On the bialgebra of functional graphs and differential algebras
Maurice Ginocchio

To cite this version:

HAL Id: hal-00955700
https://hal.inria.fr/hal-00955700
Submitted on 5 Mar 2014

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers. L’archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d’enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.
On the bialgebra of functional graphs and differential algebras

Maurice Ginocchio

Laboratoire de Physique Théorique et Mathématique, Université Paris 7, Tour Centrale -3ème étage, 2, place Jussieu, F-75251 Paris Cedex 05, France
E-Mail: mag@ccr.jussieu.fr

We develop the bialgebraic structure based on the set of functional graphs, which generalize the case of the forests of rooted trees. We use noncommutative polynomials as generating monomials of the functional graphs, and we introduce circular and arborescent brackets in accordance with the decomposition in connected components of the graph of a mapping of \([1, 2, \ldots, n]\) in itself as in the frame of the discrete dynamical systems. We give applications for differential algebras and algebras of differential operators.

Keywords: bialgebraic structure, functional graphs, noncommutative polynomials

1 Introduction

We have already described the expansion of \(\Delta = \Sigma \lambda \partial_i\), i.e. the powers of a Lie operator in any dimension, in order to find the expression of the flow of formal nonlinear evolution equations [1–3]. In the one-dimensional case, the explicit expansion can be found first in Comtet [4], and other aspects connected to the ordinary differential equations can be found in Leroux and Viennot [5] and Bergeron and Reutenauer [6]. On the other hand, Grossman and Larson [7] introduced several Hopf algebras [8–10] of forests of rooted labeled trees to express the product of finite dimensional vector fields. In this paper we concentrate us on the bialgebra \(G\) of functional graphs, i.e. graphs representing mappings of finite sets in themselves [11–15]. We give only the results without proofs. In a forthcoming paper [16], we develop Hopf algebra structures, computing the antipode and giving detailed aspects and proofs.

In Sect. 1 we consider a bialgebra structure on \(G\) and three interesting subalgebras: \(T\) the set of labeled forests; \(S\) the set of permutation graphs; and \(L\) the set of well labeled forests, i.e. with strictly decreasing labels on the chains toward the roots. Recall that the graded bialgebra \(L\) is sufficient for the calculus of the powers of one derivation [1], and it is extendable in a Hopf algebra, the element of which is known in the computer literature as ‘heap ordered trees’. This bialgebra is useful to compute products of derivations or to transform differential monomials in differential algebras [17], and it is interesting to note that the elements \(L_n\) (\(n\) edges) can be coded by the words (monomials) of the expansion of \(Q_n = q_0(q_0 + q_1) \ldots (q_0 + q_1 + \ldots q_{n-1})\), where \(Q^0 = \{q_0, q_1, \ldots\}\) is a noncommutative alphabet. We describe in particular the bialgebra \(L\), first in the polynomial form by the ‘factorial’ monoid \(L_0^0 = (L_n^0)_{n \geq 0}\), where \(L_n^0\) is the set of words in the expansion of \(Q_n\), and second, we establish the bijective correspondence between
L and L. We show that the calculus are easier with L, and that the product on L can be expressed in a very natural way. For example, \((q_0)^n = Q_n\), hence the (exponential) generating function of all the elements of L. We describe principally the formalism in the general case G, and the calculus uses the fields \(F_2 = \{0, 1\}\) as well as characteristic zero fields K.

In Sect. 2, we describe the link with the graded differential algebra \(K\{U\} = \oplus_{r \geq 0} K\{U\}_r\) and the graded algebra of differential operators \(K\{U, D\} = \oplus_{r \geq 0} K\{U\}_rD^r\), where \(U = (u_1, u_2, ...) = (u_0)_{g \geq 1} = (u_0^\alpha)_{g \geq 0, \beta_1 \geq 1}\) is a set of indeterminates, \(D \equiv \{\partial_0, \partial_1, ...\}\) and the differential indeterminates \(u_0^\alpha\sigma_1 ... \sigma_p = \partial_{\sigma_1} ... \partial_{\sigma_p} u_0^\alpha\) generate \(K\{U\}_r\) [17]. This shows that the above Q-calculus, which is a kind of ‘dissection’ on functional graphs can be used as pre-calculus in differential algebras as well as in discrete dynamical systems [18].

2 Bialgebra Based on the Semi-group of Functional Graphs

2.1 Types of Functional Graphs

In this paper, a connected functional graph will be called excycle [13, 15]. In the area of discrete dynamical systems, an excycle is known as a basis of attraction. Consider several graded and filtered sets of labeled functional graphs

(i) E (resp. G) the set of excycles (resp. functional graphs) and designated by \(G_n\) (resp. \(G(n)\)), the set of functional graphs having (resp. having at most) \(n + 1\) nodes for \(n \geq 0\).

(ii) R (resp. T) the set of labeled arborescences (resp. forests).

(iii) C (resp. S) the set of cycles (resp. permutation graphs).

(iv) A (resp. L) the set of well labeled arborescences (resp. forests), i.e. with strictly decreasing labels on the chains toward the root(s).

As in (i), we consider for (ii)–(iv) graduations and filtrations.

2.2 Free Representation by Q-polynomials

Let \(G_n\) be the semigroup of mappings of \(\{1, 2, ... n\}\) in itself (‘Semigroup of endofunctions’),

- Card \(G_n = n^n\), and the subsemigroups,
- \(T_n \equiv \{f; f \in G_n, f^n = f^{n-1}\}\) (i.e. f acyclic and \(\text{Card } T_n = (n + 1)^{n-1}\)),
- \(S_n\) the symmetric group and \(\text{Card } S_n = n!\)
- \(L_n \equiv \{f; f \in G_n, f(1) \leq i\}\) (i.e. f subdiagonal and \(\text{Card } L_n = n!\)).

We have the well known bijections \([F-F]\): \(G_n \approx G_n, T_n \approx T_n, S_n \approx S_n, L_n \approx L_n\).

Let \(Q = \{q_0, q_1, ...\}\) be a noncommutative alphabet, \(Q^0 = \{q_0\} \cup Q\) with \(q_0\) noncommuting with the \(q_i\)’s. \(Q_n = \{q_1, q_2, ... q_n\}\), \(Q^0_n = \{q_0\} \cup Q_n\) and \(Q^*\) (resp. \(Q^0_n\)), the corresponding free monoids. Taking \(F_2 = \{0, 1\}\) as the field, consider

(i) the \(G_n\) module \(F_2Q_n\) by the \(F_2\) linear incidence matrix action of \(f \in G_n\) as \(I_q = q(I_f)\) hence \(I_qI_g = I_{qg}\).
(ii) the generating monomial associated with $f$.

By morphism extension, denoted again by $l_f$, we define

$$Q_f = q_f(1) q_f(2) \ldots q_f(n) = l_f Q_w$$

(1)

where $Q_w = q_1 q_2 \ldots q_n$ is associated with the identity $t_0$ of $G_n$ and $Q_0 = 1$. One again has $l_f l_g = l_{f g}$.

For the following we consider

(iii) The graded subsets of $Q^*$ as $G \equiv (G_n)_{n \geq 0}, T \equiv (T_n)_{n \geq 0}, S \equiv (S_n)_{n \geq 0}, L \equiv (L_n)_{n \geq 0} $, respectively associated with $G, T, S$ and $L$, with $G_0 = T_0 = S_0 = L_0 = \{1\}$.

(iv) The corresponding graded $F_2$-modules: $F_2 G, F_2 T, F_2 S, F_2 L$, admit components of degree $n$ which are, respectively, $G_n, T_n, S_n, L_n$ modules, with

$$\dim F_2 G_n = n^n, \dim F_2 T_n = (n + 1)^{n-1}, \dim F_2 S_n = \dim F_2 L_n = n!$$

(v) We will denote by $R_n$ one of the above subsemi-groups of $G_n$ (or of another category).

Similarly, let $R \equiv (R_n)_{n \geq 0}, (resp. F_2 R = \oplus_{n \geq 0} F_2 R_n)$ be the corresponding graded subsets of $Q^*$ (resp. graded $F_2$-modules of $F_2 G = \oplus_{n \geq 0} F_2 G_n$).

2.3 Virtual Root and External Product

Let $f \in G_n, l_0$ be the set of fixed points of $f$ and $H_0$ a subset of $I_0$, and set $[p,q] \equiv \{r,p \leq r \leq q\}$ if $p \leq q$ and $\emptyset$ otherwise.

Define $f^0 : [1,n] \rightarrow [0,n]$ such that $f^0(i) = f(i)$ if $i \notin H_0$ and $f^0(i) = 0$ if $i \in H_0$. The ‘0’ is the label of a virtual root added to the graph representation of $f$, and we will say that $H_0$ is ‘confined in 0’, which is a fixed point of $f^0$. We call ‘extended endofunctions’ such functions $f^0$, denote by $G^0_n \equiv [0,n][1,n]$ their set, and we consider $G_n$ as a subset of $G^0_n$. Similarly, we will have $T^0_n \supset T_n, S^0_n \supset S_n, L^0_n \supset L_n$. Consequently, adding $q_0$, we get the extended graded sets $G^0 \equiv (G^0_n)_{n \geq 0}$, the extended graded $F_2$-module $F_2 G^0 = \oplus_{n \geq 0} F_2 G^0_n$ and their substructures $F_2 T^0_n, F_2 S^0_n, F_2 L^0_n$.

Now let $\phi \in G^0_n, \chi_0$ be the characteristic function of $H_0 = \phi^{-1}(0)$, and write

$$Q_\psi = l_\phi Q_w = q_\phi(1) q_\phi(2) \ldots q_\phi(n) = \prod_{i=1}^n q_{\phi(i)}$$

(cf. Figures 1 and 2).

With $\psi \in G^0_n$, consider the $F_2$-bilinear product in $F_2 G^0$ defined by

$$Q_\psi Q_\phi = Q_\psi \prod_{i=1}^n (q_{\phi(i)+n} + \chi_0(i)(q_0 + q_1 + \ldots q_{n-1}))$$

(2)

On the right-hand side we have a sum of concatenated monomials, and on the right factor the substitutions $q_0 \rightarrow q_0 + q_1 + \ldots q_n$ and $q_h \rightarrow q_{h+n}$ when $h \neq 0$. 


On the other hand, the product belongs to $F_2 G^0_{m+n}$. This external product is associated with unit 1 and $F_2 G^0$ is ‘$\cdot$’ graded. To see this consider $i, j, k$ being $\neq 0$, three homogeneous polynomials,

$$A = A(q_0; q_i) \in F_2 G^0_{m}, B = B(q_0; q_j) \in F_2 G^0_{n}, C = C(q_0; q_k) \in F_2 G^0_{p},$$

then by (2)

$$A.B = B(q_0; q_j)A(q_0 + q_1 + \ldots q_n,q_{i+n})$$

and so, using $\deg(B,C) = n + p,$

$$\langle A(q_0; q_i).B(q_0; q_j) \rangle.C = \langle B(q_0; q_j)A(q_0 + q_1 + \ldots q_n,q_{i+n}) \rangle.C$$

Moreover, because $T_n, S_n, L_n$ are subsemi-groups of $G_n$, one can see that $F_2 R^0 = F_2 T^0, F_2 S^0, F_2 L^0$ are ‘$\cdot$’ graded subalgebras of $F_2 G^0$. Hence

**Proposition 1** Let the sequence $(G^0_m)_{m \geq 1}$ of the sets of the extended endofunctions in $\{1, 2, \ldots, m\}$ and $Q^0 \equiv \{q_0, q_1, \ldots\}$ be a noncommutative alphabet. For $\phi \in G^0_m$ let $Q_\phi = \prod_{i=1}^m q_{\phi(i)}$ be the generating monomial of $\phi$ and the graded module $F_2^0 \text{-} module F_2 G^0 = \oplus_{n \geq 0} F_2 G^0_n$ on $F_2 \equiv \{0, 1\}$ generated by all the $\phi$'s.

Then $F_2 G^0$ is a graded algebra for the associative product with unit 1

$$Q_\phi Q_\psi = Q_\psi \prod_{i=1}^m (q_{\phi(i)+n} + \chi_0(i)(q_0 + q_1 + \ldots q_{n-1}))$$

where $\psi \in G^0_m$ and $\chi_0$ is the characteristic function of $\phi^{-1}\{0\}$.

Moreover, if $(R^0_n)_{m \geq 1}$ is a sequence of subsets associated with subsemi-groups of the sequence $(G^0_m)_{m \geq 1}$, then $F_2 R^0 = \oplus_{n \geq 0} F_2 R^0_n$ is a graded subalgebra of $F_2 G^0$.

### 2.4 Splitting Operator $\delta_n \in F_2 G^0$

This operator substitutes the $n$-coproduct $\Delta_n$ of the Leibniz–Lie type. Associate to $A \in Q^0$ the left linear operator $\tau_n A$ acting on $B \in Q^0$, such that, if $A \in G^0_m, B \in G^0_n$, then $[B](\tau_n A) \equiv BA$ if $\deg B = n$, and 0 otherwise, where BA is the concatenation of B and A.

(i) Now let $f \in G_m$ and $H_0$ as in Sect. 3. and note first that if

$$\tau_n f = \begin{cases} f(i) + n, & \text{for } i \in [1, m], \\ f(i), & \text{for } i \notin [1, m]. \end{cases}$$

where $f_0 = 1 - \chi_0$. According to (2), define for $\phi \in G^0_m$

$$\delta_n Q_\phi \equiv \tau_n \prod_{i=1}^m (q_{\phi(i)+n} + \chi_0(i)(q_0 + q_1 + \ldots q_{n-1}))$$

If $d_0 = \text{Card } H_0$ the expansion (4) gives a sum of $(n + 1)^{d_0}$ generating monomials of functions $\psi_{(K)}$ of $[n + 1, n + m]$ into $\{0\} \cup [n + 1, n + m]$, and the corresponding functional graphs factorized in commutative exycles.
The operator $\delta_p A$ is left linear on $F_2 G^0$, and (2) can be written

$$Q_{\phi}, Q_{\psi} = [Q_{\psi}](\delta_{\phi} Q_{\psi})$$

(5)

(ii) Moreover, $\delta_p$ is a graded antimorphism for $\cdot$.

$$\delta_p(A,B) = (\delta_p B)(\delta_{p+n} A)$$

(6)

where $n = \text{deg}B$ and $p \in \mathbb{N}$.

For this to compute with (5) and $A, B, C$ as in Sect. 3, $[C] \delta_p(A,B) = (A,B,C) = [B,C] \delta_p + n A = \{C\} (\delta_p B)(\delta_{p+n} A)$. If $p = 0$ we recover $A.B = [B][\delta_n A]$ and $[B] \delta_n A = 0$ if $k \neq \text{deg}B$.

(iii) Also, $\delta_n$ is a power

$$\delta_n = \delta^n, \delta = \delta_1, \delta^0 = 1$$

(7)

For this to compute, $\delta_p \delta_n A(q_0; q_i) = \delta_p \tau_n A(q_0 + q_1 + \ldots q_n; q_i + \ldots + q_{n+p} + q_{i+n}) = \tau_{n+p} A(q_0 + q_1 + \ldots q_n + q_{i+p} + \ldots q_{i+n}) = \delta_{n+p} A(q_0; q_i)$.

(iv) Define the left linear operator $\mu$ in $F_2 G^0$ by the expansion

$$\mu \equiv \sum_{n \geq 0} \delta^n$$

(8)

By left linear action of $\mu A$ on $F_2 G^0$, we get $A.B = [B](\mu A)$ for $A, B \in F_2 G^0$ with the antimorphism property

$$\mu(A.B) = (\mu B)(\mu A)$$

(9)

which express the associativity of $\cdot$.

**Proposition 2** Let $A \in F_2 G^0_0, B \in F_2 G^0_0$. Then the splitting linear operator $\delta_p$ defined left linearly by $[B] \delta_p A \equiv A.B$ if $p = n$, and 0 otherwise, verifies $\delta_p = \delta^n$ with $\delta = \delta_1, \delta^0 = 1$ and $\delta^n(A,B) = (\delta^n B)(\delta_{n+p} A)$. Moreover, $\mu \equiv \sum_{n \geq 0} \delta^n$ is an antimorphism in $F_2 G^0$ such that $A.B = [B](\mu A)$.

### 2.5 Exponential Generating Function of the Monomials of $L^0$

All the words of $L^0_n$ (i.e. subdiagonals) are obtained from the expansion of $Q_n \equiv q_0(q_0 + q_1) \ldots (q_0 + q_1 + \ldots q_{n-1}) \in F_2 L^0$ and $Q_0 = 1$.

By equation (3), one has $Q_{m+n} = Q_{m+n}, \text{ and if } A \in F_2 L^0_m, B \in F_2 L^0_n, \text{ we have } A.B \in F_2 L^0_{m+n}, \text{ and then }$ we recover that $F_2 L^0$ is stable for the product $\cdot$. Because $Q_1 = q_0$, the associativity gives

$$Q_n = (q_n)^n$$

(10)

With the $Q[[t]]$-modules on $L^0$, one has the exponential generating function

$$\exp \{t q_0\} = \sum_{n \geq 0} \frac{t^n}{n!}(q_0)^n = \sum_{n \geq 0} \frac{t^n}{n!} Q_n$$

(11)

$$\exp \{s q_0\} \exp \{t q_0\} = \exp \{ (s+t) q_0 \}$$
2.6 Examples
Consider equations (4) and (5) for $Q^r = q^n$.

2.6.1 Rooted Trees with $n=1$

\[
\delta_i(q^2_0q_1) = \tau q_0 + q_1 \partial q^2_2 = \tau q_0q_0q_2 + \tau q_0q_1q_2 + \tau q_1q_0q_2 + \tau q_1q_1q_2
\]

(Figure 3)

\[
(q^2_0q_1).q_0 = q_0(q_0 + q_1)^2q_2 = q^3_0q_2 + q^2_0q_1q_2 + q_0q_1q_0q_2 + q_0q_1q_1q_2
\]

(Figure 4)

2.6.2 Excycles with $n=2$

\[
\delta_i(q^3_2q_1q_0) = \tau^2q^2_2q_1q_0 + \tau^2q^3_3q_1q_0 + \tau^2q^3_3q_1q_2
\]

(Figure 5)

\[
q^3_2q_1q_0q_2 = q^3_0q^3_2q_3q_0 + q^3_0q^3_2q_3q_1 + q^3_0q^3_2q_3q_2
\]

(Figure 6)

3 Differential Algebra

3.1 Differential indeterminates
Let $D \equiv \{\delta_0, \delta_1, \ldots\}$, where $\delta_\alpha \equiv \partial/\partial \xi^\alpha$, the $\alpha^h$ canonical derivation in $S \equiv K[[\xi]]$, the algebra of formal power series in $\xi \equiv \{\xi^0, \xi^1, \ldots\}$, where $K$ is a characteristic zero field. If $S^{N \times N}$ is the set $U \equiv \{u_1, u_2, \ldots\} = (u_\beta)_{\beta \geq 1} = (u_\alpha^\beta)_{\alpha \geq 0, \beta \geq 1}$ with $u_\alpha^\beta \in S$, consider $U$ as a set of indeterminates, $u_\alpha^\beta \sigma_1 \ldots \sigma_n = \partial_\sigma_1 \ldots \partial_\sigma_n u_\alpha^\beta$ as differential indeterminates, replace $S^{N \times N}$ by $KU$, and consider the graded differential algebra $K\{U\} = +_{r \geq 0}K\{U\}, D'$.

To each $W \in F_2^K$ we associate the differential operator $W(U) \in \{U, D\}$; for example, with $W_r(U) \in K\{U\}$, one has

\[
W(U) = W_0(U) + W_1(U)^\alpha \partial_\alpha + W_2(U)^{\alpha \beta} \partial_\alpha \partial_\beta + \ldots = W_0(U) + \sum_{r \geq 1} \langle W_r(U), D' \rangle
\]

(12)

We will use now the summation convention.

3.2 Brackets in $K\{U\}$

Define for $u, v, w, \ldots \in U$ the multilinear operations valued in $K\{U\}$.

3.2.1 Arborescent Brackets (Valued in $K\{U\}_1$)

\[
\langle u, v, \ldots \rangle, \langle u, w, \ldots \rangle = \langle u, v \rangle^\beta \equiv u^\alpha v^\beta \alpha, \text{ hence } \langle \langle u, v \rangle, D \rangle = u^\alpha v^\beta \alpha \partial_\beta \text{ (1 fixed point sent to '0')} \langle \langle u, w \rangle \rangle^\gamma \equiv u^\alpha v^\beta w^\gamma \alpha \beta, \text{ hence }
\]

\[
\langle \langle u, v \rangle, D \rangle = u^\alpha v^\beta w^\gamma \alpha \beta \partial_\gamma
\]

(13)

Also, for $A \in K\{U\}_n, B \in K\{U\}_s, \langle A, B \rangle^{\beta_1 \ldots \beta_s} \equiv A^{\alpha_1 \ldots \alpha_n} B^{\beta_1 \ldots \beta_s \alpha_1 \ldots \alpha_n}$.

3.2.2 Circular Brackets (Valued in $K\{U\}_0$)

\[
\langle u \rangle \equiv u^\alpha \alpha \text{ (1 fixed point)}, \langle u \rangle (v) \equiv u^\alpha v^\beta \beta \text{ (2 fixed points)}
\]

\[
\langle u, v \rangle \equiv u^\alpha v^\beta \alpha (2\text{-cycle}), \langle u, v, w \rangle \equiv u^\alpha v^\beta w^\gamma \alpha (3\text{-cycle})
\]

(14)
3.2.3 Mixed Brackets (Valued in $K\{U\}_0$)

Let $E$ be a proper excycle (i.e. with no fixed point); we can write it $E = (A_i, A_{i2}, \ldots A_{ip})$, where the $A_i$’s are arborescences with root $i_k$. If each arborescence $A_i$ is reduced to its root $i_k$, we recover simply a cycle $E = (i_1, i_2, \ldots i_p)$.

Now let $F_i$ be the forest under $i_k$, i.e. obtained by cutting the root of $A_i$, and defined with $F_i(U) \equiv F_i(\{u_j\}; j \in N_i)$, where $N_i$ is the set of nodes of $F_i$:

$$E(U) \equiv (u_1^{(F_i)}, u_2^{(F_i)}, \ldots u_p^{(F_i)}) \equiv (F_1(U), u_1 a_0^{(F_i)}), (F_2(U), a_2^{(F_i)}), \ldots (F_p(U), a_p^{(F_i)})$$

3.3 Action of $F_2^R$

Moreover, $F_2^R$ operates $K$-linearly in $K\{U\}$ with values in $K\{U, D\}$. For this let $\phi \in G_0^i, H_j = \phi^{-1}(0)$ for $j \in [0, m], I = [1, m]$, and $H = u_{1b_1}u_{2b_2}u_{mb_m} \in U^*$, a word on $U$ of length $m$. Then the action is

$$Q_\phi = \prod_{i \in I} q_{0(i)} \rightarrow Q_\phi(H) = \prod_{j \in I} \left( \prod_{i \in I_j} \partial_{\alpha} \right) u_{a_j}^{(F_j)} \prod_{k \in H_0} \partial_{\alpha_k}$$

(15)

The differential monomial $Q_\phi(H)$ is such that $u_{\beta_j}$ is associated with $j$ in the domain $I$ of $\phi$. If $d_j$ is the degree in $q_j$ (in degree of the node labeled by ‘$j$’), then $u_{a_j}^{(F_j)}$ is derived $d_j$ times and the indices of derivation are related to the places of the $q_j$’s in the word. Similarly, the differential operator $D^j$ is characterized by the number $r$ (degree of the root) of the $q_0$’s and their places. So we can summarize: In a word $A \in R^3$ where $q_j$ is at the place $i$, then in $A(H)$ the $j$th letter of $H$ is derived according to $i$, i.e. $\partial_{\alpha_i}$ acts. One has, in particular, taking $H = u_{1}u_{2}, \ldots$:

**Arborescent brackets**

1. $q_0(U) = u_{1} a_1, \partial a_1 = (u_1, D)$
2. $q_0q_0(U) = u_{1} a_1 u_{2} a_2, \partial a_1, \partial a_2 = (u_1 u_2, D^2)$
3. $q_0q_1(U) = u_{1} a_1 u_{2} a_2, \partial a_1 = (u_1 u_2, D)$
4. $q_0q_2q_0(U) = u_{1} a_1 u_{2} a_2 u_{3} a_3, \partial a_1, \partial a_2 = (u_1 u_2, u_3, D^3)$

**Circular brackets**

1. $q_1(U) = u_{1} a_1, a_1 = (u_1)$
2. $q_1q_2(U) = u_{1} a_1 u_{2} a_2, a_2 = (u_1)(u_2)$
3. $q_2q_1(U) = u_{1} a_1 u_{2} a_2, a_2 = (u_1, u_2)$
4. $q_3q_1q_2(U) = u_{1} a_1 u_{2} a_2 a_3, a_3 = (u_1, u_2, u_3)$

3.4 Product of Differential Operators

The product (2) on words with correspondence (15) gives the product of differential operators. We state, without proof,

**Proposition 3** Let the graded differential algebra $K\{U\} = \oplus_{r \geq 0} K\{U\}_r$, and the graded algebra of differential operators $K\{U, D\} = \oplus_{r \geq 0} K\{U\}_{r} D^r$. Let $\phi \in G_0^i, I = [1, m], H_j = \phi^{-1}(j)$ for $j \in [0, m]$ and $H = u_{1b_1}u_{2b_2}u_{mb_m}$ a word on $U$ of length $m$. Then the mapping of $F_2^G$ into $K\{U, D\}$ which associates to the generating monomial $Q_\phi = \prod_{i \in I} q_{0(i)}$ of $\phi$ the differential operator $Q_\phi(H) = \prod_{j \in I} (\prod_{i \in I_j} \partial_{\alpha}) u_{a_j}^{(F_j)} \prod_{k \in H_0} \partial_{\alpha_k}$
is a morphism, such that if $\psi \in G_n^0$ and $K$ is a word on $U$ of length $n$, one has $Q_0(H)Q_\psi(K) = (Q_\psi Q_0)(KH)$, where $KH$ is the concatenation of $K$ and $H$.

**Example**

$$A = q_0, B = q_2q_1q_0, H = u_4, K = u_1u_2u_3,$$

$$AB = q_2q_1q_0(q_0 + q_1 + q_2 + q_3) = q_2q_1q_0q_0 + q_2q_1q_0q_1 + q_2q_1q_0q_2 + q_2q_1q_0q_3$$

(Figure 7)

$$A(H) = \langle u_1, D \rangle, B(K) = \langle u_1, u_2 \rangle \langle u_3, D \rangle,$$

$$A(H)B(K) = \langle u_1, u_2 \rangle \langle u_3u_4, D^2 \rangle + \langle u_4, (u_1, u_2) \rangle \langle u_3, D \rangle + \langle u_1, u_2 \rangle \langle \langle u_4, u_3 \rangle, D \rangle$$

Observe that:

$$\langle u_4, (u_1, u_2) \rangle \langle u_3, D \rangle = u_1 a_1^{a_1} a_2 a_4 u_2^{a_2} a_1 u_3^{a_3} u_4^{a_4} \partial_{a_3} + u_1 a_1^{a_1} a_2 a_4 u_2^{a_2} a_1 a_2 u_3^{a_3} u_4^{a_4} \partial_{a_3}$$

which corresponds to $q_2q_1q_0(q_1 + q_2)$, i.e. the second and third terms in the graph expansion.

**Appendix**

To view Figures 1–7, click here. To return to the main paper, click on the red box.

**References**


