corresponding to the exponents of R. We can now state the main theorem:

Theorem 1.3. *Let W be a finite Weyl group. An element $w \in W$ is rationally smooth if and only if the inversion hyperplane arrangement $\mathcal{I}(w)$ is free, and the product $\prod_i (1 + d_i)$ of the coexponents $d_1, \ldots, d_l$ is equal to the size of the Bruhat interval $[e, w]$. Furthermore, if w is rationally smooth then the coexponents $d_1, \ldots, d_l$ are equal to the exponents of w.*

When $\mathcal{I}(w)$ is free, the product $\prod_i (1+d_i) = (-1)^l \chi(\mathcal{I}(w); -1)$ is equal to the number of chambers of $\mathcal{I}(w)$. The condition that the number of chambers of $\mathcal{I}(w)$ be equal to the size of the Bruhat interval $[e, w]$ has previously been studied by Hultman, Linusson, Shareshian, and Sjöstrand (type A, [HLSS09]) and Hultman (all finite Coxeter groups, [Hul11]). Accordingly, we call this the HLSS condition (see Section 5).

As an immediate corollary of Theorem 1.3, we have:

Corollary 1.4. *Let W be a finite Weyl group. If $w \in W$ is rationally smooth, then*

$$\chi(\mathcal{I}(w); t) = \prod_{i=1}^{l} (t - m_i),$$

where $m_1, \ldots, m_l$ are the exponents of w.

In type A, Corollary 1.4 has previously been proved by Oh, Postnikov, and Yoo [OPY08]. Their proof implicitly shows that $\mathcal{I}(w)$ is free. The general case of Corollary 1.4 answers a conjecture of Yoo [Yoo11, Conjecture 1.7.3]. Oh, Postnikov, and Yoo also show, in type A, that w is rationally smooth if and only if the Poincare polynomial $P_w(q)$ is equal to the wall-crossing polynomial of $\mathcal{I}(w)$. This result has been extended to all finite-type Weyl groups by Oh and Yoo [OY10], using what we will call chain Billey-Postnikov (BP) decompositions (this is a modest variation on the terminology in [OY10]).

Inspired by [OY10], we list the rationally smooth elements in finite type which do not have a chain BP decomposition. We also show that an element w of an arbitrary finite Coxeter group has a chain BP decomposition if and only if $\mathcal{I}(w)$ has a modular coatom of a certain form. From these two results, we show that $\mathcal{I}(w)$ is inductively free when w is rationally smooth, with coexponents equal to the exponents of w. To prove that w is rationally smooth when $\mathcal{I}(w)$ is free and the HLSS condition holds, we use the root-system pattern avoidance criterion for rational smoothness due to Billey and Postnikov [BP05].

The roots in R^+ can be ordered by dominance order $\succeq$, so $\alpha \succeq \beta$ if and only if $\beta - \alpha$ is a sum of simple positive roots. Given a lower order ideal $T \subset R^+$ with respect to dominance order, let h_i be the number of roots in T of height i. Since T is a lower order ideal, we always have $h_i \geq h_{i+1}$. Let $\mathrm{Exp}(T)$ be the multiset which contains i with multiplicity $h_i - h_{i+1}$. One example of a lower order ideal is the set $-\Omega T_e X(w)$, where $\Omega T_e X(w) \subset R^-$ is the set of torus weights of the Zariski tangent space of $X(w)$ at the identity. A theorem of Akyildiz-Carrell states that when $X(w)$ is smooth (a stronger condition than being rationally smooth), the set $\mathrm{Exp}(-\Omega T_e X(w))$ is precisely the set of exponents of w [AC12]. This gives an analogue of the Shapiro-Steinberg-Kostant rule for the exponents of a smooth Schubert variety w. A theorem of Summers-Tymoszko [ST06] and Abe-Barakat-Cuntz-Hoge-Terao [ABC$^+$13] states that the arrangement $\mathcal{A}(T)$ cut out by the elements of a lower order ideal $T \subset R^+$ is always free, with coexponents equal to $\mathrm{Exp}(T)$. Thus when $T = -\Omega T_e X(w)$ and w is smooth, we get that $\mathcal{I}(w)$ and $\mathcal{A}(T)$ are free with the same coexponents. In Section 6, we show that any inversion set $I(w)$ can be transformed into a lower order ideal T using a combinatorial version of Peterson translation. In the simply-laced types and type B, if $\mathcal{I}(w)$ is free then the arrangement $\mathcal{A}(T)$ is free, and $\mathcal{I}(w)$ and $\mathcal{A}(T)$ have the same coexponents. Thus for these types we get an analogue of the Shapiro-Steinberg-Kostant rule for calculating the exponents of a rationally smooth element $w \in W$.

2 Chain Billey-Postnikov decompositions

Let S be the set of simple generators of W. Given a subset $J \subset S$, we let W_J denote the parabolic subgroup generated by J, W^J denote the set of minimal length left coset representatives, and ${}^J W$ denote the set of minimal length right coset representatives. Every element $w \in W$ can be written uniquely as $w = vu$, where $v \in W^J$ and $u \in W_J$. This factorization is called the right parabolic decomposition of w. Left parabolic decompositions are defined similarly. If $v \in W^J$, then the Poincare polynomial of v relative to J is the polynomial

$$P_v^J(q) = \sum_{x \in [e,v] \cap W^J} q^{\ell(v)}.$$

If $w = vu$ is the parabolic decomposition of w with respect to $J \subset S$, then multiplication gives an injective map

$$\left([e,v] \cap W^J\right) \times [e,u] \to [e,w]. \tag{1}$$

If $x = v_1 u_1$ is the parabolic decomposition of an element $x \in [e,w]$, then $v_1 \leq v$. However, it is not necessarily true that $u_1 \leq u$, even though $u_1 \leq w$ and $u_1 \in W_J$.

Let $S(w) \subset S$ denote the support set of an element $w \in W$ (i.e. the set of simple generators which appear in some reduced decomposition of w), and let $D_L(w)$ denote the left descent set.

Lemma 2.1 ([BP05], [OY10], [RS13]). *Let $w = vu$ be the parabolic decomposition of w with respect to J. Then the following are equivalent:*

(a) The map in equation (1) is surjective (hence bijective).

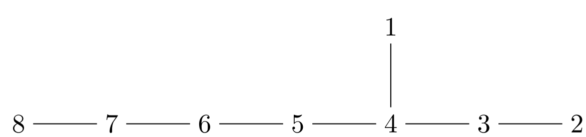
(b) u is the maximal element of $[e,w] \cap W_J$.

(c) $S(v) \cap J \subseteq D_L(u)$.

(d) $P_w(q) = P_v^J(q) P_u(q)$.

If any of the equivalent conditions of Lemma 2.1 are satisfied, then we say that $w = vu$ is a (right) Billey-Postnikov (BP) decomposition with respect to J. If in addition $[e,v] \cap W^J$ is a chain, or equivalently $P_v^J(q) = [\ell(v)+1]_q$, then we say that $w = vu$ is a chain BP decomposition. Left BP decompositions and left chain BP decompositions are defined similarly. Note that a left parabolic (resp. BP) decomposition of w is the same as a right parabolic (resp. BP) decomposition of w^{-1}.

Deep results of Gasharov [Gas98], Billey [Bil98], and Billey-Postnikov [BP05] imply that every rationally smooth element $w \in W$ has either a left or right BP decomposition. In addition, most rationally smooth elements have a chain BP decomposition. In this section, we list the few exceptions. To do so, we use the following labelling for the Dynkin diagram of E_8:



We write elements of E_8 as products of the simple generators s_i, $i = 1 \ldots, 8$, where s_i corresponds to the node in the Dynkin diagram labelled by i. For this section, we let $S_k = \{s_1, \ldots, s_k\}$, and take the convention that E_6 and E_7 are embedded in E_8 as W_{S_6} and W_{S_7} respectively. Finally, we let $J_k = S_k \setminus \{s_2\}$, $\tilde{u}_k$ be the maximal element of W_{J_k}, and $\tilde{v}_k$ be the maximal element of $W_{S_k}^{J_k}$.

Theorem 2.2. *Suppose that $w \in W$ is a rationally smooth element of a finite Weyl group W, that $\ell(w) \geq 2$, and that w has no (left or right) chain BP decomposition. Then w is one of the following elements:*

- *The maximal element of D_n, $n \geq 4$.*
- *The maximal element of E_n, $n = 6, 7, 8$.*
- *The element $w_{kl} = \tilde{v}_l \tilde{u}_k$ in E_8, or its inverse w_{kl}^{-1}, where $5 \leq l < k \leq 8$.*
- *The maximal element of F_4.*

For the proof of Theorem 2.2 in the classical types, we refer to [Gas98] and [Bil98], as well as the summary of this work in [OY10]. For the exceptional types it is possible to check Theorem 2.2 by computer. We give a human-readable proof based on the existence theorems of [BP05] (which also use computer verification) in [Slo13b].

Theorem 2.2 allows us to give a direct proof of Theorem 1.1. If w is rationally smooth and has a chain BP decomposition $w = vu$ or $w = uv$, then $P_w(q) = [\ell(v) + 1]_q P_u(q)$. The element u is also rationally smooth, and we can proceed by induction. As mentioned in the introduction, Theorem 1.1 for maximal elements is well-known. Furthermore, $P_w(q) = P_{w^{-1}}(q)$, so we only need to check Theorem 1.1 for the elements w_{kl}. If $\tilde{w}_l$ is the maximal element of W_{S_l}, then $P_{w_{kl}}(q) = P_{\tilde{v}_l}^{J_l}(q) P_{\tilde{u}_k}(q) = P_{\tilde{w}_l}(q) P_{\tilde{u}_k}(q) P_{\tilde{u}_l}^{-1}(q)$, where the last equality uses the fact that $\tilde{w}_l = \tilde{v}_l \tilde{u}_l$. The Poincare polynomials $P_{\tilde{w}_l}(q)$ and $P_{\tilde{u}_k}(q)$ are well-known, so it is easy to check that $P_{w_{kl}}(q)$ is a product of q-integers as desired. For example, the exponents of w_{87} are $1, 6, 7, 9, 11, 11, 13, 17$.

3 The HLSS condition and nbc-sets

In this section we give some background on the Hultman-Linusson-Shareshian-Sjöstrand (HLSS) condition which is necessary for the proof of Theorem 1.3. If $s \in S$ is a simple generator, let α_s denote the corresponding simple root. Conversely, let $t_\alpha \in W$ denote the reflection corresponding to a root $\alpha \in R$. Given an order $<$ on a set T of vectors in V, a broken circuit is defined to be an ordered subset $\{v_1 < \ldots < v_k\}$ of T such that there is $v_{k+1} > v_k$ for which $\{v_1, \ldots, v_{k+1}\}$ is a minimal linearly dependent set in T. An nbc-set is an ordered subset of T which does not contain a broken circuit. The number of nbc-sets is equal to the number of chambers in the arrangement $\mathcal{A}(T)$. In particular, the number of nbc-sets does not depend on the chosen order.

If $s_1 \cdots s_k$ is a reduced expression for an element $w \in W$, we can order the inversion set $I(w)$ by $\beta_1 < \cdots < \beta_{\ell(w)}$, where $\beta_i = s_1 \cdots s_{i-1} \alpha_{s_i}$. A total order on $I(w)$ constructed is this way is called a convex order. Let $2^{I(w)}$ denote the power set of $I(w)$. Given a convex order, we can define a surjective map

$$\phi : 2^{I(w)} \to [e, w] : \{\beta_1, \ldots, \beta_k\} \mapsto t_{\beta_1} \cdots t_{\beta_k} w, \text{ where } \beta_1 < \cdots < \beta_k.$$

Theorem 3.1 (Hultman-Linusson-Shareshian-Sjöstrand [HLSS09])**.** *Choose a convex order for $I(w)$, and let $\mathrm{nbc}(I(w))$ denote the set of nbc-sets of $I(w)$ with respect to the chosen order. Then the restriction of ϕ to $\mathrm{nbc}(I(w))$ is injective.*

In particular, the number of nbc-sets of $I(w)$ is less than the size of the Bruhat interval. The restriction of ϕ to $\mathrm{nbc}(I(w))$ will be surjective if and only if the number of nbc-sets is equal to the size of the Bruhat interval. As mentioned in the introduction, when the restriction of ϕ is surjective we say that w

satisfies the HLSS condition. A theorem of Hultman-Linusson-Shareshian-Sjöstrand (type A [HLSS09]) and Hultman (all finite Coxeter groups [Hul11]) characterizes when this condition holds in terms of the directed Bruhat graph of $[e, w]$. In particular, the HLSS condition is weaker than being rationally smooth:

Theorem 3.2 ([Hul11])**.** *If w is rationally smooth, then w satisfies the HLSS condition.*

4 Chain BP decompositions and modular flats

Given an arrangement $\mathcal{A}$, let $L(\mathcal{A})$ denote the intersection lattice of $\mathcal{A}$. By convention, the maximal element of $\mathcal{A}$ is the center $\bigcap_{H \in \mathcal{A}} H$ of $\mathcal{A}$. A coatom is a flat of $L(\mathcal{A})$ of rank one more than the center. If $X \in L(\mathcal{A})$, the localization of $\mathcal{A}$ at X is the arrangement $\mathcal{A}_X$ containing all hyperplanes H of $\mathcal{A}$ such that $X \subset H$. We let $\mathcal{A}^X$ denote the restriction of $\mathcal{A}$ to X, and $\mathcal{A} \setminus H$ the deletion by H, which is the arrangement containing all hyperplanes of $\mathcal{A}$ except H. An arrangement is said to be inductively free if either (a) $\mathcal{A}$ contains no hyperplanes (in which case all coexponents are zero) or (b) there is some hyperplane $H \subset \mathcal{A}$ such that $\mathcal{A} \setminus H$ is inductively free with coexponents $d_1, \ldots, d_l - 1$, and $\mathcal{A}^H$ is inductively free with coexponents $d_1, \ldots, d_{l-1}$. Any inductively free arrangement is free. The following lemma gives a useful sufficient criterion for $\mathcal{A}$ to be inductively free.

Lemma 4.1. *If X is a modular coatom of $L(\mathcal{A})$, and $\mathcal{A}_X$ is inductively free with coexponents $0, m_1, \ldots, m_{l-1}$, then $\mathcal{A}$ is inductively free with coexponents $m_1, \ldots, m_{l-1}$, $m_l = |\mathcal{A}| - |\mathcal{A}_X|$.*

Here $|\mathcal{A}|$ denotes the number of hyperplanes in $\mathcal{A}$. We use the following characterization of modular coatoms:

Lemma 4.2 ([CDF+09], Lemma 3.20)**.** *Let $\mathcal{A}$ be an arrangement, and for each hyperplane $H \in \mathcal{A}$ let α_H be a normal vector to H. Let $X \subset L(\mathcal{A})$ be a coatom. Then X is modular if and only if for every distinct pair $H_1, H_2 \notin \mathcal{A}_X$, there is $H_3 \in \mathcal{A}_X$ such that $\alpha_{H_1}, \alpha_{H_2}, \alpha_{H_3}$ are linearly dependent.*

Recall that V is the ambient Euclidean space containing R. Given $J \subset S$, let $V_J \subset V$ be the subspace spanned by $\{\alpha_s : s \in J\}$, and let $R_J = R \cap V_J$ be the root system for W_J. The following lemma is easy to prove:

Lemma 4.3. *The linear span of the inversion set $I(w)$ in V is $V_{S(w)}$, where $S(w)$ is the support set of w. The center of the inversion hyperplane arrangement $\mathcal{I}(w)$ is the orthogonal complement of $V_{S(w)}$.*

Lemma 4.3 implies that the rank of $\mathcal{I}(w)$ is the size of the support set $S(w)$, so a coatom of $\mathcal{I}(w)$ is an element of $L(\mathcal{I}(w))$ of rank $|S(w)| - 1$. If $w = uv$ is a left parabolic decomposition, then the inversion set $I(w)$ is the disjoint union of $I(u)$ and $uI(v)$, and

$$X = \bigcap_{\alpha \in I(u)} \ker \alpha \tag{2}$$

is a flat of $L(\mathcal{I}(w))$. This flat has rank $|S(u)|$, and hence X will be a coatom if and only if $|S(u)| = |S(w)| - 1$.

Theorem 4.4. *Suppose that $w = uv$ is a left parabolic decomposition with respect to J, so $u \in W_J$ and $v \in {}^J W$. Let $X \in L(\mathcal{I}(w))$ be defined as in Equation (2). Then $w = uv$ is a chain BP decomposition if and only if X is a modular coatom of $L(\mathcal{I}(w))$.*

Theorem 4.4 holds for any element $w \in W$, where W is an arbitrary finite Coxeter group. However, the proof is lengthy, and will be left for [Slo13b]. We give a shorter proof of Theorem 4.4 when w and u both satisfy the HLSS condition, as this assumption is sufficient to prove Theorem 1.3.

Proof of Theorem 4.4. If $w = uv$ is a chain BP decomposition, then $|S(w) \cap J| = |S(w)| - 1$, and $S(v) \cap J \subset S(u)$, so $|S(u)| = |S(w)| - 1$. Hence we can assume throughout that X is a coatom. The set $I(u) = I(w) \cap V_J$, and $w \in {}^J W$ if and only if $I(w) \cap V_J$ is empty. The hyperplanes of $\mathcal{I}(w)$ which do not contain X correspond precisely to the roots of $I(w)$ in $uI(v)$.

Suppose that $w = uv$ is a chain BP decomposition, and that u satisfies the HLSS condition. Order $I(w)$ so that all the elements of $I(u)$ come after the elements of $uI(v)$. Every element $\alpha \in uI(v)$ is independent from the span of $I(u)$. Thus if $\{\gamma_1, \dots, \gamma_k\}$ is an nbc-set for $I(u)$, then $\{\alpha, \gamma_1, \dots, \gamma_k\}$ is an nbc-set for $I(w)$. Consequently

$$|\operatorname{nbc}(I(w))| \geq (1 + \ell(v)) \cdot |\operatorname{nbc}(I(u))|.$$

Since u satisfies the HLSS condition, $|\operatorname{nbc}(I(u))| = |[e, u]|$, while $|\operatorname{nbc}(I(w))| \leq |[e, w]|$. But $w = uv$ is a chain BP decomposition, so

$$|[e, w]| = P_w(1) = (1 + \ell(v))|[e, u]| = (1 + \ell(v)) \cdot |\operatorname{nbc}(I(u))|.$$

We conclude that all nbc-sets of $I(w)$ are either nbc-sets of $I(u)$, or of the form $\{\alpha, \gamma_1, \dots, \gamma_k\}$ for $\alpha \in uI(v)$ and $\{\gamma_1, \dots, \gamma_k\}$ an nbc-set of $I(u)$.

In particular, if $\alpha, \beta \in uI(v)$, $\alpha < \beta$, then $\{\alpha, \beta\}$ is not an nbc-set, and hence there must be some $\gamma \in I(w)$, $\gamma > \beta$, such that α, β, γ is linearly dependent. If $\gamma \notin I(u)$ we can repeat this process by replacing α, β with β, γ until we find $\gamma' \in I(u)$ such that β, γ, γ' is linearly dependent. This implies that γ' is in the span of β and γ, and since γ is in the span of α and β, we get that α, β, γ' is linearly dependent. By Lemma 4.2, X is modular.

Now suppose that X is modular. If we assume that u and w satisfy the HLSS condition, then

$$|[e, w]| = |\operatorname{nbc}(I(w))| = |\operatorname{nbc}(I(u))| \cdot (|\mathcal{A}| - |\mathcal{A}_X| + 1) = |[e, u]| \cdot (\ell(v) + 1).$$

On the other hand, $|[e, w]| \geq |[e, u]| \cdot |[e, v] \cap {}^J W|$, and $|[e, v] \cap {}^J W| \geq \ell(v) + 1$. So we must have $|[e, v] \cap {}^J W| = \ell(v) + 1$. But $[e, v] \cap {}^J W$ contains a chain of size $\ell(v) + 1$, so $[e, v] \cap {}^J W$ is a chain. Furthermore, the multiplication map $[e, u] \times ([e, v] \times {}^J W) \to [e, w]$ will be surjective, so $w = uv$ is a BP decomposition. □

We now prove one direction of Theorem 1.3:

Corollary 4.5. *If $w \in W$ is rationally smooth then $\mathcal{I}(w)$ is inductively free, and the coexponents of $\mathcal{I}(w)$ are equal to the exponents of w.*

Proof. The proof is by induction on $|S(w)|$. Clearly the corollary is true if $|S(w)| \leq 1$. Suppose w has a chain BP decomposition. The element w is rationally smooth if and only if w^{-1} is rationally smooth, and since $I(w^{-1}) = -w^{-1} I(w)$, the arrangements $\mathcal{I}(w)$ and $\mathcal{I}(w^{-1})$ are linearly equivalent. Thus we can assume without loss of generality that w has a left chain BP decomposition $w = uv$. Then u is also rationally smooth, and $P_w(q) = [\ell(v) + 1]_q P_u(q)$, so if the exponents of u are $0, m_1, \dots, m_{l-1}$, then the exponents of w are $m_1, \dots, m_{l-1}$, $m_l = \ell(v)$. The coatom X corresponding to $\mathcal{I}(u)$ is modular by

Theorem 4.4, and the arrangement $\mathcal{I}(w)_X$ is simply $\mathcal{I}(u)$, which by induction is inductively free with coexponents $0, m_1, \ldots, m_{l-1}$. Finally, $|\mathcal{I}(w)| - |\mathcal{I}(w)_X| = \ell(w) - \ell(u) = \ell(v)$. By Lemma 4.1, the arrangement $\mathcal{I}(w)$ is inductively free with coexponents equal to $m_1, \ldots, m_l$.

This leaves the possibility that w is one of the elements listed in Theorem 2.2. If w is the maximal element of D_n, E_n, or F_4, then Barakat and Cuntz have shown that $\mathcal{I}(w)$ is inductively free [BC12]. Again, without loss of generality we only need to check that the corollary holds for the elements w_{kl}, and this is done on a computer (we defer to [Slo13b] for details of the computation). □

5 The flattening map

Let U be a subspace of V. The intersection $R_U = R \cap U$ is also a root system, with positive and negative roots $R_U^+ = R^+ \cap U$ and $R_U^- = R^- \cap U$ respectively. Let W_U be the Weyl group of R_U. Note that W_U is the parabolic subgroup of W generated by the reflections t_β for $\beta \in R_U^+$, or equivalently is the subgroup of W which acts identically on the orthogonal complement of U in V.

A subset $I \subset R^+$ is convex if $\alpha, \beta \in I$, $\alpha + \beta \in R^+$ implies that $\alpha + \beta \in I$. The subset I is coconvex if $R^+ \setminus I$ is convex, and I is biconvex if it is both convex and coconvex. A subset I is biconvex if and only if it is the inversion set $I(w)$ for some $w \in W$. Since biconvexity is a linear condition, the intersection $I(w) \cap U$ is biconvex, and hence there is an element $w' \in W_U$ such that $I(w') = I(w) \cap U$. The element w' is called the flattening of w, and is denoted by $\mathrm{fl}_U(w)$ [BP05]. If $U = V_J$, then $\mathrm{fl}_U(w) = u$, where $w = uv$ is the left parabolic decomposition of W with respect to J. We use the following lemma:

Lemma 5.1 ([BB03]). *If $u \in W_U$, $w \in W$, then $\mathrm{fl}_U(uw) = u\,\mathrm{fl}_U(w)$.*

Recall from the definition of the HLSS condition that a convex order on an inversion set $I(w)$ is an order coming from a reduced expression for w. An arbitrary total order $<$ on $I(w)$ is convex if and only if it satisfies two conditions [Pap94]:

- if $\alpha < \beta$ and $\alpha + \beta \in R^+$, then $\alpha < \alpha + \beta < \beta$, and
- if $\alpha \in I(w)$, $\beta \notin I(w)$, and $\alpha - \beta \in R^+$, then $\alpha - \beta < \alpha$.

Because these conditions are linear, we immediately get the following lemma:

Lemma 5.2. *If $<$ is a convex order on $I(w)$, then the induced order on $I(\mathrm{fl}_U(w)) = I(w) \cap U$ is also convex.*

Proposition 5.3. *Let $U \subset V$ be any subspace. If w satisfies the HLSS condition, then so does $\mathrm{fl}_U(w)$.*

Proof. The absolute length $\ell'(w)$ of an element $w \in W$ is the smallest integer k such that w can be written as a product of k reflections. If $w = t_{\beta_1} \cdots t_{\beta_m}$, then clearly the fixed point space of w contains the orthogonal complement of $\mathrm{span}\{\beta_1, \ldots, \beta_m\}$. A theorem of Carter states that the fixed point space of w is equal to the orthogonal complement of $\mathrm{span}\{\beta_1, \ldots, \beta_m\}$ if and only if $\ell'(w) = m$, and furthermore $\ell'(w) = m$ if and only if $\beta_1, \ldots, \beta_m$ are linearly independent [Car72].

Choose a convex order $<$ on $I(w)$, and take the induced convex order on $I(\mathrm{fl}_U(w))$. If $x \in [e, \mathrm{fl}_U(w)]$, we can always find $u = t_{\beta_1} \cdots t_{\beta_m}$, where $\beta_1 < \ldots < \beta_m$ in $I(\mathrm{fl}_U(w))$ such that $x = u\,\mathrm{fl}_U(w)$. To show that $\mathrm{fl}_U(w)$ satisfies the HLSS condition, we want to show that we can take $\{\beta_1 < \ldots < \beta_m\}$ to be an nbc-set with respect to the given convex order. Now $x' = uw$ is less than w in Bruhat order, and since w satisfies the HLSS condition, we can find an nbc-set $\{\gamma_1 < \ldots < \gamma_m\}$ such that $x' = yw$, where

$y = t_{\gamma_1} \cdots t_{\gamma_k}$. Let V_0 denote the fixed point space of y. Then $\{\gamma_1, \ldots, \gamma_m\}$ is linearly independent, so V_0 is the orthogonal complement of $\operatorname{span}\{\gamma_1, \ldots, \gamma_m\}$. Since $yw = uw$, we have $y = u$, so the orthogonal complement of $\operatorname{span}\{\beta_1, \ldots, \beta_k\}$ is contained in V_0. It follows that $\operatorname{span}\{\gamma_1, \ldots, \gamma_m\} \subset \operatorname{span}\{\beta_1, \ldots, \beta_k\}$, and hence $\gamma_1, \ldots, \gamma_m \in R_U$. We conclude that $y \in W_U$.

Now $\{\gamma_1 < \ldots < \gamma_k\}$ is an nbc-set in $I(\mathrm{fl}_U(w))$, and $x = u\,\mathrm{fl}_U(w) = \mathrm{fl}_U(uw) = \mathrm{fl}_U(yw) = y\,\mathrm{fl}_U(w)$ by Lemma 5.1. We conclude that the map from $2^{I(\mathrm{fl}_U(w))} \to [e, \mathrm{fl}_U(w)]$ restricts to a surjective map on $\mathrm{nbc}(I(\mathrm{fl}_U(w)))$, and hence $\mathrm{fl}_U(w)$ satisfies the HLSS condition. □

Given an arrangement $\mathcal{A}$ in V and a subspace U_0 of the center of $\mathcal{A}$, we let $\mathcal{A}/U_0$ denote the quotient arrangement in V/U_0. It is easy to see that $\mathcal{A}$ is free if and only if $\mathcal{A}/U_0$ is free.

Proposition 5.4. *Let $U \subset V$ be any subspace. If $\mathcal{I}(w)$ is free, then so is $\mathcal{I}(\mathrm{fl}_U(w))$.*

Proof. Let $U^\perp$ be the orthogonal complement to U, and let

$$X = \bigcap_{\alpha \in I(w) \cap U} \ker \alpha \in L(\mathcal{I}(w)).$$

Then $U^\perp \subset X$, and $\mathcal{I}(\mathrm{fl}_U(w))$ is isomorphic to the localization $\mathcal{I}(w)_X/U^\perp$. It is well-known that localization preserves freeness (see [OT92, Theorem 4.37]), so if $\mathcal{I}(w)$ is free then $\mathcal{I}(w)_X$ is free, and consequently $\mathcal{I}(w)_X/U^\perp$ is free. □

Let R' be another root system with Weyl group $W(R')$, and let $w' \in W(R')$. An element $w \in W$ is said to contain the pattern (w', R') if there is a subspace $U \subset V$ such that R_U is isomorphic to R', and $\mathrm{fl}_U(w) = w'$ when R_U is identified with R'. If this does not happen for any subspace U, then w is said to avoid (w', R'). This notion of root system pattern avoidance due to Billey and Postnikov generalizes the usual notion of pattern avoidance for permutations [BP05]. We can now prove the main theorem:

Proof of Theorem 1.3. One direction of the theorem has already been proved in Corollary 4.5. Suppose that $\mathcal{I}(w)$ is free, and the product of the coexponents $\prod_i (1 + d_i)$ is equal to the size of the Bruhat interval. This latter condition implies that w satisfies the HLSS condition. We want to show that w is rationally smooth. The main result of [BP05] states that w is rationally smooth if and only if w avoids a finite list of bad patterns in the root systems $R' = A_3$, B_3, C_3, and D_4. If (w', R') is a pattern in this list (there are 17 bad patterns, since the patterns for B_3 and C_3 are equivalent), then either $\mathcal{I}(w')$ is not free, or w' does not satisfy the HLSS condition. By Propositions 5.3 and 5.4, $\mathcal{I}(\mathrm{fl}_U(w))$ is free and $\mathrm{fl}_U(w)$ satisfies the HLSS condition for any subspace $U \subset V$. We conclude that w must avoid all the bad patterns, and hence w is rationally smooth. □

6 Peterson translation

In this section we assume that R has no components of type C or F_4. Given $\alpha \in R^+$, an α-string is a subset of R^+ of the form $\{\beta, \beta + \alpha, \ldots, \beta + k\alpha$, where $\beta - \alpha \notin R^+$ and $\beta + (k+1)\alpha \notin R^+$. The set of α-strings partitions R^+. The Peterson translate of a subset $T \subset R^+$ compresses each α-string:

Definition 6.1. *Given $T \subset R^+$, $\alpha \in R^+$, we define the Peterson translate $\tau(T, \alpha)$ of T by α as follows:*

- *If T is a subset of an α-string $\{\beta, \beta + \alpha, \beta + k\alpha\}$, so $T = \{\beta + i_1\alpha, \ldots, \beta + i_r\alpha\}$, then $\tau(S, \alpha) = \{\beta, \beta + \alpha, \ldots, \beta + (r-1)\alpha\}$.*

- *For a general subset T of R^+, let $T = \bigcup T_i$ be the partition of T induced by the partition of R^+ into α-strings. Then $\tau(T,\alpha) = \bigcup \tau(T_i,\alpha)$.*

This definition is equivalent to the geometric Peterson translate defined by Carrell and Kuttler [CK03].

Theorem 6.2. *Let T be a coconvex set in R^+, where R has no components of type C or F_4. Then:*

(a) The Peterson translate $\tau(T,\alpha)$ is coconvex for every $\alpha \in R^+$.

(b) If $\mathcal{A}(T)$ is free and $\alpha \in R^+$ then $\mathcal{A}(\tau(T,\alpha))$ is free with the same coexponents as $\mathcal{A}(T)$.

(c) If T is not a lower order ideal, then there is $\alpha \in T$ such that $\tau(T,\alpha)$ is not equal to T.

Proof sketch. Part (c) of the theorem follows from the definition of coconvex set and lower order ideal. We use a computer to check that parts (a) and (b) hold for the root systems $R = A_3$, B_3, and G_2. The key idea of the proof for general R is that Peterson translation is local, in the sense that if U is a subspace of V, and $\alpha \in U$, then $\tau(T,\alpha) \cap U = \tau(T \cap U, \alpha)$, where the latter refers to Peterson translation in R_U. Since part (a) holds for all root systems of rank ≤ 3, and $T \cap U$ is a coconvex subset of R_U for any subspace U, we conclude that $\tau(T,\alpha) \cap U$ is coconvex for all subspaces $U \subseteq V$ of rank 3 with $\alpha \in U$. But to check that $\tau(T,\alpha)$ is coconvex, we only need to check that $\tau(T,\alpha) \cap U$ is coconvex for subspaces $U' \subset V$ of rank 2, so part (a) holds for all root systems.

If an arrangement $\mathcal{A}$ is free, then the Ziegler multiarrangement $\tilde{\mathcal{A}}^H$ is free for any hyperplane $H \in \mathcal{A}$ [Zie89]. Abe and Yoshinaga have recently proved a converse to Zeigler's theorem: if $\tilde{\mathcal{A}}^H$ is free for some $H \in \mathcal{A}$, then $\mathcal{A}$ is free if and only if $\mathcal{A}_X$ is free for every flat $X \subset H$ of corank 3.

Now given a subspace U in V spanned by elements of T, we can take the flat $X \in L(\mathcal{A}(T))$ cut out by the elements of $U \cap T$. The corank of X is the rank of U, and $X \subset \ker \alpha$ if and only if $\alpha \in U$. If $\mathcal{A}(T)$ is free, then the Zeigler multirestriction of $\mathcal{A}(T)$ to $\ker \alpha$ is also free. Since the elements of T and $\tau(T,\alpha)$ only differ by translation by α, the Zeigler multirestriction of $\mathcal{A}(T)$ to $\ker \alpha$ is isomorphic to the Zeigler multirestriction of $\mathcal{A}(\tau(T,\alpha))$. Since part (b) holds for every root system of rank ≤ 3, we know that $\mathcal{A}(\tau(T,\alpha))_X = \mathcal{A}(\tau(T,\alpha) \cap U)$ is free for every corank 3 flat $X \subset \ker \alpha$. Hence we can apply Abe and Yoshinaga's theorem to prove that $\mathcal{A}(\tau(T,\alpha))$ is free. The coexponents of $\mathcal{A}(\tau(T,\alpha))$ can easily be recovered from the coexponents of the Ziegler multirestriction, so $\mathcal{A}(\tau(T,\alpha))$ has the same coexponents as $\mathcal{A}(T)$. ☐

Peterson translation decreases the heights of roots, so part (c) of Theorem 6.2 implies that we can repeatedly translate any coconvex set T until we get an order ideal T'. As mentioned in the introduction, $\mathcal{A}(T')$ is free with coexponents $\mathrm{Exp}(T')$ [ST06] [ABC+13], and Theorem 6.2 implies that if $\mathcal{A}(T)$ is free, then $\mathcal{A}(T)$ and $\mathcal{A}(T')$ have the same exponents. If $T = I(w)$ is the inversion set of a rationally smooth element w, then T is a coconvex set and $\mathcal{A}(T)$ is free, so we get an analogue of the Shapiro-Steinberg-Kostant rule for calculating the exponents of w.

Example 6.3. *Let $w = s_1 s_2 s_3 s_2$ in the Weyl group of A_3, where s_1 and s_3 are the simple generators corresponding to the leaves of the Dynkin diagram. If α_i is the simple root corresponding to s_i, then*

$$I(w) = \{\alpha_1, \alpha_3, \alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + \alpha_3\},$$

so $I(w)$ is not an order ideal. But

$$T = \tau(I(w), \alpha_1) = \{\alpha_1, \alpha_2, \alpha_3, \alpha_2 + \alpha_3\}$$

is an order ideal, with $\mathrm{Exp}(T) = \{1, 1, 2\}$. *Since* w *is rationally smooth, we conclude that the exponents of* w *are* $1, 1, 2$.

If $X(w)$ is smooth, it follows from [CK03] that there is a sequence of translations that sends $I(w)$ to the lower order ideal $-\Omega T_e X(w)$, and thus we can use Theorem 6.2 to directly compare $\mathcal{I}(w)$ and $\mathcal{A}(-\Omega T_e X(w))$. In [Slo13a], Theorem 6.2 is used to give a root-system pattern avoidance criterion for the arrangement $\mathcal{A}(T)$ to be free, assuming that T is coconvex.

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References

[ABC+13] Takuro Abe, Mohamed Barakat, Michael Cuntz, Torsten Hoge, and Hiroaki Terao. The freeness of ideal subarrangements of Weyl arrangements. 2013. arXiv:1304.8033.

[AC12] Ersan Akyildiz and James Carrell. Betti numbers of smooth Schubert varieties and the remarkable formula of Kostant, Macdonald, Shapiro, and Steinberg. *Michigan Math. J.*, 61(3):543–553, 2012.

[BB03] Sara Billey and Tom Braden. Lower bounds for Kazhdan-Lusztig polynomials from patterns. *Transformation Groups*, 8(4):321–332, 2003.

[BC12] Mohamed Barakat and Michael Cuntz. Coxeter and crystallographic arrangements are inductively free. *Advances in Mathematics*, 229(1):691–709, 2012.

[Bil98] Sara C. Billey. Pattern avoidance and rational smoothness of Schubert varieties. *Adv. Math.*, 139(1):141–156, 1998.

[BP05] Sara Billey and Alexander Postnikov. Smoothness of Schubert varieties via patterns in root subsystems. *Adv. in Appl. Math.*, 34(3):447–466, 2005.

[Car72] R. W. Carter. Conjugacy classes in the Weyl group. *Compositio Math.*, 25:1–59, 1972.

[Car94] James B. Carrell. The Bruhat graph of a Coxeter group, a conjecture of Deodhar, and rational smoothness of Schubert varieties. In *Algebraic groups and their generalizations: classical methods (University Park, PA, 1991)*, volume 56 of *Proc. Sympos. Pure Math.*, pages 53–61. Amer. Math. Soc., Providence, RI, 1994.

[CDF+09] Dan Cohen, Graham Denham, Michael Falk, Hal Scheck, Alex Suciu, Hiro Terao, and Sergey Yuzvinsky. *Complex Arrangements: Algebra, Geometry, Topology*. 2009. In preparation.

[CK03] James B. Carrell and Jochen Kuttler. Smooth points of T-stable varieties in G/B and the Peterson map. *Inventiones Mathematicae*, 151(2):353–379, 2003.

[Gas98] Vesselin Gasharov. Factoring the Poincaré polynomials for the Bruhat order on S_n. *J. Combin. Theory Ser. A*, 83(1):159–164, 1998.

[HLSS09] Axel Hultman, Svante Linusson, John Shareshian, and Jonas Sjöstrand. From Bruhat intervals to intersection lattices and a conjecture of Postnikov. *J. Combin. Theory Ser. A*, 116(3):564–580, 2009.

[Hul11] Axel Hultman. Inversion arrangements and Bruhat intervals. *J. Combin. Theory Ser. A*, 118(7):1897–1906, 2011.

[OPY08] Suho Oh, Alexander Postnikov, and Hwanchul Yoo. Bruhat order, smooth Schubert varieties, and hyperplane arrangements. *J. Combin. Theory Ser. A*, 115(7):1156–1166, 2008.

[OT92] Peter Orlik and Hiroaki Terao. *Arrangements of hyperplanes*. Springer, July 1992.

[OY10] Suho Oh and Hwanchul Yoo. Bruhat order, rationally smooth Schubert varieties, and hyperplane arrangements. *DMTCS Proceedings*, FPSAC, 2010.

[Pap94] Paolo Papi. A characterization of a special ordering in a root system. *Proc. AMS*, 120(3):661–665, 1994.

[RS13] Edward Richmond and William Slofstra. Billey-Postnikov decompositions and the fibre bundle structure of Schubert varieties. 2013. In preparation.

[Slo13a] William Slofstra. Pattern avoidance criterion for free inversion arrangements. 2013. In preparation.

[Slo13b] William Slofstra. Rationally smooth Schubert varieties and inversion hyperplane arrangements. 2013. arXiv:1312.7540.

[ST06] Eric Sommers and Julianna Tymoczko. Exponents for B-stable ideals. *Trans. AMS*, 358(8):3493–3509, 2006.

[Ter81] Hiroaki Terao. Generalized exponents of a free arrangement of hyperplanes and Shepherd-Todd-Brieskorn formula. *Inv. math.*, 63(1):159–179, 1981.

[Yoo11] Hwanchul Yoo. *Combinatorics in Schubert varieties and Specht modules*. PhD thesis, MIT, 2011.

[Zie89] G. M. Ziegler. Multiarrangements of hyperplanes and their freeness. In *Singularities (Iowa City, IA, 1986)*, pages 345–359. Amer. Math. Soc, 1989.