The arithmetic Tutte polynomials of the classical root systems

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Abstract. Many combinatorial and topological invariants of a hyperplane arrangement can be computed in terms of its Tutte polynomial. Similarly, many invariants of a hypertoric arrangement can be computed in terms of its *arithmetic* Tutte polynomial.

We compute the arithmetic Tutte polynomials of the classical root systems A_n, B_n, C_n, and D_n with respect to their integer, root, and weight lattices. We do it in two ways: by introducing a *finite field method* for arithmetic Tutte polynomials, and by enumerating signed graphs with respect to six parameters.

Résumé. De nombreux invariants combinatoires et topologiques d'un arrangement d'hyperplans peuvent être calculées en fonction de son polynôme de Tutte. De même, de nombreux invariants d'un arrangement hypertoric peuvent être calculés en termes de Tutte arithmétique polynomiale. Nous calculons les polynômes de Tutte arithmétiques des systèmes racinaires classiques A_n, B_n, C_n, D_n et par rapport à leur entier, racine, et le treillis de poids. Nous le faisons de deux facons: par l'introduction d'une méthode de champ fini de polynômes de Tutte arithmétiques, et en énumérant graphiques signés à l'égard de six paramètres.

Keywords: Root systems, Toric arrangements, Tutte polynomials, Rogers-Ramanujan function, Signed graphs

1 Introduction

There are numerous constructions in mathematics which associate a combinatorial, algebraic, geometric, or topological object to a list of vectors A. It is often the case that important invariants of those objects (such as their size, dimension, Hilbert series, Betti numbers) can be computed directly from the *matroid* of A, which only keeps track of the linear dependence relations between vectors in A. Sometimes such invariants depend only on the (arithmetic) Tutte polynomial of A, a two-variable polynomial defined below.

It is therefore of great interest to compute (arithmetic) Tutte polynomials of vector configurations. Welsh and Whittle [15], and later the first author [1], gave a finite field method for computing Tutte

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polynomials. In this paper we present an analogous method for computing arithmetic Tutte polynomials, a version of which was first published by Branden and Moci [3]. We cannot expect miracles from this method; computing Tutte polynomials is #P-hard in general and we cannot overcome that difficulty. However, this finite field method is extremely successful when applied to some vector configurations of interest.

Arguably the most important vector configurations in mathematics are the irreducible root systems, which play a fundamental role in many fields. The first author [1] used the finite field method to compute the Tutte polynomial of the classical root systems $\Phi = A_n, B_n, C_n, D_n$.

The main goal of this paper is to compute the **arithmetic** Tutte polynomial of the classical root systems $\Phi = A_n, B_n, C_n, D_n$. In doing so, we obtain combinatorial formulas for various quantities of interest, such as:

The volume and number of (interior) lattice points, of the zonotopes $Z(\Phi)$.

Various invariants associated to the hypertoric arrangement $\mathcal{T}(\Phi)$ in a compact, complex, or finite torus.

The dimension of the Dahmen-Micchelli space $DM(\Phi)$ from numerical analysis.

The dimension of the De Concini-Procesi-Vergne space $DPV(\Phi)$ coming from index theory.

Our formulas are given in terms of the *deformed exponential function*, which is the following evaluation of the three variable Rogers-Ramanujan function:

$$F(\alpha, \beta) = \sum_{n \geq 0} \frac{\alpha^n \, \beta^{\binom{n}{2}}}{n!}.$$

As a corollary, we obtain simple formulas for the characteristic polynomials of the classical root systems. In particular, we discover a surprising connection between the arithmetic characteristic polynomial of the root system A_n and the enumeration of cyclic necklaces.

2 Preliminaries

2.1 Tutte polynomials and hyperplane arrangements

Given a vector configuration A in a vector space V over a field $\mathbb{F}$, the *Tutte polynomial* of A is defined to be

$$T_A(x, y) = \sum_{B \subseteq A} (x-1)^{r(A)-r(B)}(y-1)^{|B|-r(B)}$$

where, for each $B \subseteq X$, the *rank* of B is $r(B) = \dim \operatorname{span} B$. The Tutte polynomial carries a tremendous amount of information about A. Three prototypical theorems are the following.

Let $V^* = \mathrm{Hom}(V, \mathbb{F})$ be the dual space of linear functionals from V to $\mathbb{F}$. Each vector $a \in A$ determines a normal hyperplane

$$H_a = \{x \in V^* \, : \, x(a) = 0\}$$

Let

$$\mathcal{A}(A) = \{H_a \, : \, a \in A\}, \qquad V(A) = V \setminus \bigcup_{H \in \mathcal{A}(A)} H$$

be the *hyperplane arrangement* of A and its complement. There is little harm in thinking of $\mathcal{A}(A)$ as the arrangement of hyperplanes perpendicular to the vectors of A, but the more precise definition will be useful in the next section.

Theorem 2.1 ($\mathbb{F} = \mathbb{R}$) *(Zaslavsky) [16] Let $\mathcal{A}(A)$ be a **real** hyperplane arrangement in $\mathbb{R}^n$. The complement $V(A)$ consists of $|T_A(2,0)|$ regions.*

Theorem 2.2 ($\mathbb{F} = \mathbb{C}$) *(Goresky-MacPherson, Orlik-Solomon) [9, 12] Let $\mathcal{A}(A)$ be a **complex** hyperplane arrangement in $\mathbb{C}^n$. The cohomology ring of the complement $V(A)$ has Poincaré polynomial*

$$\sum_{k \geq 0} \operatorname{rank} H^k(V(A), \mathbb{Z}) q^k = (-1)^r q^{n-r} T_A(1-q, 0).$$

Theorem 2.3 *($\mathbb{F} = \mathbb{F}_q$: Finite field method) (Crapo-Rota, Athanasiadis, Ardila, Welsh-Whittle) [1, 2, 5, 15] Let $\mathcal{A}(A)$ be a hyperplane arrangement in $\mathbb{F}_q^n$ where $\mathbb{F}_q$ is the finite field of q elements for a prime power q. Then the complement $V(A)$ has size*

$$|V(A)| = (-1)^r q^{n-r} T_A(1-q, 0)$$

and, furthermore,

$$\sum_{p \in \mathbb{F}_q^n} t^{h(p)} = (t-1)^r q^{n-r} T_A\left(\frac{q+t-1}{t-1}, t\right)$$

where $h(p)$ is the number of hyperplanes of $\mathcal{A}(A)$ that p lies on.

2.2 Arithmetic Tutte polynomials and hypertoric arrangements

If our vector configuration A lives **in a lattice** Λ, then the *arithmetic Tutte polynomial* is

$$M_A(x,y) = \sum_{B \subseteq A} m(B)(x-1)^{r(A)-r(B)}(y-1)^{|B|-r(B)}$$

where, for each $B \subseteq A$, the *multiplicity* $m(B)$ of B is the index of $\mathbb{Z}B$ as a sublattice of $\operatorname{span} B \cap \Lambda$. The arithmetic Tutte polynomial also carries a great amount of information about A, but it does so in the context of *toric arrangements*.

Definition 2.4 *Let $T = Hom(\Lambda, \mathbb{F}^*)$ be the character group, consisting of the group homomorphisms from Λ to the multiplicative group $\mathbb{F}^* = \mathbb{F} \backslash \{0\}$ of the field $\mathbb{F}$. We might also consider the unitary characters $T = Hom(\Lambda, \mathbb{S}^1)$ where $\mathbb{S}^1$ is the unit circle in $\mathbb{C}$. It is easy to check that T is isomorphic to $\mathbb{F}^*$ and to $\mathbb{S}^1$, respectively.*

Each element $a \in A$ determines a hypertorus

$$T_a = \{t \in T \,:\, t(a) = 1\}$$

in T. For instance $a = (2, -3, 5)$ gives the hypertorus $x^2 y^{-3} z^5 = 1$. Let

$$\mathcal{T}(A) = \{T_a \,:\, a \in A\}, \qquad R(A) = T \setminus \bigcup_{T \in \mathcal{T}(A)} T$$

be the toric arrangement *of A and its complement, respectively.*

Theorem 2.5 ($\mathbb{F}^* = \mathbb{S}^1$) *(Moci) Let $\mathcal{T}(A)$ be a **real** toric arrangement in the compact torus $T \cong (\mathbb{S}^1)^n$. Then $\mathcal{R}(A)$ consists of $|M_A(1,0)|$ regions.*

Theorem 2.6 ($\mathbb{F}^* = \mathbb{C}^*$) *(Moci) Let $\mathcal{T}(A)$ be a **complex** toric arrangement in the torus $T \cong (\mathbb{C}^*)^n$. The cohomology ring of the complement $\mathcal{R}(A)$ has Poincaré polynomial*

$$\sum_{k\geq 0} \operatorname{rank} H^k(\mathcal{R}(A), \mathbb{Z})q^k = q^n M_A\left(\frac{2q+1}{q}, 0\right)$$

For finite fields we prove the following result (also proved independently by Brandren and Moci [3]), which is also one of our main tools for computing arithmetic Tutte polynomials.

Theorem 2.7 *($\mathbb{F}^* = \mathbb{F}_{q+1}^*$: Finite field method) (Branden-Moci, Ardila-Castillo-Henley)Let $\mathcal{T}(A)$ be a toric arrangement in the torus $T \cong (\mathbb{F}_{q+1}^*)^n$ where $\mathbb{F}_{q+1}$ is the finite field of $q+1$ elements for a prime power $q+1$. Assume that $m(B)|q$ for all $B \subseteq A$. Then the complement $\mathcal{R}(A)$ has size*

$$|\mathcal{R}(A)| = (-1)^r q^{n-r} M_A(1-q, 0)$$

and, furthermore,

$$\sum_{p\in T} t^{h(p)} = (t-1)^r q^{n-r} M_A\left(\frac{q+t-1}{t-1}, t\right)$$

where $h(p)$ is the number of hypertori of $\mathcal{T}(A)$ that p lies on.

The second statement of Theorem 4.1 is significantly stronger than the first because it involves two different parameters; so if we are able to compute the left hand side, we will have computed the whole arithmetic Tutte polynomial. For that reason, we regard this as a *finite field method* for arithmetic Tutte polynomials.

There are several other reasons to care about the arithmetic Tutte polynomial of A; we refer the reader to the references for the relevant definitions.

Theorem 2.8 *Let A be a vector configuration in a lattice Λ.*

- *The volume of the zonotope $Z(A)$ is $M_A(1,1)$. [13].*
- *The Ehrhart polynomial of the zonotope $Z(A)$ is $q^n M(1+\frac{1}{q}, 1)$. [13, 11]*
- *The dimension of the Dahmen-Micchelli space $DM(A)$ is $M_A(1,1)$. [6, 10]*
- *The dimension of the De Concini-Procesi-Vergne space $DPV(A)$ is $M_A(2,1)$. [6, 4]*

2.3 Root systems and lattices

We will pay special attention to the four infinite families of finite root systems, known as the *classical root systems*:

$$\begin{aligned}
A_{n-1} &= \{e_i - e_j, : 1 \leq i < j \leq n\} \\
B_n &= \{e_i - e_j, e_i + e_j \,:\, 1 \leq i < j \leq n\} \cup \{e_i \,:\, 1 \leq i \leq n\} \\
C_n &= \{e_i - e_j, e_i + e_j \,:\, 1 \leq i < j \leq n\} \cup \{2e_i \,:\, 1 \leq i \leq n\} \\
D_n &= \{e_i - e_j, e_i + e_j \,:\, 1 \leq i < j \leq n\}
\end{aligned}$$

The arithmetic Tutte polynomial of a vector configuration A depends on the lattice where A lives. For a root system Φ in $\mathbb{R}^v$ there are at least three natural choices: the integer lattice $\mathbb{Z}^v$, the weight lattice Λ_W, and the root lattice Λ_R. The second is the lattice generated by the roots, while the third is the lattice generated by the fundamental weights.

3 Results

We give explicit formulas for the arithmetic Tutte polynomials of the classical root systems. Our results are most cleanly expressed in terms of the *(arithmetic) coboundary polynomial*, which is the following simple transformation of the (arithmetic) Tutte polynomial:

$$\overline{\chi}_{\mathcal{A}}(X,Y) = (y-1)^{r(\mathcal{A})} T_{\mathcal{A}}(x,y), \qquad \psi_{\mathcal{A}}(X,Y) = (y-1)^{r(\mathcal{A})} M_{\mathcal{A}}(x,y)$$

where

$$x = \frac{X+Y-1}{Y-1}, \quad y = Y, \qquad \text{and} \qquad X = (x-1)(y-1), \quad Y = y.$$

Clearly, the (arithmetic) Tutte polynomial can be recovered readily from the (arithmetic) coboundary polynomial. Throughout the paper, we will continue to use the variables X, Y for coboundary polynomials and x, y for Tutte polynomials.

Our formulas are conveniently expressed in terms of the exponential generating functions for the coboundary polynomials:

Definition 3.1 *For the infinite families $\Phi = B, C, D$, of classical root systems, let the* Tutte generating function *and the* arithmetic Tutte generating function[(i)] *be*

$$\overline{X}_{\Phi}(X,Y,Z) = \sum_{n\geq 0} \overline{\chi}_{\Phi_n}(X,Y)\frac{Z^n}{n!}, \qquad \Psi_{\Phi}(X,Y,Z) = \sum_{n\geq 0} \psi_{\Phi_n}(X,Y)\frac{Z^n}{n!},$$

respectively; and for $\Phi = A$ let them be

$$\overline{X}_{A}(X,Y,Z) = 1 + X\sum_{n\geq 1} \overline{\chi}_{A_{n-1}}(X,Y)\frac{Z^n}{n!}, \quad \Psi_{A}(X,Y,Z) = 1 + X\sum_{n\geq 1} \psi_{A_{n-1}}(X,Y)\frac{Z^n}{n!}.$$

For $\Phi = A$ we need the extra factor of X, since the root system A_{n-1} is of rank $n-1$ inside $\mathbb{Z}^n$.

Our formulas are given in terms of the following functions:

Definition 3.2 *Let the* three variable Rogers-Ramanujan function *be*

$$\widetilde{R}(\alpha,\beta,q) = \sum_{n\geq 0} \frac{\alpha^n \beta^{\binom{n}{2}}}{(1+q)(1+q+q^2)\cdots(1+q+\cdots+q^{n-1})}$$

and the deformed exponential function *be*

$$F(\alpha,\beta) = \sum_{n\geq 0} \frac{\alpha^n \beta^{\binom{n}{2}}}{n!} = \widetilde{R}(\alpha,\beta,1).$$

(i) It might be more accurate to call it the *arithmetic coboundary generating function*, but we prefer this name because the Tutte polynomial is much more commonly used than the coboundary polynomial.

We denote the arithmetic Tutte generating functions of the root systems with respect to the integer, weight, and root lattices by Ψ_Φ, Ψ_Φ^W, and Ψ_Φ^R, respectively. Tutte (in Type A) and the first author (in types A, B, C, D) computed the ordinary Tutte generating functions for the classical root systems:

Theorem 3.3 *[1] The Tutte generating functions of the classical root systems are*

$$\begin{aligned} X_A &= F(Z,Y)^X \\ X_B &= F(2Z,Y)^{(X-1)/2} F(YZ,Y^2) \\ X_C &= F(2Z,Y)^{(X-1)/2} F(YZ,Y^2) \\ X_D &= F(2Z,Y)^{(X-1)/2} F(Z,Y^2) \end{aligned}$$

In this paper we compute the **arithmetic** Tutte polynomials of the classical root systems. Our main results are the following:

Theorem 3.4 *The arithmetic Tutte generating functions of the classical root systems* ***in their integer lattices*** *are*

$$\begin{aligned} \Psi_A &= F(Z,Y)^X \\ \Psi_B &= F(2Z,Y)^{\frac{X}{2}-1} F(Z,Y^2) F(YZ,Y^2) \\ \Psi_C &= F(2Z,Y)^{\frac{X}{2}-1} F(YZ,Y^2)^2 \\ \Psi_D &= F(2Z,Y)^{\frac{X}{2}-1} F(Z,Y^2)^2 \end{aligned}$$

Theorem 3.5 *The arithmetic Tutte generating functions of the classical root systems* ***in their root lattices*** *are*

$$\begin{aligned} \Psi_A^R &= F(Z,Y)^X \\ \Psi_B^R &= F(2Z,Y)^{\frac{X}{2}-1} F(Z,Y^2) F(YZ,Y^2) \\ \Psi_C^R &= \frac{1}{2} F(2Z,Y)^{\frac{X}{2}-1} \left[F(2Z,Y) + F(YZ,Y^2)^2 \right] \\ \Psi_D^R &= \frac{1}{2} F(2Z,Y)^{\frac{X}{2}-1} \left[F(2Z,Y) + F(Z,Y^2)^2 \right] \end{aligned}$$

Theorem 3.6 *The arithmetic Tutte generating functions of the classical root systems* ***in their weight lattices*** *are*

$$\Psi_A^W = \sum_{n \in \mathbb{N}} \varphi(n) \left(\left[F(Z,Y) F(\omega_n Z, Y) F(\omega_n^2 Z, Y) \cdots F(\omega_n^{n-1} Z, Y) \right]^{X/n} - 1 \right)$$

where $\varphi(n) = \#\{m \in \mathbb{N} \,:\, 1 \leq m \leq n,\ (m,n) = 1\}$ *is Euler's totient function and* ω_n *is a primitive* n*th root of unity for each* n*,*

$$\begin{aligned} \Psi_B^W &= F(2Z,Y)^{\frac{X}{4}-1} F(Z,Y^2) F(YZ,Y^2) \left[F(2Z,Y)^{\frac{X}{4}} + F(-2Z,Y)^{\frac{X}{4}} \right] \\ \Psi_C^W &= F(2Z,Y)^{\frac{X}{2}-1} F(YZ,Y^2)^2 \\ \Psi_D^W &= F(2Z,Y)^{\frac{X}{4}-1} F(Z,Y^2)^2 \left[F(2Z,Y)^{\frac{X}{4}} + F(-2Z,Y)^{\frac{X}{4}} \right] \end{aligned}$$

This also allows us to give formulas for the respective arithmetic characteristic polynomials

$$\chi_A^\Lambda(q) = (-1)^r q^{n-r} M_A^\Lambda(1-q, 0).$$

The following formula is quite different from the classical one.

Theorem 3.7 *The arithmetic characteristic polynomials of the root systems A_{n-1}* ***in their weight lattices*** *are given by*

$$\chi^W_{A_{n-1}}(q) = \frac{n!}{q} \sum_{m|n} (-1)^{n-\frac{n}{m}} \varphi(m) \binom{q/m}{n/m}$$

In particular, when $n \geq 3$ is prime,

$$\chi^W_{A_{n-1}}(q) = (q-1)(q-2)\cdots(q-n+1) + (n-1)(n-1)!.$$

When n is odd and $n|q$ we obtain an intriguing combinatorial interpretation:

Theorem 3.8 *If n, q are integers with n odd and $n|q$, then $\chi^W_{A_{n-1}}(q)/n!$ equals the number of cyclic necklaces with n black beads and $q-n$ white beads.*

3.1 Comparing the two methods.

For all but one of the formulas above, we will give one "finite field" proof and one "graph enumeration" proof. Each method has its advantages. When the underlying lattice is $\mathbb{Z}^n$, the finite field method seems preferrable, as it gives more straightforward proofs than the graph enumeration method. However, this is no longer the case with more complicated lattices. In particular, we only have one proof for the formula for Ψ^W_A, using graph enumeration. There should also be a "finite field method" proof for this result, but it seems more difficult and less natural.

4 Computing Tutte polynomials using the finite field method

Recall that the *arithmetic Tutte polynomial* of a vector configuration $A \subseteq \Lambda$ is

$$M_A(x, y) = \sum_{B \subseteq A} m(B)(x-1)^{r(A)-r(B)}(y-1)^{|B|-r(B)}$$

where, for each $B \subseteq A$, $m(B)$ is the index of $\mathbb{Z}B$ as a sublattice of span$B \cap \Lambda$.

4.1 Statement of the theorem

We start by restating Theorem 4.1 more explicitly. We now omit the first statement in the previous formulation, which follows from the second one by setting $t = 0$.

Theorem 4.1 *($\mathbb{F}^* = \mathbb{F}^*_{q+1}$: Finite field method) Let A be a collection of vectors in a lattice Λ of rank n. Let $q+1$ be a prime power such that $m(B)|q$ for all $B \subseteq A$ and consider the torus $T = Hom(\Lambda, \mathbb{F}^*_{q+1}) \cong (\mathbb{F}^*_{q+1})^n$. Let $\mathcal{T}(A)$ be the corresponding arrangement of hypertori in T. Then*

$$\sum_{p \in T} t^{h(p)} = (t-1)^r q^{n-r} M_A\left(\frac{q+t-1}{t-1}, t\right)$$

where $h(p)$ is the number of hypertori of $\mathcal{T}(A)$ that p lies on.

Remark: The finite field method is cleanly expressed in terms of the arithmetic coboundary polynomial:

$$\sum_{p \in (\mathbb{F}_{X+1}^*)^n} Y^{h(p)} = X^{n-r}\psi(X,Y)$$

whenever $X+1$ is a prime power with $m(B)|X$ for all $B \subseteq A$.

A key ingredient of the proof is the following observation, interesting on its own:

Lemma 4.2 *For any $B \subseteq \Lambda$ and any q such that $m(B)|q$, we have $\left|\bigcap_{b\in B} T_b\right| = m(B)q^{n-rk(B)}$, where $T_b = \{t \in T \,:\, t(b) = 1\}$ is the hypertorus associated to b.*

4.2 Using the theorem

Let's be see more detail on how root systems define toric arrangements. For concreteness we set $\mathbb{F} = \mathbb{C}$. We begin with the easiest lattice: $\mathbb{Z}^n$. The torus is

$$\text{Hom}(\mathbb{Z}^n, \mathbb{C}^*) \cong (\mathbb{C}^*)^n$$

Any element in $\mathbb{Z}^n$ acts on the torus by evaluation and the hypertorus it defines its just its kernel. For example if $(1,-1,0,0,\cdots) = e_1 - e_2 \in \mathbb{Z}^n$ the corresponding hypertorus is the solution in $(\mathbb{C}^*)^n$ to the equation $x_1x_2^{-1} = 1$ where x_i are the coordinates of $\mathbb{C}^n$. All lattices are isomorphic to $\mathbb{Z}^n$ once you fix a basis, but for our applications we won't have explicit bases, that's why we consider abstract lattices. Nonetheless expressing some relation with $\mathbb{Z}^n$ is useful, as seen in the following example.

Theorem 4.3 *The arithmetic Tutte generating functions of the classical root system of type B* ***in its weight lattice*** *is*

$$\Psi_B^R \quad = \quad F(2Z,Y)^{\frac{X}{2}-1}F(Z,Y^2)F(YZ,Y^2)$$

Proof: *Sketch*
First of all some arithmetic considerations. By the statement of 4.1, we need to consider X such that $X+1$ is prime and $m(B)$ divides X for any subset B of roots. The multiplicities of the subsets of the complete set of roots B_n, will always divide the multiplicity of the whole set B_n, which is a power of two, i.e. 2^k for some k . So we need to focus our attention on X such that $X+1 \equiv 1 \mod 2^k$ is a prime. Dirichlet's theorem [7] says there are infinitely many such X's. Notice that since $k \geq 2$ we have $X+1 \equiv 1 \mod 4$, so -1 has a square root mod $X+1$, a fact that is crucial to the computations of proof.

Consider the lattice Λ_W in this case

$$\mathbb{Z}^n = \mathbb{Z}\{e_1,\ldots,e_n\} \subset \Lambda_W = \mathbb{Z}\{e_1,\ldots,e_n,(e_1+\cdots+e_n)/2\}$$

and the corresponding tori

$$T_{\mathbb{Z}} = \text{Hom}(\mathbb{Z}^n, \mathbb{F}_{X+1}^*), \qquad T_W = \text{Hom}(\Lambda_W, \mathbb{F}_{X+1}^*).$$

In order to use Theorem 4.1, we are interested in computing

$$\sum_{p\in T_W} Y^{h(p)}$$

But we can compute a simpler quantity. The inclusion $i : \mathbb{Z}^n \to \Lambda_W$ gives a map $i^* : T_W \to T_{\mathbb{Z}}$. Clearly,

$$\mathrm{Im}\, i^* = T^0_W := \{f \in T_{\mathbb{Z}} \,:\, f(e_1)f(e_2)\cdots f(e_n) \text{ is a square.}\}$$

and every $f \in T^0_W$ with $f(e_1)\cdots f(e_n) = x^2$ is the image of exactly two maps $f^{\pm 1}$. Therefore

$$\psi^W_{B_n}(X,Y) = 2\sum_{p\in T^0_{\mathbb{Z}}} Y^{h(p)}$$

Note that $T^0_{\mathbb{Z}}$ can be thought as the subset of $(\mathbb{F}^*)^n$ where the product of all the coordinates is a square. If $\mathbb{F} = \mathbb{C}$ this won't be interesting, but using finite fields give a proper subset, since not every element has a square root.

Choose $a_1 = 1, a_2 = -1, a_3, \dots, a_{X/4+1}, a_{X/4+2}, \dots a_{X/2}$ such that for any $a \in \mathbb{F}^*_{X+1}$, there exists k such that $a = a_k$ or $a = a_k^{-1}$. Furthermore, since X is a multiple of 4, -1 is a square, and we can assume that $a_1, a_2, \dots, a_{X/4} + 1$ and their inverses are squares, while $a_{X/4} + 2, \dots a_{X/2}$ and their inverses are non-squares.

Again, for each $p \in (\mathbb{F}^*_{X+1})^n$, let $P_k = \{j \in [n] \mid\ p(e_j) = a_k \ \text{ or }\ p(e_j) = a_k^{-1}\}$ for $k = 1, 2, \dots, X/2+1$. We still have that

$$h(p) = |P_1|^2 + |P_2|(|P_2| - 1) + \binom{|P_3|}{2} + \cdots + \binom{|P_{X/2+1}|}{2}.$$

Also, for each partition $[n] = P_1 \sqcup \cdots \sqcup P_{X/2+1}$ there are $2^{|P_3|+\cdots+|P_{X/2+1}|}$ points p assigned to it.

However, we are now interested only in those p such that $p(e_1)\cdots p(e_n)$ is a square. Since the product of non-squares is a square, this holds if and only $|P_{X/4+2}| + \cdots + |P_{X/2} + 1|$ is even. Therefore $\psi^W_{B_n}(X,Y)/2$ equals

$$\sum_{\substack{[n]=P_1\sqcup P_2\sqcup\cdots\sqcup P_{X/2+1}\\ |P_{X/4+2}|+\cdots+|P_{X/2}+1| \text{ is even.}}} Y^{|P_1|^2} Y^{|P_2|(|P_2|-1)} 2^{|P_3|} Y^{\binom{|P_3|}{2}} \cdots 2^{|P_{X/2+1}|} Y^{\binom{|P_{X/2+1}|}{2}}.$$

Now, we use the computational formula and some algebraic manipulation, we can pick out only the "even" terms in Ψ_B by means of the following generating function:

$$\begin{aligned} \frac{\Psi^W_B}{2} &= \left(\sum_{n\geq 0} Y^{n^2}\frac{Z^n}{n!}\right)\cdot\left(\sum_{n\geq 0} Y^{n(n-1)}\frac{Z^n}{n!}\right)\cdot\left(\sum_{n\geq 0} 2^n\, Y^{\binom{n}{2}}\frac{Z^n}{n!}\right)^{X/4-1} \cdot \\ &\quad \cdot\frac{1}{2}\left[\left(\sum_{n\geq 0} 2^n\, Y^{\binom{n}{2}}\frac{Z^n}{n!}\right)^{X/4} + \left(\sum_{n\geq 0} (-1)^n\, 2^n\, Y^{\binom{n}{2}}\frac{Z^n}{n!}\right)^{X/4}\right]. \end{aligned}$$

In the last factor, by introducing minus signs appropriately, we are eliminating the odd terms (which do not contribute to Ψ^W_B) and doubling the even terms (which do contribute). Dividing by 2, we get the desired formula. □

This is the general approach that we use in other cases, but some lattices are more complicated.

5 Computing arithmetic Tutte polynomials by counting graphs

The key observation is that these computations are closely related to the enumeration of (signed) graphs. To find the arithmetic Tutte polynomials of A_n, B_n, C_n, and D_n with respect to the various lattices of interest, it becomes necessary to count (unsigned / signed) graphs according to (three / six) different parameters. We do this by proving a very general six-parameter generating function.

5.1 Statement of the theorem

Our goal in this section is to compute the *master generating function* for signed graphs.

Definition 5.1 *[18] A* signed graph *is a set of vertices, together with a set of positive edges, negative edges, and loops connecting them. A* positive edge *(resp.* negative edge*) is an edge between two vertices, labelled with a* $+$ *(*resp. *with a* $-$*). A* loop *connects a vertex to itself; we regard it as a negative edge. There is at most one positive and one negative edge connecting a pair of vertices, and there is at most one loop connecting a vertex to itself.*

Definition 5.2 *A signed graph G is* connected *if and only if its underlying graph $\overline{G}$ (ignoring signs) is connected. The* connected components *of G correspond to those of $\overline{G}$. A* cycle *in G corresponds to a cycle of $\overline{G}$; we call it* balanced *if it contains an even number of negative edges, and* unbalanced *otherwise. We say that G is* balanced *if all its cycles are balanced.*

Definition 5.3 *Let $s(c_+, c_-, c_0, l, e, v)$ be the number of signed graphs with c_+ balanced components, c_- unbalanced components with no loops, c_0 components with loops (which are necessarily unbalanced), l loops, e (non-loop) edges, and v vertices. Let*

$$S(t_+, t_-, t_0, x, y, z) = \sum s(c_+, c_-, c_0, l, e, v)\, t_+^{c_+}\, t_-^{c_-}\, t_0^{c_0}\, x^l y^e \frac{z^v}{v!}$$

be the master generating function *for signed graphs.*

The main theorem is

Theorem 5.4 *The master generating function for signed graphs is*

$$S(t_+, t_-, t_0, x, y, z) = F(2z, 1+y)^{\frac{1}{2}(t_+ - t_-)} F(z, (1+y)^2)^{t_- - t_0} F((1+x)z, (1+y)^2)^{t_0}$$

where $F(\alpha, \beta)$ is the deformed exponential function of Definition 3.2.

5.2 Using the theorem

The first step is to relate signed graphs with root systems. We'll sketch the case of type B in its weight lattice. To each signed graph G associate the following list of vectors

$$\begin{aligned} B_G &= \{e_i - e_j \,:\, ij \text{ is a positive edge of } G,\ i<j\} \\ &\cup\ \{e_i + e_j \,:\, ij \text{ is a negative edge of } G\} \\ &\cup\ \{e_i \,:\, i \text{ is a loop of } v\} \subseteq B_v, \end{aligned}$$

The next step is to be able to read the multiplicities form the information of the graph:

Lemma 5.5 $(\mathbf{B_v})$*: For any signed graph G with $c_-(G)$ unbalanced loopless components,*

$$m_W(B_G) = 2^{c_-(G)}$$

if G has odd balanced components, and

$$m_W(B_G) = 2^{c_-(G)+1}$$

otherwise

And finally to use the master generating function

Proof *sketch*: Let a *nobc graph* be a signed graph with no odd balanced components. By Lemma 5.5 we have

$$M^W_{B_v}(x,y) = \sum_{G \text{ signed}} 2^{c_-}(x-1)^{c_+}(y-1)^{l+e-v+c_+} + \sum_{G \text{ nobc}} 2^{c_-}(x-1)^{c_+}(y-1)^{l+e-v+c_+}$$

so $\sum_{v\geq 0} M^W_{B_v}(x,y)\frac{z^v}{v!}$ equals

$$S(X,2,1,Y-1,Y-1,Z) + Q(X,2,1,Y-1,Y-1,Z)$$

where Q enumerates nobc graphs. We know S from Theorem 5.4 and to compute Q consider

$$CQ_+ = \sum_G y^e \frac{z^v}{n!}$$

Where the sum is over all balanced signed graphs with an even number of connected components

Notice that

$$2CQ_+(y,z) = CS_+(y,z) + CS_+(y,-z)$$

where CS_+ is the generating function for connected balanced signed graphs. Also denote by CS_- the generating function of connected unbalanced signed graphs with no loops and CS_0 the generating function of connected unbalanced signed graphs with loops. In the course of the proof on Theorem 5.4 we prove

$$\begin{aligned} CS_+(y,z) &= \frac{1}{2}\log F(2z,1+y) \\ CS_-(y,z) &= \log F(z,(1+y)^2) - \frac{1}{2}\log F(2z,1+y) \\ CS_0(x,y,z) &= \log F((1+x)z,(1+y)^2) - \log F(z,(1+y)^2) \end{aligned}$$

For the first one note that it suffices to find a formula for B, the generating function of balanced signed graphs, and then by the compositional formula $B = e^{CS_+}$ or $\log B = CS_+$. One more application of the compositional formula gives

$$Q = e^{t_+ CQ_+ + t_- CS_- + t_0 CS_0}$$

Putting all the equations together we get the desired result. □

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