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Expanding Hall-Littlewood and related polynomials as sums over Yamanouchi words

Austin Roberts*

University of Washington& Seattle, WA, USA

Abstract. This paper uses the theory of dual equivalence graphs to give explicit Schur expansions to several families of symmetric functions. We begin by giving a combinatorial definition of the modified Macdonald polynomials and modified Hall-Littlewood polynomials indexed by any diagram $\delta \subset \mathbb{Z} \times \mathbb{Z}$, written as $\widetilde{H}_\delta(X;q,t)$ and $\widetilde{P}_\delta(X;t)$, respectively. We then give an explicit Schur expansion of $\widetilde{P}_\delta(X;t)$ as a sum over a subset of the Yamanouchi words, as opposed to the expansion using the charge statistic given in 1978 by Lascoux and Schüztenberger. We further define the symmetric function $R_{\gamma,\delta}(X)$ as a refinement of $\widetilde{P}_\delta$ and similarly describe its Schur expansion. We then analysize $R_{\gamma,\delta}(X)$ to determine the leading term of its Schur expansion. To gain these results, we associate each Macdonald polynomial with a signed colored graph $\mathcal{H}_\delta$. In the case where a subgraph of $\mathcal{H}_\delta$ is a dual equivalence graph, we provide the Schur expansion of its associated symmetric function, yielding several corollaries.

Résumé. Ce document utilise la théorie des graphes double équivalence pour donner expansions de Schur explicites à plusieurs familles de fonctions symétriques. Nous commençons par donner une définition combinatoire des polynômes de Macdonald modifiés et polynômes de Hall-Littlewood modifiés indexés par tout schèma $\delta \subset \mathbb{Z} \times \mathbb{Z}$, écrit $\widetilde{H}_\delta(X,q,t)$ et $\widetilde{P}_\delta(X,t)$, respectivement. Nous donnons ensuite une expansion de Schur explicite de $\widetilde{P}_\delta(X,t)$ comme une somme sur un sous-ensemble des mots Yamanouchi, plutôt que l'expansion en utilisant la statistique de charge donnée en 1978 par Lascoux et Schüztenberger . Nous définissons davantage la fonction symétrique $R_{\gamma,\delta}(X)$ comme un raffinement de $\widetilde{P}_\delta$ et décrire même son expansion de Schur . Nous analysons puis $R_{\gamma,\delta}(X)$ afin de dèterminer le premier terme de son expansion de Schur. pour obtenir ces résultats, nous associons chaque polynôme Macdonald avec un graphique coloré signé $\mathcal{H}_\delta$. En le cas où un sous-graphe de $\mathcal{H}_\delta$ est un graphe dual équivalence, nous fournissons l'expansion de Schur de sa fonction symétrique associée, ce qui donne plusieurs corollaires.

Keywords: Dual Equivalence Graph, Hall-Littlewood Polynomials, Macdonald Polynomials, quasisymmetric functions, symmetric functions

1 Introduction

Adriano Garsia posed the question, when can the modified Hall-Littlewood polynomials $\widetilde{P}_\mu(X;t)$ be expanded into the Schur functions as a particular sum over the Yamanouchi words, and is there a way to

*Email: `austinis@math.washington.edu`. Partially supported by DMS-1101017 from the NSF.

fix the expansion when it is not? The results of this paper are in direct response to Garsia's question. In fact, the results we found proved to be more general than the question as originally posed.

In this paper, we will concentrate on three main families of polynomials. First, the Macdonald polynomials were introduced in Macdonald (1988) and are often defined as the set of q, t-symmetric functions satisfying certain orthogonality and triangularity conditions. Macdonald polynomials were shown to be Schur positive by Mark Haiman via representation-theoretic and geometric means in Haiman (2001). Macdonald polynomials also specialize to several well known functions, including Hall-Littlewood polynomials and Jack polynomials. A combinatorial descriptions of the Schur expansion of Macdonald polynomials remains elusive outside of some special cases.

As just noted, Macdonald polynomials specialize to Hall-Littlewood polynomials. Hall-Littlewood polynomials, in turn, specialize to the Schur functions as well as the monomial symmetric functions. They were first studied by Paul Hall in relation to the Hall algebra in Hall (1957), though their current definition is due to D.E. Littlewood in Littlewood (1961). It should be noted that the earliest known work on Hall-Littlewood polynomials actually dates back to the lectures of Ernst Steinitz in Steinitz (1901). Expanding Hall-Littlewood polynomials into Schur functions can be achieved via the charge statistic, as found in Lascoux and Schützenberger (1978), though we will present a new expansion in this paper.

We use the statistics defined in Haglund et al. (2005), to generalize the definition for the modified Macdonald polynomials $\widetilde{H}_\mu(X; q, t)$ and the modified Hall-Littlewood polynomials $\widetilde{P}_\mu(X; t)$ to any diagram $\delta \subset \mathbb{Z} \times \mathbb{Z}$, giving the functions $\widetilde{H}_\delta(X; q, t)$ and $\widetilde{P}_\delta(X; t)$. We may then write $\widetilde{P}_\delta(X; t)$ in terms of the refinement polynomials $R_{\gamma,\delta}(X)$, defined via row reading words of fillings of δ with a fixed descent set $\gamma \subset \delta$. We will discuss these polynomials in the general context of diagrams, though the reader with a refined taste for the specific is free to replace δ with a partition shape. We may then write the main theorem of this paper as follows.

Theorem 1.1. *If γ and δ are any diagrams such that $\gamma \subset \delta$, then*

$$\widetilde{P}_\delta(X; t) = \sum_{\lambda \vdash |\delta|} \sum_{\substack{w \in \mathrm{Yam}_\delta(\lambda) \\ \mathrm{inv}_\delta(w)=0}} t^{\mathrm{maj}_\delta(w)} s_\lambda, \quad \text{and} \quad R_{\gamma,\delta}(X) = \sum_{\lambda \vdash |\delta|} \sum_{\substack{w \in \mathrm{Yam}_\delta(\lambda) \\ \mathrm{inv}_\delta(w)=0 \\ \mathrm{Des}_\delta(w)=\gamma}} s_\lambda.$$

Here, $\mathrm{Yam}_\delta(\lambda)$ is the subset of the Yamanouchi words with content λ whose elements, when thought of as row reading words of a tableau of shape δ, never have the j^{th} from last i in the same pistol of δ as the $j + 1^{th}$ from last $i + 1$. The above definitions and notation will be given a more thorough treatment in Section 2.

The main tool used in the proof of Theorem 1.1 is the theory of dual equivalence graphs. Building on work in Haiman (1992), Sami Assaf introduced the theory of dual equivalence graphs in her Ph.D. dissertation Assaf (2007) and later preprint Assaf (2013). The theory was further advanced by the author in Roberts (2013), from which we will derive the definition of dual equivalence graph used in this paper. In these papers, a dual equivalence graph is associated to a symmetric function so that each component of the graph corresponds to a single Schur function. Thus, the Schur expansion of said symmetric function is described by a sum over the the set of components of the graph.

This paper will focus on dual equivalence graphs that emerge as components of a larger family of graphs. The involution $D_i^\delta \colon S_n \to S_n$ was first introduced in Assaf (2007) and can be used to define the edge sets of a signed colored graph $\mathcal{H}_\delta$ with vertex set S_n and vertices labeled by the signature function σ, which is defined via the inverse descent sets of permutations. We may then associate $\widetilde{P}_\mu(X; t)$ and $R_{\gamma,\delta}(X)$ to subgraphs of $\mathcal{H}_\delta$. We show that these two subgraphs are dual equivalence graphs in Theorem 3.4. The main contribution of this paper to the theory of dual equivalence graphs can then be stated in the following theorem.

Theorem 1.2. *Let δ be a diagram of size n, and let $\mathcal{G} = (V, \sigma, E)$ be a dual equivalence graph such that $\mathcal{G}$ is a component of $\mathcal{H}_\delta$ and $\mathcal{G} \cong \mathcal{G}_\lambda$. Then there is a unique vertex of V in $\mathrm{SYam}_\delta(\lambda)$, and $V \cap \mathrm{SYam}_\delta(\mu) = \emptyset$ for all $\mu \neq \lambda$.*

This paper is organized as follows. We begin with the necessary material from the literature in Section 2, discussing tableaux, symmetric functions, and dual equivalence graphs. In Section 3, we show that the signed colored graphs associated to $R_{\gamma,\delta}(X)$ and $\widetilde{P}_\delta(X;t)$ are dual equivalence graphs. We then sketch the proof of Theorem 1.2 followed by Theorem 1.1. Section 4 is dedicated to further analysis of $\widetilde{H}_\mu(X;q,t)$, $\widetilde{P}_\mu(X;t)$, and $R_{\gamma,\delta}(X)$. After classifying when $\widetilde{H}_\mu(X;q,t)$ and $\widetilde{P}_\mu(X;t)$ expand via Yamanouchi words in Corollary 4.1 and Proposition 20, we then end this section by classifying when $R_{\gamma,\delta}(X) = 0$ in Proposition 4.4 and giving a description of the leading term in the Schur expansion of $R_{\gamma,\delta}(X)$ in Proposition 4.6.

2 Preliminaries

2.1 Tableaux and Permutations

By a *diagram* δ, we mean a subset of $\mathbb{Z} \times \mathbb{Z}$. A *partition* λ is a weekly decreasing finite sequence of nonnegative integers $\lambda_1 \geq \ldots \geq \lambda_k \geq 0$. We write $|\lambda| = n$ or $\lambda \vdash n$ if $\sum \lambda_i = n$. We will give the diagram of a partition in french notation by drawing left justified rows of boxes, where λ_i is the number of boxes in the i^{th} row, from bottom to top, with bottom left cell at the origin, as in the left diagram of Figure 1.

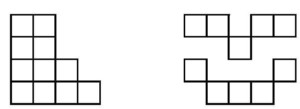

Fig. 1: The diagrams for (4,3,2,2) and an arbitrary subset $\delta \subset \mathbb{Z} \times \mathbb{Z}$.

A *tableau* is a function that takes each cell of a diagram δ to a positive integer. We express a tableau visually by writing the value assigned to a cell inside of the cell. A *standard* tableau uses each value in some $[n] = \{1, \ldots, n\}$ exactly once. Given a standard tableau T, define the *shape* of T, $\mathrm{sh}(T)$, to be the shape of the underlying diagram of T, and we define $\mathrm{ST}(\delta)$ as the set of standard tableaux with shape δ. That is $\mathrm{sh}(T) = \delta$ for all $T \in \mathrm{ST}(\delta)$. A *Young tableau* is a tableau in which all values are required to be increasing up columns and across rows from left to right. A *standard Young tableau* is a Young tableau that is also a standard tableau. The set of all standard Young tableaux on diagrams of partition shape λ is denoted by $\mathrm{SYT}(\lambda)$, and the union of $\mathrm{SYT}(\lambda)$ over all $\lambda \vdash n$ is denoted $\mathrm{SYT}(n)$. For more information, see (Fulton, 1997, Part I), (Sagan, 2001, Ch. 3), or (Stanley, 1999, Ch. 7).

Define the *row reading word* of a tableau T, denoted $\mathrm{rw}(T)$, by reading across rows from left to right, starting with the top row and working down, as in Figure 2. The row reading word of a standard tableau is necessarily a permutation. By a *pistol* of a diagram δ or tableau T, we mean a set of cells, in row reading order, between some cell c and the cell directly below c, inclusive. We will often conflate cells of a tableau

Fig. 2: On the left, a standard tableau with row reading word 483691257. On the right, a standard tableau with row reading word 3214.

T with indices of its row reading word. In particular, the cells of a pistol of δ give a set of indices of w when thought of as the row reading word of a tableau T of shape δ. We will refer to the indices of w as forming a *pistol* of δ if they correspond to the cells of of a pistol of δ in this fashion.

Fig. 3: Four pistols filled with bullets.

Given a permutation w in one-line notation, the signature of w is a string of 1's and -1's, or +'s and $-$'s for short, where there is a $+$ in the i^{th} position if and only if i comes before $i+1$ in w. Notice that a word is one entry longer than its signature. The signature of a standard tableau T is defined as $\sigma(T) := \sigma(\mathrm{rw}(T))$. As an example, the signatures of the tableaux in Figure 2 are $+--+-+-+$ and $--+$, respectively.

We may *standardize* a word w with positive integer values by replacing the values in w with the values in $[n]$ while respecting the relative order of the values in w, treating each occurrence of the value i as less than any later occurrence of the value i in w. We denote the resulting permutation as $\mathrm{st}(w)$. If w is a permutation, we will sometimes *unstandardize* w as $\mathrm{unst}(w)$, which is the result of replacing each value in w with the smallest possible positive integer while respecting the relative order just described. Specifically, if w is a permutation, then i and $i+1$ are taken to the same value if i occurs before $i+1$. Otherwise, $i+1$ is taken to the value that is one lager than that of i. If w is a permutation, the signature of w uniquely determines $\mathrm{unst}(w)$, and $\mathrm{st}(\mathrm{unst}(w)) = w$.

Next, we define a useful subset of S_n. For $\lambda \vdash n$, let U_λ be the standard tableau of shape λ given by assigning the numbers in $[n]$ in order across the first row, then across the second row, and so on. Now define $\mathrm{SYam}(\lambda) := \{w \in S_n\colon\ P(w) = U_\lambda\}$, where $P(w)$ is the *insertion tableau* as given by the Robinson-Schensted-Knuth (R-S-K) correspondence. Call this set of permutations the *standardized Yamanouchi words of shape* λ. We may generate $\mathrm{SYam}(\lambda)$ more directly by considering all words of length n such that there are never more $i+1$'s than i's while reading from right to left. We further require that i occurs λ_i times. Any such word is called a Yamanouchi word, and the set of such words is denoted $\mathrm{Yam}(\lambda)$. It then follows that $\mathrm{SYam}(\lambda) = \{\mathrm{st}(w)\colon w \in \mathrm{Yam}(\lambda)\}$. Similarly, $\mathrm{Yam}(\lambda) = \{\mathrm{unst}(w)\colon w \in \mathrm{SYam}(\lambda)\}$.

Definition 2.1. Let δ be any diagram. Given a word w of length n, we say that w *jams* δ if there exists some i and some j such that the j^{th} from last i in $\mathrm{unst}(w)$ shares a pistol of δ with the $j+1^{th}$ from last $i+1$ in $\mathrm{unst}(w)$.

We may then define

$$\mathrm{Yam}_\delta(\lambda) \quad := \{w \in \mathrm{Yam}(\lambda)\colon w \text{ does not jam } \delta\}, \tag{1}$$
$$\mathrm{SYam}_\delta(\lambda) \quad := \{w \in \mathrm{SYam}(\lambda)\colon w \text{ does not jam } \delta\}, \tag{2}$$

with examples of each set given in Figure 4.

$$\mathrm{Yam}_{(3,3)}(2,2,2) = \left\{\mathrm{rw}\left(\begin{array}{|c|c|c|}\hline 3&2&1\\\hline 3&2&1\\\hline\end{array}\right), \mathrm{rw}\left(\begin{array}{|c|c|c|}\hline 3&2&3\\\hline 1&2&1\\\hline\end{array}\right)\right\}, \quad \mathrm{rw}\left(\begin{array}{|c|c|c|}\hline 3&3&\mathbf{2}\\\hline 1&2&\mathbf{1}\\\hline\end{array}\right), \mathrm{rw}\left(\begin{array}{|c|c|c|}\hline \mathbf{3}&2&3\\\hline \mathbf{2}&1&1\\\hline\end{array}\right) \notin \mathrm{Yam}_{(3,3)}(2,2,2)$$
$$\mathrm{SYam}_{(3,3)}(2,2,2) = \left\{\mathrm{rw}\left(\begin{array}{|c|c|c|}\hline 5&3&1\\\hline 6&4&2\\\hline\end{array}\right), \mathrm{rw}\left(\begin{array}{|c|c|c|}\hline 5&3&6\\\hline 1&4&2\\\hline\end{array}\right)\right\}, \quad \mathrm{rw}\left(\begin{array}{|c|c|c|}\hline 5&6&\mathbf{3}\\\hline 1&4&\mathbf{2}\\\hline\end{array}\right), \mathrm{rw}\left(\begin{array}{|c|c|c|}\hline \mathbf{5}&3&6\\\hline \mathbf{4}&1&2\\\hline\end{array}\right) \notin \mathrm{SYam}_{(3,3)}(2,2,2)$$

Fig. 4: At left, the sets $\mathrm{Yam}_{(3,3)}(2,2,2)$ and $\mathrm{SYam}_{(3,3)}(2,2,2)$. At right, examples of words in $\mathrm{Yam}(2,2,2)$ and $\mathrm{SYam}(2,2,2)$ that jam $(3,3)$.

Remark 2.2.

1. The usual method for listing Yamanouchi words is to begin with the number 1 and add numbers to the left of it, as allowed by the description of Yamanouchi words in Section 2.1. The condition that a word not jam δ means that upon adding the $j+1^{th}$ $i+1$, this $i+1$ may not be in a pistol with the j^{th} i. Similarly, we may check if the addition of each new value creates an inversion pair or an inversion triple. That is, the process is readily integrated into the procedure for generating Yamanouchi words.

2. For the reader that prefers permutations, we may describe $\mathrm{SYam}_\delta(\lambda)$ as follows. Consider the result of right justifying the tableau U_λ, and let S be the set of pairs of values in cells that are touching on a southeasterly diagonal. Now treat $w \in \mathrm{SYam}(\lambda)$ as a row reading word of δ. Then $w \in \mathrm{SYam}_\delta(\lambda)$ if and only if no pairs in S are in a pistol of δ. See Figure 5 for an example.

$$\begin{array}{|c|c|ccc}\cline{1-2} 8&9&&&\\\cline{1-2} 6&7&&&\\\hline 1&2&\multicolumn{1}{c|}{3}&\multicolumn{1}{c|}{4}&\multicolumn{1}{c|}{5}\\\hline\end{array} \qquad \begin{array}{ccc|c|c|}\cline{4-5} &&&8&9\\\cline{4-5} &&&6&7\\\hline \multicolumn{1}{|c|}{1}&\multicolumn{1}{c|}{2}&3&4&5\\\hline\end{array} \qquad S = \{(6,5),(8,7)\} \qquad \begin{array}{|c|c|c|}\hline \mathbf{8}&6&9\\\hline \mathbf{7}&1&2\\\hline 3&4&5\\\hline\end{array}$$

Fig. 5: From the left, $U_{(5,2,2)}$, followed by the result of right justifying, followed by S, followed by a tableau of shape $\mu = (3,3,3)$ by $w \in \mathrm{SYam}((5,2,2))$ such that the pair $(8,7) \in S$ is in a pistol of μ. Thus $w \notin \mathrm{SYam}_\mu((5,2,2))$.

3. The set $\mathrm{Yam}(\lambda)$ is a Knuth class. The set $\mathrm{Yam}_\delta(\lambda)$ is necessarily a subset of this class, and so can be expressed via some set of recording tableaux of a given shape. Finding a more explicit way of generating all such recording tableaux remains an open problem.

2.2 Symmetric Functions

We will take the unorthodox approach of defining several symmetric functions via the fundamental quasisymmetric functions.

Definition 2.3. Given any signature $\sigma \in \{\pm 1\}^{n-1}$, define the *fundamental quasisymmetric function* $F_\sigma(X) \in \mathbb{Z}[x_1, x_2, \ldots]$ by

$$F_\sigma(X) := \sum_{\substack{i_1 \leq \cdots \leq i_n \\ i_j = i_{j+1} \Rightarrow \sigma_j = +1}} x_{i_1} \cdots x_{i_n}.$$

We may now use the previous definition to define the Schur functions, relying on a result of Ira Gessel. While it is not the standard definition, it is the most functional for our purposes.

Definition 2.4. Gessel (1984) Given any partition λ, define

$$s_\lambda := \sum_{T \in \mathrm{SYT}(\lambda)} F_{\sigma(T)}(X), \tag{3}$$

where s_λ is a Schur function of shape λ.

In order to define Macdonald polynomials, we first need to define some statistics, relying on the results in Haglund et al. (2005) for our definitions. Let T be a tableau of shape δ. Given a cell $c \in \delta$, let $T(c)$ denote the value of T in cell c. A descent of T is a cell c in δ such that $T(c) > T(d)$, where d is the cell directly below and adjacent to c. We denote the set of descents of T as $\mathrm{Des}(T)$. If, in addition, w is the row reading word of T, we will say that c is a *descent of w in δ*, taking care to differentiate it from other common meanings of the word descent. Given a cell c of δ, define $\mathrm{leg}(c)$ as the number of cells in δ strictly above and in the same column as c. We may then define

$$\mathrm{maj}_\delta(w) := \mathrm{maj}(T) := \mathrm{maj}_\delta(\mathrm{Des}(T)) := \sum_{c \in \mathrm{Des}(T)} 1 + \mathrm{leg}(c). \tag{4}$$

Let c, d, and e be cells of δ in row reading order. Then c, d, and e form a ***triple*** if c and d are in the same row and e is the cell immediately below c, as in Figure 6. If, in addition, T is a tableau of shape δ, then c, d, and e form an *inversion triple* of T if $T(e) < T(d) < T(c)$, $T(c) \leq T(e) < T(d)$, or $T(d) < T(c) \leq T(e)$. As a mnemonic, in each sequence of inequalities, the three cells are presented in a counterclockwise order. If either c or e is not in δ, then the remaining two cells form an *inversion pair* of T if either $T(c) > T(d)$ or $T(d) > T(e)$. See Figure 6 for an example of each of these types of inversions. We may now define the final statistic as

$$\mathrm{inv}_\delta(w) := \mathrm{inv}(T) := |\{\text{inversion triples of } T\}| + |\{\text{inversion pairs of } T\}|. \tag{5}$$

$$\begin{array}{|c|c|c|}\hline c & & d \\ \hline e \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|}\hline 1 & & 2 \\ \hline 1 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|}\hline 2 & & 1 \\ \hline 2 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|}\hline 3 & & 2 \\ \hline 1 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|}\hline 2 & & 1 \\ \hline \times \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|}\hline \times & & 2 \\ \hline 1 \\ \cline{1-1}\end{array}$$

Fig. 6: From left, a generic triple, three inversion triples, and then two inversion pairs, where $\times$ denotes the lack of a cell.

We are now able to define a generalized version of the modified Macdonald polynomials and Hall-Littlewood polynomials (respectively) as,

$$\widetilde{H}_\delta(X; q, t) := \sum_{T \in \mathrm{ST}(\delta)} q^{\mathrm{inv}(T)} t^{\mathrm{maj}(T)} F_{\sigma(\mathbf{T})}, \qquad \widetilde{P}_\delta(X; t) := \widetilde{H}_\delta(X; 0, t) = \sum_{\substack{T \in \mathrm{ST}(\delta) \\ \mathrm{inv}(T) = 0}} t^{\mathrm{maj}(T)} F_{\sigma(T)}. \tag{6}$$

Further, if we let $\gamma \subset \delta$, we may define

$$R_{\gamma,\delta}(X) = \sum_{\substack{T \in \mathrm{ST}(\delta) \\ \mathrm{inv}(T)=0 \\ \mathrm{Des}(T)=\gamma}} F_{\sigma(T)}. \tag{7}$$

It follows immediately from these definitions that

$$\widetilde{P}_{\delta}(X;t) = \sum_{\gamma \subset \delta} t^{\mathrm{maj}_{\delta}(\gamma)} R_{\gamma,\delta}(X). \tag{8}$$

2.3 Dual Equivalence Graphs

Definition 2.5 (Haiman (1992))**.** Given a permutation in S_n expressed in one-line notation, define an *elementary dual equivalence* as an involution d_i that interchanges the values $i-1, i$, and $i+1$ as

$$d_i(\ldots i \ldots i-1 \ldots i+1 \ldots) = (\ldots i+1 \ldots i-1 \ldots i \ldots), \tag{9}$$
$$d_i(\ldots i-1 \ldots i+1 \ldots i \ldots) = (\ldots i \ldots i+1 \ldots i-1 \ldots), \tag{10}$$

and that acts as the identity if i occurs between $i-1$ and $i+1$. Two words are $dual\ equivalent$ if one may be transformed into the other by successive elementary dual equivalences.

For example, 21345 is dual equivalent to 41235 because $d_3(d_2(21345)) = d_3(31245) = 41235$.

We may also let d_i act on the entries of a tableau by applying them to the row reading word. It is not hard to check that the result is again a tableau of the same shape. The transitivity of this action is described in the following theorem.

Theorem 2.6 (Haiman (1992), Prop. 2.4)**.** *Two standard Young tableaux on partition shapes are dual equivalent if and only if they have the same shape.*

By definition, d_i is an involution, and so we define a graph on standard Young tableaux by letting each nontrivial orbit of d_i define an edge colored by i. By Theorem 2.6, the graph on $\mathrm{SYT}(n)$ with edges labeled by $1 < i < n$ has connected components with vertices in $\mathrm{SYT}(\lambda)$ for each $\lambda \vdash n$. We may further label each vertex with its signature to create a *standard dual equivalence graph* that we will denote $\mathcal{G}_\lambda$. See Figure 7 for an example.

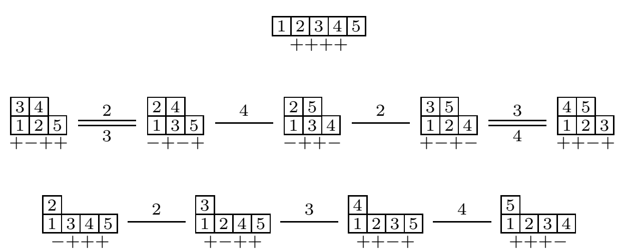

Fig. 7: Some standard dual equivalence graphs on $\lambda \vdash 5$.

Definition 2.4 and Theorem 2.6 determine the connection between Schur functions and dual equivalence graphs as highlighted in (Assaf, 2013, Cor. 3.10). Given any standard dual equivalence graph $\mathcal{G}_\lambda = (V, \sigma, E)$,

$$\sum_{v \in V} F_{\sigma(v)} = s_\lambda. \tag{11}$$

Here, $\mathcal{G}_\lambda$ is an example of the following broader class of graphs.

Definition 2.7. A *signed colored graph* consists of the following data:

1. *a finite vertex set* V*,*
2. *a signature function* $\sigma : V \to \{\pm 1\}^{n-1}$ *for some fixed positive integer* n*,*
3. *a collection* E_i *of unordered pairs of distinct vertices in* V *for each* $i \in \{2, \ldots, n-1\}$ *and the same positive integer* n*.*

We denote a signed colored graph by $\mathcal{G} = (V, \sigma, E_2 \cup \cdots \cup E_{n-1})$ or simply $\mathcal{G} = (V, \sigma, E)$.

In order to give an abstract definition of dual equivalence graphs, we will need definitions for isomorphisms and restrictions.

Definition 2.8. Given two signed colored graphs $\mathcal{G}(V, \sigma, E)$ and $\mathcal{H}(V', \sigma', E')$, an *isomorphism* $\phi \colon \mathcal{G} \to \mathcal{H}$ is a bijective map from V to V' such that both ϕ and ϕ^{-1} preserve colored edges and signatures.

Definition 2.9. Given a signed colored graph $\mathcal{G} = (V, \sigma, E)$ and an interval of nonnegative integers I, define the restriction of $\mathcal{G}$ to I, denoted $\mathcal{G}|_I$, as the signed colored graph $\mathcal{H} = (V, \sigma', E')$, where

1. $\sigma'(v)_i = \sigma(v)_{\min(I)+i-1}$ when $i \in \{1, \ldots, |I| - 1\}$, and $\sigma_{\min(I)+i-1}$ is defined.
2. $E'_i = E_{\min(I)+i-1}$ when $i \in \{2, 3, \ldots, |I| - 1\}$, and $E_{\min(I)+i-1}$ is defined.

We now proceed to the definition of a dual equivalence graph. Here, we use a result of Roberts (2013) as our definition. For more general definitions, see Assaf (2013) and Roberts (2013).

Definition 2.10. A signed colored graph $\mathcal{G} = (V, \sigma, E)$ is a *dual equivalence graph* if the following properties hold:

(P1): If I is any interval of integers with $|I| = 6$, then $\mathcal{G}|_I \cong \mathcal{G}_\lambda$ for some partition λ.
(P2): If $\{v, w\} \in E_i$ and $\{w, x\} \in E_j$ for some $|i-j| > 2$, then there exists $y \in V$ such that $\{v, y\} \in E_j$ and $\{x, y\} \in E_i$.

Theorem 2.11 (Assaf (2013), Thm. 3.9)**.** *A connected component of a signed colored graph is a dual equivalence graph if and only if it is isomorphic to a unique* $\mathcal{G}_\lambda$*.*

Next, we will associate to every Macdonald polynomial and Hall-Littlewood polynomial a signed colored graph. To do this, we need to define an involution D_i^δ to provide the edge sets of a signed colored graph, as defined originally in Assaf (2013). First let $\tilde{d}_i : S_n \to S_n$ be the involution that permutes the values $i-1, i$, and $i+1$ as

$$\tilde{d}_i(\ldots i \ldots i-1 \ldots i+1 \ldots) = (\ldots i-1 \ldots i+1 \ldots i \ldots), \tag{12}$$

$$\tilde{d}_i(\ldots i \ldots i+1 \ldots i-1 \ldots) = (\ldots i+1 \ldots i-1 \ldots i \ldots), \tag{13}$$

and that acts as the identity if i occurs between $i-1$ and $i+1$. For example, $\tilde{d}_3 \circ \tilde{d}_2(4123) = \tilde{d}_3(4123) = 3142$.

We now define the desired involution. Given a word w of length n and a diagram δ of size n,

$$D_i^\delta(w) := \begin{cases} \tilde{d}_i(w) & \text{if the values } i-1,\, i, \text{ and } i+1 \text{ occur in a pistol of } \delta. \\ d_i(w) & \text{otherwise.} \end{cases} \tag{14}$$

As an example, we may take $w = 534826179$ and δ as in Figure 8. Then $D_3^\delta(w) = \tilde{d}_3(w) = 542836179$ and $D_5^\delta(w) = d_5(w) = 634825179$.

$$\begin{array}{|c|c|c|cc}\hline 5&3&4&\multicolumn{1}{c}{}&\\\hline 8&2&6&\multicolumn{1}{c}{}&\\\hline \multicolumn{1}{c}{}&\multicolumn{1}{c|}{}&1&\multicolumn{1}{c|}{7}&\\\cline{3-4} \multicolumn{1}{c}{}&\multicolumn{1}{c}{}&\multicolumn{1}{c|}{}&\multicolumn{1}{c|}{9}&\\\cline{4-4}\end{array} \qquad \begin{array}{|c|c|c|cc}\hline 5&4&2&\multicolumn{1}{c}{}&\\\hline 8&3&6&\multicolumn{1}{c}{}&\\\hline \multicolumn{1}{c}{}&\multicolumn{1}{c|}{}&1&\multicolumn{1}{c|}{7}&\\\cline{3-4} \multicolumn{1}{c}{}&\multicolumn{1}{c}{}&\multicolumn{1}{c|}{}&\multicolumn{1}{c|}{9}&\\\cline{4-4}\end{array} \qquad \begin{array}{|c|c|c|cc}\hline 6&3&4&\multicolumn{1}{c}{}&\\\hline 8&2&5&\multicolumn{1}{c}{}&\\\hline \multicolumn{1}{c}{}&\multicolumn{1}{c|}{}&1&\multicolumn{1}{c|}{7}&\\\cline{3-4} \multicolumn{1}{c}{}&\multicolumn{1}{c}{}&\multicolumn{1}{c|}{}&\multicolumn{1}{c|}{9}&\\\cline{4-4}\end{array}$$

Fig. 8: Three standard tableaux of shape δ. At left, a tableau with row word $w = 534826179$ followed by $D_3^\delta(w)$ and then $D_5^\delta(w)$.

Given some δ of size n, we may then define the signed colored graph $\mathcal{H}_\delta = (V, \sigma, E)$ with vertex set $V = S_n$ and edge sets E_i defined via the nontrivial orbits of D_i^δ. It is readily shown that the action of D_i^δ on w preserves $\text{inv}_\delta(w)$, $\text{Des}_\delta(w)$, and $\text{maj}_\delta(w)$. Thus, these functions are all constant on components of $\mathcal{H}_\delta$. We may study $\widetilde{P}_\delta(X;t)$ by restricting our attention to components of $\mathcal{H}_\delta$ where inv_δ is zero, as in the following definition.

Definition 2.12. Let γ and δ be diagrams such that $\gamma \subset \delta$ and $|\delta| = n$. Define $\mathcal{R}_{\gamma,\delta} = (V, \sigma, E)$ as the signed colored graph with $V = \{w \in S_n\colon\ \text{inv}_\delta(w) = 0, \text{Des}_\delta(w) = \gamma\}$ and E_i defined via the nontrivial orbits of D_i^δ on V. Define $\mathcal{P}_\delta = (V', \sigma, E')$ as the signed colored graph with $V' = \{w \in S_n\colon\ \text{inv}_\delta(w) = 0\}$ and E_i' defined via the nontrivial orbits of D_i^δ on V'.

Notice that $\mathcal{R}_{\gamma,\delta} = (V, \sigma, E)$ is a subgraph of $\mathcal{P}_\delta = (V', \sigma, E')$, which in turn is a subgraph of $\mathcal{H}_\delta = (V'', \sigma, E'')$, each respectively comprised of connected components in $\mathcal{H}_\delta$, and that

$$R_{\gamma,\delta}(X) = \sum_{v\in V} F_{\sigma(v)}, \tag{15}$$

$$\widetilde{P}_\delta(X;t) = \sum_{v\in V'} t^{\text{maj}_\delta(v)} F_{\sigma(v)}, \tag{16}$$

$$\widetilde{H}_\delta(X;q,t) = \sum_{v\in V''} q^{\text{inv}_\delta(v)} t^{\text{maj}_\delta(v)} F_{\sigma(v)}. \tag{17}$$

3 Dual Equivalence graphs in $\mathcal{H}_\delta$

In this section we develop the key results of the paper. The following lemma demonstrates the importance of the set $\text{SYam}_\delta(\lambda)$ in the study of the graph $\mathcal{H}_\delta$ and is crucial to the proof of Theorem 1.2.

Lemma 3.1. *Let $\mathcal{C}$ and $\mathcal{D}$ be connected components of $\mathcal{H}_\gamma$ and $\mathcal{H}_\delta$, respectively. Further suppose that there exists an isomorphism $\phi\colon \mathcal{C} \to \mathcal{D}$, and let w be a vertex of $\mathcal{C}$. Then $w \in \mathrm{SYam}_\gamma(\lambda)$ if and only if $\phi(w) \in \mathrm{SYam}_\delta(\lambda)$, for all partitions shapes λ.*

Proof of Theorem 1.2: By Theorem 2.11, we may assume the existence of an isomorphism $\phi\colon \mathcal{G} \to \mathcal{G}_\mu$ for some $\mu \vdash n$. By conflating tableaux in $\mathcal{G}_\mu$ with their row reading words, we may consider $\mathcal{G}_\mu$ as a component of $\mathcal{H}_\gamma$, where γ is the subset of the vertical-axis $\{(0,i) \in \mathbb{Z}\times\mathbb{Z}\colon 0 \le i < n\}$. Hence,

$\mathrm{SYam}_\gamma(\lambda) = \mathrm{SYam}(\lambda)$. The unique standardized Yamanouchi word in $\mathcal{G}_\lambda$ is the row reading word of U_λ. Applying Lemma 3.1, there is a unique permutation in $V \cap \mathrm{SYam}_\delta(\lambda)$ corresponding to $\phi^{-1}(U_\lambda)$. Also by Lemma 3.1, $V \cap \mathrm{SYam}_\delta(\mu) = \emptyset$ when $\mu \neq \lambda$.

Corollary 3.2. *If $\mathcal{G} = (V, \sigma, E)$ is a dual equivalence graph contained in $\mathcal{H}_\delta$, then*

$$\sum_{v \in V} F_{\sigma(v)}(X) = \sum_{\lambda \vdash n} |V \cap \mathrm{SYam}_\delta(\lambda)| \cdot s_\lambda \tag{18}$$

Remark 3.3. For each partition λ and diagram δ, there exists a set S_δ defined as the intersection of $\mathrm{SYam}_\delta(\lambda)$ with the set of permutations of length $|\delta|$ whose component in $\mathcal{H}_\delta$ is a dual equivalence graph. That is, if $\mathcal{G} = (V, \sigma, E)$ is a component of $\mathcal{H}_\delta$, then $|V \cap S_\delta(\lambda)| = 1$ if $\mathcal{G} \cong \mathcal{G}_\lambda$, and $|V \cap S_\delta(\lambda)| = 0$ otherwise. Finding a more direct way to generate $S_\delta(\lambda)$, however, is an open problem.

In order to apply the theory of dual equivalence graphs to $\widetilde{P}_\mu(X;t)$ and $R_{\gamma,\delta}(X)$, we need the following important result about their associated signed colored graphs.

Theorem 3.4. *If γ and δ are any diagrams such that $\gamma \subset \delta$, then $\mathcal{R}_{\gamma,\delta}$ and $\mathcal{P}_\delta$ are dual equivalence graphs.*

Proof: We only sketch the proof here. Using Definition 2.10, it suffices to only consider δ such that $|\delta| = 6$. Since the action of D_i^δ is determined by the pistols of δ considered as subsets of the indices of permutations, we may reduce our consideration to the finitely many ways that δ may define these subsets of indices. We may then use a computer to classify a finite list of components of $\mathcal{H}_\delta$ that are not dual equivalence graphs. By analyzing this finite list, it is straight forward to show that if v is a vertex of a component of $\mathcal{H}_\delta$ that is not a dual equivalence graph, then $\mathrm{inv}_\delta(v) > 0$. Hence, v is not a vertex of $\mathcal{P}_\delta$.

Proof of Theorem 1.1: Expressing $R_{\gamma,\delta}(X)$ and $\widetilde{P}_\delta(X;t)$ as in (15) and (16), the result follows from (11), Theorems 3.4, Corollary 3.2, and the fact that the maj_δ statistic is constant on components of $\mathcal{P}_\delta$.

We conclude this section with a discussion of some computational considerations. Using the results of this paper, we may compute $\widetilde{P}_\mu(X;t)$ by making a tree — proceeding as mentioned in Part 1 of Remark 2.2 by filling μ in reverse row reading order and checking that there are no inversions, that we still have a Yamanouchi word, and that no pistol is jammed with the addition of each new entry. In fact, it is readily shown that upon filling the bottom three rows, such a tableau must be one of the three possibilities in Figure 9. That is, the bottom row must be all 1's, the second row starts with k many 2's followed by all 1's, and then the third row has $j \leq k$ many 3's followed by one of three options. Either the rest of the third row is 1's, there or are $k - j$ 1's followed by all 2's, or the rest may be all 2's if the result is still a Yamanouchi word. It is, in theory, possible to precompute more rows in this fashion at the expense of more complicated rules. In this way, we may skip the creation of the beginning of the tree, and save computation time.

3	3	1	1	1	1	1
2	2	2	2	1	1	1
1	1	1	1	1	1	1

3	3	1	1	2	2	2	2
2	2	2	2	1	1	1	1
1	1	1	1	1	1	1	1

3	3	2	2	2	2	2
2	2	2	2	1	1	1
1	1	1	1	1	1	1

Fig. 9: The three types of tableaux of the first three rows of μ when generating $\widetilde{P}_\mu(X;t)$.

It should be noted that the tree described above may still have dead ends. In that respect, a key open problem is to find an algorithm that avoids any dead ends in order to maximize efficiency. Such an algorithm was provided for the Littlewood-Richardson coefficients in Remmel and Whitney (1984), suggesting that it may be possible in this case as well.

4 Further Applications to Symmetric Functions

In this section, we build on the results of Section 3. In particular, we can now explicitly answer the question of Garsia mentioned in Section 1. We also provide the analogous result for Macdonald polynomials. In the following two results, notice that we have shifted our focus from $\mathrm{Yam}_\delta(\lambda)$ to $\mathrm{Yam}(\lambda)$.

Corollary 4.1. *Given a partition μ, the following equality holds if and only if μ does not contain $(3,3,3)$ as a subdiagram.*

$$\widetilde{P}_\mu(X;t) = \sum_{\lambda \vdash |\mu|} \sum_{\substack{w \in \mathrm{Yam}(\lambda) \\ \mathrm{inv}_\mu(w)=0}} t^{\mathrm{maj}_\mu(w)} s_\lambda. \tag{19}$$

Proposition 4.2. *Given a partition μ, the following equality holds if and only if μ does not contain (4) or $(3,3)$ as a subdiagram.*

$$\widetilde{H}_\mu(X;q,t) = \sum_{\lambda \vdash |\mu|} \sum_{w \in \mathrm{Yam}(\lambda)} q^{\mathrm{inv}_\mu(w)} t^{\mathrm{maj}_\mu(w)} s_\lambda. \tag{20}$$

We now turn our attention to $R_{\gamma,\delta}(X)$. In particular, we analyze when $R_{\gamma,\delta}(X) = 0$ and find the leading term of the Schur expansion of $R_{\gamma,\delta}(X)$. In order to do this, we will need the following definition.

Definition 4.3. Given diagrams γ and δ such that $\gamma \subset \delta$, γ is a *fillable subset* of δ if the following hold.

1. If $(x,y) \in \gamma$, then $(x, y-1) \in \delta$,
2. If $x_1 < x_2$ are any integers and I is any integer interval such that $(x_1, I \setminus \max(I)) \subset \delta$ and $(x_2, \min(I)) \notin \gamma$, then $|\gamma \cap (x_1, I)| \geq |\gamma \cap (x_2, I)|$.

Fig. 10: Diagrams with bullets representing γ and boxes representing δ. From the left, three examples where γ is not a fillable subset of δ, then three examples where γ is a fillable subset of δ.

Proposition 4.4. *Given diagrams γ and δ, then $R_{\gamma,\delta}(X) = 0$ if and only if γ is not a fillable subset of δ.*

Definition 4.5. Given diagrams γ and δ such that γ is a fillable subset of δ, define the *leading Yamanouchi word* of $R_{\gamma,\delta}(X)$ as the row reading word of the tableau of shape δ achieved by placing a 1 in all cells in $\delta \setminus \gamma$ and defining the values in γ as one larger than the the value in the cell immediately below it in δ.

Notice that the leading Yamanouchi word of $R_{\gamma,\delta}(X)$ is indeed a Yamanouchi word. We can then use the leading Yamanouchi word to provide the leading term in the expansion of $R_{\gamma,\delta}(X)$ into Schur functions.

Proposition 4.6. *Given diagrams γ and δ such that γ be a fillable subset of δ, let $R_{\gamma,\delta}(X) = \sum c_\lambda s_\lambda$ for some nonzero integers c_λ, and let $w \in \mathrm{Yam}(\mu)$ be the leading Yamanouchi word of $R_{\gamma,\delta}(X)$, then*

1. $c_\lambda = 1$ *if* $\lambda = \mu$,
2. $c_\lambda = 0$ *if* $\lambda > \mu$ *in lexicographic order.*

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