On the partial categorification of some Hopf algebras using the representation theory of towers of $\mathcal{J}$-trivial monoids and semilattices

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Abstract. This paper considers the representation theory of towers of algebras of $\mathcal{J}$-trivial monoids. Using a very general lemma on induction, we derive a combinatorial description of the algebra and coalgebra structure on the Grothendieck rings G_0 and K_0.

We then apply our theory to some examples. We first retrieve the classical Krob-Thibon's categorification of the pair of Hopf algebras QSym/**NCSF** as representation theory of the tower of 0-Hecke algebras. Considering the towers of semilattices given by the permutohedron, associahedron, and Boolean lattices, we categorify the algebra and the coalgebra structure of the Hopf algebras **FQSym**, **PBT**, and **NCSF** respectively. Lastly we completely describe the representation theory of the tower of the monoids of Non Decreasing Parking Functions.

Résumé. Cet article traite de la théorie des représentations des tours d'algèbres de monoïdes $\mathcal{J}$-triviaux. Nous introduisons un lemme général d'induction, duquel nous déduisons une description combinatoire des algèbres et cogèbres des groupes de Grothendieck G_0 et K_0.

Nous appliquons ensuite notre théorie pour retrouver le théorème de Krob-Thibon qui catégorifie la paire QSym/**NCSF** comme les algèbres de Hopfs duales K_0 et G_0 de la tour des algèbres 0-Hecke. En considérant les tours de semi-treillis du permutohedron, associahedron et booléen, nous catégorifions les structures d'algèbre et de cogèbre des algèbres de Hopf **FQSym**, **PBT** et **NCSF**. Enfin, nous décrivons complètement la théorie des représentations de la tour des monoïdes des fonctions de parking croissantes.

Keywords: Algebraic combinatorics, combinatorial representation theory, combinatorial Hopf algebras, lattices, categorification

1 Introduction

Since Frobenius it has been known that the self-dual Hopf algebra of symmetric functions encodes the representation theory of the tower of symmetric groups Sym through the Frobenius character map. Namely, Sym is isomorphic to the Grothendieck group of the category of simple modules of

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the symmetric groups, and the product and the coproduct of Schur functions encode respectively the induction and the restriction rule for simple modules (see e.g. [Gei77]). In [KT97], Krob and Thibon discovered that the same construction for the tower of Hecke algebras at $q = 0$ gives rise to the pair of dual Hopf algebras **NCSF** and QSym (since the algebras are not semisimple, one needs to consider both the categories of simple and projective modules, which gives two Grothendieck rings $G_0(A)$ and $K_0(A)$). This sparked a keen interest in studying Grothendieck rings arising from towers of algebras ([HNT06], [Kho99], [SY13]...).

A natural but long running open question is that of categorification:

Problem 1.1 *Which (pairs of dual) combinatorial Hopf algebras can be recovered as Grothendieck groups of some tower of algebras?*

In particular, is it possible to categorify the combinatorial Hopf algebras of Free Quasi Symmetric Functions, or the Planar Binary Tree algebra of Loday Ronco.

In [BL09], Bergeron and Li propose an axiomatic definition of towers of algebras which guarantees that the associated Grothendieck rings are Hopf algebras. In [BLL12] Bergeron, Lam, and Li proves further that those axioms are very strong: namely the tower of algebras is necessarily of graded dimension $r^n n!$.

In order to explore a larger setting which includes our favorite examples, we drop axioms (4) and (5), and weaken axiom (3) not to be necessarily two sided. On the other hand we focus on towers of algebras of $\mathcal{J}$-trivial monoids in order to take advantage of recent progress in the representation theory of those monoids which is very combinatorial (see e.g. [DHS$^+$11]).

In Section 2, we specify the axiomatic definition of *towers of algebras* we will be working with, and recall some results on the representation theory of $\mathcal{J}$-trivial monoids and semilattices, and in particular the description of simple and projective modules.

We proceed in Section 3 with a general formula for inducing a quotient of an idempotent-generated module from an algebra to a super algebra, and specialize it in Section 4 to derive a combinatorial description of the induction rule of simple modules for a tower A of $\mathcal{J}$-trivial monoids, that is the product in $G_0(A)$. Similarly, we give a combinatorial description of the product in $K_0(A)$ and of the coproduct in $G_0(A)$. As an example, we recover Krob-Thibon's theorem: for the tower $A := (H_n(0))_n$ of 0-Hecke algebras, $G_0(A)/K_0(A)$ forms the pair of dual Hopf algebras of Quasi Symmetric Functions and Non Commutative Symmetric Functions. Note however that, in most other cases, the coproduct is *not* compatible with the product, so that we do not get Hopf algebras.

In Section 4.3, we further specialize those results to towers of join-semilattices (that is commutative and idempotent $\mathcal{J}$-trivial monoids). The theory is particularly simple in this case since semilattices are semisimple – so that $G_0(A)$ and $K_0(A)$ coincide – and the induction rule admits a purely order-theoretical description.

Despite the apparent simplicity of this setting, we show in Section 5 that the towers of semilattices given respectively by the permutohedrons, the Tamari, and the Boolean lattices can be used to partially categorify the Hopf algebras of Free Quasi Symmetric Functions (**FQSym**), Planar Binary Trees (**PBT**), and Non Commutative Symmetric Functions (**NCSF**). Namely, in each case the induction and restriction rules are described respectively by the product and the coproduct in one of the natural bases of those Hopf algebras. However the basis for the product does not coincide with the basis for the coproduct, and hence does not give a full categorification

of the dual Hopf algebra structure. It is to be noted that adding some radical to the semilattices has the effect, on G_0, of deforming the product without altering the coproduct; a work in progress is to try to use this trick to recover the full categorification.

Finally, in Section 6, we give a complete combinatorial description of the Grothendieck rings for the tower of the monoids of *Non Decreasing Parking Functions*. We obtain two copies of **NCSF** on different bases, including the well known ribbon basis R_I.

2 Preliminaries

2.1 Towers of algebras

In [BL09], Bergeron and Li propose an axiomatic definition of towers of algebras which guarantees that the Grothendieck rings of their categories of simple and projective modules give a pair of dual Hopf algebras. They further prove in [BLL12] that those axioms are very strong, implying that the tower of algebras is of graded dimension $r^n n!$. We recall here the axioms of [BL09]:

Definition 2.1 *Let $(A_i)_{i \geqslant 0}$ be a family of associative algebras endowed with a collection of algebra morphisms $(\rho_{m,n} : A_m \otimes A_n \hookrightarrow A_{m+n})_{m,n \geqslant 0}$ satisfying the following axioms:*

(1) For $i \geqslant 0$, A_i is a finite dimensional algebra with unit 1_i, and $A_0 \approx \mathbb{K}$.

(2) The multiplication homomorphisms $\rho_{m,n}$ are injective and associative (in the sense that the external multiplication morphism *they implement on the direct sum $\bigoplus_{i \geqslant 0} A_i$ is associative).*

(3) For $m, n \geqslant 0$ the algebra A_{m+n} is a two-sided projective $A_m \otimes A_n$-module.

(4) A relation between the decomposition of A_{m+n} as a left $A_m \otimes A_n$-module and as a right $A_m \otimes A_n$-module holds.

(5) An analogue of Mackey's formula relating induction and restriction holds.

Bergeron and Li then define a *tower of algebras* as a family of algebras as stated above verifying the five previous axioms.

In order to explore a larger setting which includes our favorite examples, we drop axioms 4 and 5 and weaken axiom 3 into the following:

(3') for $m, n \geqslant 0$, the algebra A_{m+n} is a right (resp. left) projective $A_m \otimes A_n$-module.

Definition 2.2 *A right (resp. left) tower of algebra A is a family $(A_i)_{i \geqslant 0}$ of algebras as above satisfying axioms (1), (2) and (3').*

For any field $\mathbb{K}$ of characteristic 0, the $\mathbb{K}$-algebra $\mathbb{K}M$ of a monoid $(M, \cdot)$, is the $\mathbb{K}$-algebra with basis $\{\epsilon_m\}_{m \in M}$, and multiplication obtained by linearization of the product of M:

$$\epsilon_{m_1} \epsilon_{m_2} = \epsilon_{m_1 \cdot m_2}.$$

A *tower of monoids* is a family of monoids $(M_i)_{i \geqslant 0}$ together with a collection of morphisms such that $(\mathbb{K}M_i)_{i \geqslant 0}$ is a *tower of algebras* for the corresponding embedding.

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We recall the definition of the *Grothendieck groups* G_0 and K_0 of an associative finite dimensional algebra F (see [CR90]). For a category $\mathcal{F}$ of finitely generated left F-modules, the Grothendieck group $\mathbf{G}(\mathcal{F})$ is the abelian group generated by symbols $[M]$, one for every isomorphism class of modules M in $\mathcal{F}$ and relations $[M] = [L] + [N]$ for any short exact sequence $0 \to L \to M \to N \to 0$ in $\mathcal{F}$. We let $G_0(F)$ be the Grothendieck group of the category of finitely generated simple F-modules and $K_0(F)$ the Grothendieck group of the category of finitely generated projective F-modules. More combinatorially, it is easy to prove that $G_0(F)$ is the free $\mathbb{Z}$-module with basis $\{[S_i]\}_{i \in I}$, where $(S_i)_{i \in I}$ is a complete set of non pairwise isomorphic simple F-modules. For an F-module M, we can decompose $[M] = \sum_{i \in I} c_i [S_i]$ where c_i is the multiplicity of S_i in M. Continuing the same way, the set $\{[P_i]\}_{i \in I}$ of projective covers of the simple modules form a basis of the $\mathbb{Z}$-module $K_0(F)$.

Let A be a tower of algebras. Axiom (1) ensures that the *Grothendieck groups*

$$G_0(A) = \bigoplus_{n \geqslant 0} G_0(A_n) \qquad \text{and} \qquad K_0 = \bigoplus_{n \geqslant 0} K_0(A_n)$$

are graded connected. Axioms (2) and (3) allows us to define induction and restriction functors on G_0 and K_0, endowing them with a multiplication and a comultiplication. For M and A_m-module and N an A_n-module, the product and coproduct of their classes $[M]$ and $[N]$ in G_0 (or in K_0) are given respectively by

$$[M][N] = [\mathrm{Ind}_{A_m \otimes A_n}^{A_{m+n}} M \otimes N] \qquad \text{and} \qquad \Delta([M]) = \sum_{i+j=n} [\mathrm{Res}_{A_i \otimes A_j}^{A_n} M].$$

The two Grothendieck rings are closely related thanks to the natural pairing $\langle \ , \ \rangle$ defined on $P \in K_0(A_m)$ and $M \in G_0(A_n)$ by

$$\langle [P], [M] \rangle = \left\{ \begin{array}{ll} \dim_{\mathbb{K}}(\hom_{A_m}(P, M)) & \text{if } m = n\,, \\ 0 & \text{if } m \neq n\,. \end{array} \right.$$

In particular, for $\{P_1, \ldots, P_n\}$ a complete set of indecomposable projective modules and $\{S_1, \ldots, S_n\}$ their associated simple irreducible module, we have $\langle [P_i], [S_j] \rangle = \delta_{i,j}$. With the three axioms given above, G_0 and K_0 are dual graded free $\mathbb{Z}$-modules with product and coproduct.

Induction on K_0 and restriction on G_0 are related thanks to *Frobenius reciprocity*.

Theorem 2.3 (Frobenius reciprocity) Ind *is left adjoint for* Res.

The right adjoint of Res is called *coinduction* and is noted Coind. We have an equality between Res and Coind for groups and this equality also holds in the semilattice case. However, these functors are not equal in the general case, for example for the tower of NDPF that we introduce in Section 6.

2.2   Categorification

We will use the definition of *naive categorification* as defined in [Sav14].

2.3 $\mathcal{J}$-trivial monoids, semilattices, and their representation theory

We recall here some facts about $\mathcal{J}$-trivial monoids and their representations. For details, see for example [Pin11] and [DHS$^+$11] respectively. Let M be a monoid, and write $E(M)$ for the set of idempotents of M. In the sequel, we always assume M to be finite. Define the $\mathcal{J}$ preorder $\leqslant_{\mathcal{J}}$ on M by $x \leqslant_{\mathcal{J}} y$ if $x \in MyM$. This preorder induces an equivalence relation, namely $x\mathcal{J}y$ if and only if $MxM = MyM$. The equivalence classes are called $\mathcal{J}$-classes.

Definition 2.4 *A monoid M is $\mathcal{J}$-trivial if all its $\mathcal{J}$-classes are of cardinality 1.*

Equivalently, M is $\mathcal{J}$-*trivial* if $\leqslant_{\mathcal{J}}$ is a partial order. In this case, we write $\mathcal{J}_<(x)$ the set of all elements strictly smaller than x for the $\mathcal{J}$-order. We define similarly the *right preorder* $\leqslant_{\mathcal{R}}$ by $x \leqslant_{\mathcal{R}} y$ if $x \in yM$. This preorder gives $\mathcal{R}$-*classes* and we call $\mathcal{R}$-trivial a monoid for which $\leqslant_{\mathcal{R}}$ is a partial order. The symmetric definition on the left side leads to $\mathcal{L}$-trivial monoids. A monoid is $\mathcal{J}$-trivial if it is both $\mathcal{L}$-trivial and $\mathcal{R}$-trivial. For an element $x \in M$, we denote by $\mathcal{R}_<(x)$ the set of all elements of M strictly smaller than x for the $\mathcal{R}$-order. For a tower of monoids $(A_i) = (\mathbb{K}M_i)$ and $x \in M_i$, we will note $\mathcal{R}_<^{A_i}(\epsilon_x)$ the subspace of A_i generated by $\mathcal{R}_<(x)$.

The representation theory of $\mathcal{J}$-trivial monoids is essentially independent of the ground field $\mathbb{K}$. The simple modules admit an easy description. Namely, for $x \in M$, define $S_x = \mathbb{K}\left(xM/\mathcal{R}_<(x)\right)$. It's a right module of dimension 1 where, denoting the single basis element by ϵ_x, the action is given by $\epsilon_x \cdot m = \epsilon_x$ if $xm = x$ and $\epsilon_x \cdot m = 0$ otherwise.

Theorem 2.5 *Let M be $\mathcal{J}$-trivial monoid. The set $(S_e)_{e \in E(M)}$ is a complete set of pairwise non-isomorphic simple $\mathbb{K}M$-modules.*

We now turn to the description of the indecomposable projective modules. For $x \in M$, set:

$$\mathrm{rfix}(x) = \min_{\mathcal{J}}\{e \in E(M) : xe = x\} \quad \text{and} \quad \mathrm{lfix}(x) = \min_{\mathcal{J}}\{e \in E(M) : ex = x\}.$$

For e an idempotent, one can define the module $P_e = \mathbb{K}\left(eM/\{m \in M : \mathrm{lfix}(m) <_{\mathcal{J}} (e)\}\right)$. Its basis is indexed by the family $\{x : \mathrm{lfix}(x) = e\}$.

Theorem 2.6 ([DHS$^+$11], Corollary 3.22) *Let M be a $\mathcal{J}$-trivial monoid. The set $(P_e)_{e \in E(M)}$ is a complete set of pairwise non-isomorphic indecomposable projective $\mathbb{K}M$-modules.*

The radical of the algebra of an $\mathcal{J}$-trivial monoid viewed as a module on itself is given by its non idempotent elements. It is thus natural to consider the semisimple case, when all the elements of M are idempotent. It turns out that M is then necessarily commutative thanks to the following theorem.

Theorem 2.7 ([Pin11]) *The class of idempotent (or equivalently semisimple) $\mathcal{J}$-trivial monoids coincide with the class of finite semilattices $(\mathcal{L}, \vee)$.*

In particular, a good source of examples is to take one's favorite finite lattice, and consider the monoid given by its join (resp. its meet) operation, together with its smallest (resp. largest) element as identity.

3 Induction lemma

We now introduce our key lemma.

Lemma 3.1 *Let $B \subseteq A$ two $\mathbb{K}$-algebras, $f \in B$ an idempotent and $U \subseteq fB$ a right B-module. We have the following A-mod isomorphism:*

$$\mathrm{Ind}_B^A(fB)/U \approx (fA)/(UA) \, .$$

4 Representation theory of towers of $\mathcal{J}$-trivial monoids

4.1 General case

In the following section, we fix a tower of monoids $(M_i)_{i \geqslant 0}$ with $A = (A_i)_{i \geqslant 0}$ the associated tower of algebras. Thanks to $\mathcal{J}$-triviality, the representation theory of such a monoid is combinatorial. In order to describe the general rules, we need to expand the previous definitions of rfix and lfix to tensorial algebras. For $x \in M_{m+n}$ set

$$\mathrm{rfix}_{M_m \times M_n}(x) = \min_{\mathcal{J}}\{e \in M_m \times M_n : \rho_{m,n}(e) \in E(M_{m+n}) \text{ and } x\rho_{m,n}(e) = x\} \, ,$$

and define similarly $\mathrm{lfix}_{M_m \times M_n}(x)$. We can therefore state the following proposition:

Proposition 4.1 *Let $(A_i)_{i \geqslant 0} = (\mathbb{K}M_i)_{i \geqslant 0}$ be a tower of monoid algebras. Take $x \in M_{m+n}$, and let S_x be the associated A_{m+n}-simple module, then $\mathrm{Res}_{A_m \otimes A_n}^{A_{m+n}} S_x = S_e$ where $e = \mathrm{rfix}_{M_m \times M_n}(x)$.*

Thanks to Frobenius Reciprocity (Theorem 2.3) we directly get the product rule in K_0.

Proposition 4.2 *Let e be an idempotent of $M_m \otimes M_n$ and P_e be the projective module associated to e. Then,*

$$\mathrm{Ind}_{A_m \otimes A_n}^{A_{m+n}} P_e = \sum_{f \in C} P_f \quad with \quad C = \{g \in E(A_{m+n}) : \mathrm{rfix}_{M_m \times M_n}(g) = e\} \, .$$

We can now study how this lemma applies in the case of a tower of $\mathcal{J}$-trivial monoids. Each simple module M can be interpreted as an element x of a graded component A_n of the tower, and the action is characterized by the partial ordering. Indeed, A_n acts by 1 on M for all elements $\{z \geqslant_{\mathcal{R}} x\}$ and by 0 otherwise. Given two simple modules S_f and S_g of A_m and A_n, the tensor product $S_f \otimes S_g$ is a simple two sided $(A_m \otimes A_n)$-module. Thanks to lemma 3.1, the induced module on A_{m+n} is the quotient $\mathbb{K}(f \cdot g)A_{m+n}/(\mathcal{R}_<(f) \cdot \mathcal{R}_<(g)A_{m+n})$.

Let S_e and S_f be two simple modules of respectively A_m and A_n. Let $X(e, f)$ denote the subset of M_{m+n} containing all elements in $\rho_{m,n}(e, 1)\,\rho_{m,n}(1, f)M_{m+n}$ which are not in $\rho_{m,n}(\mathcal{R}_<(e), 1)\rho_{m,n}(1, \mathcal{R}_<(f))$. Namely, by identifying M_m and M_n with their copies embedded in M_{m+n} we have

$$X(e, f) = efM_{m+n} \quad \backslash \bigcup_{\substack{e' \in eM_m, f' \in fM_n \\ (e', f') \neq (e, f)}} e'f'M_{m+n} \, .$$

The following theorem describes combinatorially the induction rule for simple modules:

Theorem 4.3 (Induction rule for $\mathcal{J}$-trivial monoids) *Let $M = (M_i)$ be a tower of $\mathcal{J}$-trivial monoids and $A = (A_i)$ the related tower of algebras. With the above notations, the induction rule for two simple modules S_e and S_f is given by*

$$\left[\mathrm{Ind}_{M_m \otimes M_n}^{M_{m+n}} S_e \otimes S_f\right] = \sum_{x \in X(e,f)} \left[S_{\mathrm{lfix}(x)}\right].$$

Proof: Straightforward application of Lemma 3.1 on the construction of the simple modules given in Theorem 2.5. □

4.2   Categorification of the pair of Hopf Algebras $\mathrm{QSym}/\mathbf{NCSF}$

We recover Krob-Thibon's Theorem [KT97] using Theorem 4.3. It is well known that the 0-Hecke algebra $\mathcal{H}_n$ at $q = 0$ is the algebra of the 0-Hecke monoid $H_n(0)$ which is $\mathcal{J}$-trivial. The idempotents are naturally indexed by the subsets I of $\{1, \ldots, n-1\}$, with $\mathrm{lfix}(\pi_\sigma)$ and $\mathrm{rfix}(\pi_\sigma)$ given respectively by the left and right descent sets $D_L(\sigma)$ and $D_R(\sigma)$ of σ.

Let us first recover the product rule in G_0. Each simple module S_I can be constructed as $\pi_\sigma H_n(0)/\mathcal{R}_<(\pi_\sigma)$, where σ is any permutation with left descent set I. Here we choose $\pi_I = \pi_{\sigma_I}$, where σ_I is the maximal element of the parabolic subgroup $\mathfrak{S}_I$.

Consider a simple $(\mathcal{H}_m \otimes \mathcal{H}_n)$-module $S_I \otimes S_J$. It can be written as

$$S_I \otimes S_J = (\pi_I \otimes \pi_J)(\mathcal{H}_m \otimes \mathcal{H}_n) \,/\, \mathcal{R}_<^{\mathcal{H}_n \otimes \mathcal{H}_m}(\pi_I \otimes \pi_J) = \pi_\mu(\mathcal{H}_m \otimes \mathcal{H}_n) \,/\, Q,$$

where $\mu \in \mathfrak{S}_{m+n}$ is such that $\pi_\mu = \pi_I \otimes \pi_J$, and $Q = \mathcal{R}_<^{\mathcal{H}_n \otimes \mathcal{H}_m}(\pi_\mu)$. Using Lemma 3.1, the induced module on $\mathcal{H}_{m+n}$ is

$$\mathrm{Ind}_{\mathcal{H}_m \otimes \mathcal{H}_n}^{\mathcal{H}_{m+n}} S_I \otimes S_J = \pi_\mu \mathcal{H}_{m+n} \,/\, Q\mathcal{H}_{m+n}.$$

Note that $Q = \{\pi_\mu \pi_i \pi_\nu : i \notin \mathrm{Des}_R \mu, i \neq m, \pi_\nu \in \rho_{m,n}(H_m(0) \times H_n(0))\}$. Therefore,

$$Q\mathcal{H}_{m+n} = \{\pi_\mu \pi_i \pi_\nu : i \notin \mathrm{Des}_R(\mu), i \neq m, \pi_\nu \in \mathcal{H}_{m+n}\}.$$

and it follows that

$$\begin{aligned} \pi_\mu \mathcal{H}_{m+n} \backslash Q\mathcal{H}_{m+n} &= \{\pi_\mu \pi_\nu : \mathrm{Des}_R(\nu) \subseteq \mathrm{Des}_R(\mu) \cup \{m\}\} \\ &= \{\pi_\mu \pi_\nu : \mathrm{Des}_R(\nu) \subseteq \{m\}\}. \end{aligned}$$

It is well known that the permutations ν with $\mathrm{Des}_R(\nu) \subseteq \{m\}$ are the permutations appearing in the shuffle product $\mathrm{id}_m \shuffle \mathrm{id}_n$. At the level of descents we recover the shuffle product of **NCSF**.

For the coproduct in G_0, we need to study the restrictions of each simple module S_I of $\mathcal{H}_{m+n}$ on $\mathcal{H}_m \otimes \mathcal{H}_n$. In terms of descent sets, $\mathrm{lfix}_{H_m(0) \times H_n(0)}(\pi_I)$ amounts to removing m from I and shifting the entries greater than m by $-m$; this is exactly the shifted deconcatenation rule of the fundamental basis of **NCSF**.

Altogether we proved that G_0 is isomorphic to **NCSF** in the fundamental basis. By adjunction, we get the product in K_0, then we use our knowledge of the pair $\mathrm{QSym}/\mathbf{NCSF}$ and the $\mathbb{Z}$-module duality to conclude.

4.3 Representation theory of towers of semilattices

For a tower of semilattices $(L_m)_{m \geqslant 0}$, each L_i is semisimple. Then we have $G_0(\mathbb{K}L) = K_0(\mathbb{K}L)$.

Because we combinatorially described the induction and restriction in $G_0(A)$ for any algebra tower of $\mathcal{J}$-trivial monoids, we have a complete combinatorial description of both Grothendieck rings of any tower of semilattices.

5 Partial categorification of **FQSym**, **PBT**, and **NCSF**

In the section, we assume that the reader is familiar with both Malvenuto-Reutenauer algebra **FQSym** ([MR95], [DHT02]) and the Loday-Ronco *Planar Binary Tree* Hopf algebra ([LR98]) and how it relates with **FQSym** ([HT09]).

5.1 The tower of permutohedron lattices

We first consider the tower of permutohedrons. Namely, for each n, take the left weak order $<_L$ on the n-th symmetric group $\mathfrak{S}_n$. A potential definition for the weak order on permutation is that $\sigma \leqslant \mu$ if and only if $\mathrm{inv}(\sigma) \subseteq \mathrm{inv}(\mu)$. It is well known that it endows $\mathfrak{S}_n$ with a lattice structure, and we consider the semilattice $\mathcal{P}_n = (\mathfrak{S}_n, \vee)$, with the trivial representation as identity element. Remind that the $\mathcal{J}$-order on $\mathcal{P}_n$ is the reverse order of $<_L$; the identity is the largest element.

Let $\rho_{m,n} : \mathbb{K}\mathcal{P}_m \otimes \mathbb{K}\mathcal{P}_n \to \mathbb{K}\mathcal{P}_{m+n}$ be the linear extension of the shifted concatenation on $\mathcal{P}_m \times \mathcal{P}_n$. The morphism ρ is obviously injective. Thus we obtain a tower of semilattices $\mathbb{K}\mathcal{P} = \bigoplus_{n \geqslant 0} \mathbb{K}\mathcal{P}_n$. It follows that $\mathrm{inv}(\rho_{m,n}(\sigma, \mu)) = \mathrm{inv}(\sigma) \bar{\cup} \mathrm{inv}(\mu)$ where $\bar{\cup}$ is the shifted union.

We start by describing the product rule in $G_0(\mathcal{P})$, that is the induction rule of simple modules. Let us fix a simple $\mathbb{K}(\mathcal{P}_m \otimes \mathcal{P}_n)$-module $S_{\sigma \otimes \mu}$. This simple module is the quotient

$$\mathbb{K}(\sigma \otimes \mu)(\mathcal{P}_m \otimes \mathcal{P}_n)/\mathcal{R}_<(\sigma \otimes \mu) = \mathbb{K}(\sigma \otimes \mu)(\mathcal{P}_m \otimes \mathcal{P}_n)/(\mathcal{R}_<(\sigma) \otimes \mathcal{R}_<(\mu))$$

by definition of the product order. By Lemma 3.1, we have:

$$\mathrm{Ind}_{\mathbb{K}(\mathcal{P}_m \otimes \mathcal{P}_n)}^{\mathbb{K}\mathcal{P}_{m+n}} S_{\sigma \otimes \mu} = (\sigma^{\bar{\cdot}} \mu)\mathcal{P}_{m+n}/(\mathcal{R}_<(\sigma \otimes \mu)\mathcal{P}_{m+n}).$$

This quotient is a subalgebra of $\mathbb{K}\mathcal{R}_<(\sigma^{\bar{\cdot}} \mu)$ and is quite easy to describe. The submonoid $\mathcal{R}_<(\sigma \otimes \mu)\mathcal{P}_{m+n}$ is exactly the set of all permutations $w \in \mathcal{P}_{m+n}$ such that $\mathrm{inv}(w) \subset \mathrm{inv}(\sigma) \bar{\cup} \mathrm{inv}(\mu)$. Since $\mathcal{R}_<(\sigma^{\bar{\cdot}} \mu)$ consists of all permutations ν such that $\mathrm{inv}(\nu) \subseteq \mathrm{inv}(\sigma)^{\bar{\cdot}} \mathrm{inv}(\mu)$, we can state the following proposition:

Proposition 5.1 *In the permutohedron tower, the induction rule for the simple modules is*

$$\mathrm{Ind}_{\mathbb{K}(\mathcal{P}_m \otimes \mathcal{P}_n)}^{\mathbb{K}\mathcal{P}_{m+n}} S_{\sigma \otimes \mu} = \sum_{\substack{\nu \in \mathcal{P}_{m+n} \\ \mathrm{inv}(\nu) = \mathrm{inv}(\sigma) \bar{\cup} \mathrm{inv}(\mu)}} S_\nu.$$

This gives the following product rule in $G_0(\mathcal{P})$:

$$[S_\sigma] \cdot [S_\mu] = \sum_{\nu \in \sigma \, \square \, \nu} [S_{\nu^{-1}}].$$

Example 5.2 $[S_{(21)}] \cdot [S_{(231)}] = [S_{(21453)}] + [S_{(31452)}] + [S_{(32451)}] + [S_{(41352)}] + [S_{(42351)}] + [S_{(52341)}] + [S_{(43251)}] + [S_{(51342)}] + [S_{(53241)}] + [S_{(54231)}].$

Using Frobenius formula, we directly obtain

$$\Delta([S_\sigma]) = \sum_{\substack{u \bar{\cdot} v = \sigma \\ \mathrm{inv}(\mu) = \mathrm{inv}(u) \\ \mathrm{inv}(\nu) = \mathrm{inv}(v)}} [S_\mu] \otimes [S_\nu] = \sum_{u \bar{\cdot} v = \sigma} [S_{\mathrm{std}(u)}] \otimes [S_{\mathrm{std}(v)}].$$

Example 5.3 $\Delta([S_{(2413)}]) = [1] \otimes [S_{(2413)}] + [S_1] \otimes [S_{(312)}] + [S_{(12)}] \otimes [S_{(12)}] + [S_{(312)}] \otimes [S_1] + [S_{(2413)}] \otimes [1].$

Corollary 5.4 *The map $[S_\sigma] \mapsto \mathbf{G}_\sigma$ is an algebra isomorphism between $G_0(\mathcal{P})$ and $\mathbf{FQSym}$.*
The map $[S_\sigma] \mapsto \mathbf{F}_\sigma$ is a coalgebra isomorphism between $G_0(\mathcal{P})$ and $\mathbf{FQSym}$.

Each $\mathbf{F}_\sigma$ or $\mathbf{G}_\sigma$ induced a $\mathbf{FQSym}$-module morphism by left-multiplication. We obtain a categorification of the algebra $(\mathbf{FQSym}, \mathbf{G}_\sigma)$ and a categorification of the coalgebra $(\mathbf{FQSym}, \mathbf{F}_\sigma)$.

Note that the product and the coproduct are not compatible as bialgebras, so in particular we did not categorify $\mathbf{FQSym}$ as an auto-dual Hopf algebra.

5.2 The tower of Tamari lattices

We now turn to the tower of the *Tamari lattice*. The Tamari lattice is the lattice of binary trees ordered by tree rotations. We note $\mathcal{T}_n$ the Tamari lattice of binary trees with n edges; $\mathcal{T}_n$ has cardinality C_n the nth Catalan number. In this paper, we will note the elements of the Tamari lattice as 132-avoiding permutations.

We construct a tower of semilattices by the embedding $* : \mathcal{T}_m \times \mathcal{T}_n \hookrightarrow \mathcal{T}_{m+n}$ which consist in taking the Sylvester class of the shifted concatenation.

Example 5.5 $(312) * (21) = (53214).$

The product and the coproduct in the Grothendieck rings of $\mathbb{K}\mathcal{T}$ can be computed from our previous construction on $\mathbf{FQSym}$. Therefore, the maps $[\mathcal{T}_\sigma] \mapsto \mathbf{P}_\sigma$ and $[\mathcal{T}_\sigma] \mapsto \mathbf{Q}_\sigma$ are respectively algebra and coalgebra isomorphisms from $G_0(\mathcal{T})$ to $\mathbf{PBT}$ making both diagrams of the definition of naive categorification commute.

Once again, we constructed categorifications of $(\mathbf{PBT}, \mathbf{P}_T)$ and $(\mathbf{PBT}, \mathbf{Q}_T)$. But we do not get a full Hopf categorification of the pair $(\mathbf{PBT}, \mathbf{PBT}^*)$.

5.3 The tower of Boolean lattices

Finally, let $\mathcal{B}_n$ be the Boolean lattice of subsets of $[\![1, n]\!]$. We write the elements of $\mathcal{B}_n$ as binary words of size n. The concatenation embedding is an injective morphism making a tower $\mathcal{B} = \bigoplus_{m \geqslant 0} \mathcal{B}_n$ of semilattices. The product in $G_0(\mathcal{B})$ easily follows from the remark $\mathcal{R}_<(u) \cdot \mathcal{R}_<(v) = \mathcal{R}_<(u \otimes v)$ and Theorem 4.3 gives that

$$\mathrm{Ind}_{\mathbb{K}\mathcal{B}_m \otimes \mathbb{K}\mathcal{B}_n}^{\mathbb{K}\mathcal{B}_{m+b}} S_u \otimes S_v = S_{u \cdot v}.$$

By duality, or using that $\mathcal{B}_n$ is the semisimple quotient of $H_n(0)$, the coalgebra structure on $G_0(\mathcal{B})$ is isomorphic to QSym on the fundamental basis:

$$\Delta(S_w) = \sum_{0 \leqslant i \leqslant m+n} [S_{w_{[\![1,i]\!]}}] \otimes [S_{w_{[\![i+1,m+n]\!]}}].$$

Unsurprisingly, the product and coproduct are again not compatible.

6 Non Decreasing Parking Functions

Definition 6.1 *For $n \geqslant 1$, let NDPF_n be the monoid of the non decreasing and regressive functions from $[\![1, n]\!]$ onto itself, endowed with the composition product. Denote a function $f : i \mapsto a_i \in \mathrm{NDPF}$, by the sequence $(a_1, a_2, \ldots, a_n)$; it satisfies $1 = a_1 \leqslant a_2 \leqslant \cdots \leqslant a_n$ and $a_i \leqslant i$.*

This monoid and its representation theory, were studied in [HT09, DHS$^+$11]. It's cardinality is the n-th Catalan number, and it is minimally generated by the idempotents $\pi_i = (1, 2, \ldots, i, i, i+2, \ldots, n-1)$ for $i \in [\![1, n-1]\!]$.

The exterior product $\rho_{m,n} : (a_1, \ldots, a_m) \cdot (b_1, \ldots, b_n) \mapsto (a_1, \ldots, a_m, b_1 + m, \ldots, b_m + n)$ defines an associative and injective embedding from $\mathrm{NDPF}_m \otimes \mathrm{NDPF}_n$ to NDPF_{m+n}. Thus we note $\mathrm{NDPF} = \bigoplus_{n \geqslant 0} \mathrm{NDPF}_n$, and the tower $\mathbb{K}\,\mathrm{NDPF}$ satisfy Axioms (1) and (2). Verifying the third axiom is more tricky.

Proposition 6.2 NDPF_{m+n} *is a left projective* $(\mathrm{NDPF}_m \otimes \mathrm{NDPF}_n)$*-module.*

Proof: We construct an explicit decomposition of NDPF_{m+n} as $\mathrm{NDPF}_m \otimes \mathrm{NDPF}_n$-module, and prove bijectively that each piece is isomorphic to a projective $\mathrm{NDPF}_m \otimes \mathrm{NDPF}_n$ module. $\qquad\square$

Proposition 6.3 ([DHS$^+$11]) *The monoid* NDPF_n *is* $\mathcal{J}$*-trivial for all* n*. In particular its simple modules are all one dimensional.*

The irreducible left $\mathbb{K}\,\mathrm{NDPF}_n$-modules are thus entirely characterized by the action of the idempotent generators of $\mathbb{K}\mathrm{NDPF}_n$. The eigenvalues of S on π_i is 0 or 1 so we have 2^{n-1} simple $\mathbb{K}\,\mathrm{NDPF}_n$-modules indexed by compositions. From now on, we will identify a simple $\mathbb{K}\,\mathrm{NDPF}_n$-module by the sequence $(b_1, \ldots, b_n)$ of his ordered eigenvalues on the generators (π_i). We will note $() = (1^0)$ the unique irreducible NDPF_1-module. Note that an irreducible NDPF_n-module is described by $n-1$ eigenvalues.

Proposition 6.4 *The* $\mathcal{J}$*-order on* $E(\mathrm{NDPF}_n)$ *is the Boolean lattice of* 2^n *elements.*

The product in $K_0(\mathrm{NDPF})$ is quite general thanks to [DHS$^+$11]. We again use Lemma 3.1.

Thanks to Theorem 2.6 we can explicit the product in $K_0(\mathrm{NDPF})$. Let e_I and e_J be two idempotents in respectively NDPF_m and NDPF_n indexed with compositions I of $m-1$ and J of $n-1$. We note $P_I = \mathrm{NDPF}e_I$ the projective module associated with S_e. By applying Lemma 3.1 we have:

$$
\begin{aligned}
[P_{e_I}] \cdot [P_{e_J}] &= \left[(\mathrm{NDPF}_m \otimes \mathrm{NDPF}_n)\,(e_I \otimes e_J) \bigotimes \mathrm{NDPF}_{m+n} \right] \\
&= \left[(\mathrm{NDPF}_m \otimes \mathrm{NDPF}_n)\,(e_I \otimes 1)(1 \otimes e_J) \bigotimes \mathrm{NDPF}_{m+n} \right] \\
&= \left[\mathrm{NDPF}_{m+n}\,(e_I \otimes 1)(1 \otimes e_J) \right].
\end{aligned}
$$

By Theorem 2.6 the projective module we obtain is:

$$
\left\{ f \in \mathrm{NDPF} : \begin{array}{ccc} f(1 \otimes e_J) &=& f \\ f(e_I \otimes 1) &=& f \end{array} \right\} .
$$

We deduce that $[P_{e_I}][P_{e_J}] = \left\{ f : \begin{array}{ccc} \mathrm{rfix}(f)_{[\![1,m]\!]} & = & e_I \\ \mathrm{rfix}(f)_{[\![m+2,m+n]\!]} & = & e_J \end{array} \right\}$, so that $[P_I][P_J] = [P_{I \cup J}] + [P_{I \cup \{n+1\} \cup J}]$. We just proved:

Proposition 6.5 *The map $K_0(\mathrm{NDPF}) \to \mathbf{NCSF}$ defined by $[P_I] \mapsto R_I$ is an algebra isomorphism.*

The product in $G_0(\mathrm{NDPF})$ is quite tedious to write. To avoid a straightforward but technical proof, we use our knowledge of the algebra structure of $K_0(\mathrm{NDPF})$, and take advantage of the fact that the *Cartan operator* $C : K_0(\mathrm{NDPF}) \to G_0(\mathrm{NDPF})$ is an isomorphism in our case. The Cartan operator is the $\mathbb{Z}$-module morphism defined by the Cartan matrix. More precisely, for P a projective module, $C([P]) = \sum_i c_i [S_i]$ where $c_i = \dim(\hom(P_i, P))$; that is $C([P])$ gives the composition factors of P, with multiplicities. Thanks to Theorem 2.6, it admits an explicit description for $\mathcal{J}$-trivial monoids: $C([P_{e_I}]) = \sum_{\mathrm{rfix}(x) = e_I} [S_x]$.

Lemma 6.6 *For $k \geqslant 0$ and $l \geqslant 0$ we have:*

$$[(1^k)][(0^l)] = \left\{ [(c_1, c_2, \ldots, c_{k+l+1})] : \sum_i c_i \in \{k, k+1\} \right\} .$$

Theorem 6.7 *Let $S_a = (a_1, a_2, \ldots, a_i, 1^k) \in G_0(\mathrm{NDPF}_m)$ and $S_b = (0^l, b_j, b_{j+1}, \ldots, b_{n-1}) \in G_0(\mathrm{NDPF}_n)$, then*

$$[S_a][S_b] = \left\{ [(a_1, \ldots, a_i, c_1, \ldots, c_{k+l+1}, b_j, \ldots, b_{n-1})] : \sum_i c_i \in \{k, k+1\} \right\} .$$

Theorem 6.8 *The algebra $G_0(\mathrm{NDPF})$ is the free graded algebra with generators $(), (0), (00), \ldots$.*

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