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Bott-Samelson Varieties, Subword Complexes and Brick Polytopes

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Abstract. Bott-Samelson varieties factor the flag variety $G/B$ into a product of $\mathbb{CP}^1$'s with a map into $G/B$. These varieties are mostly studied in the case in which the map into $G/B$ is birational; however in this paper we study fibers of this map when it is not birational. We will see that in some cases this fiber is a toric variety. In order to do so we use the moment map of a Bott-Samelson variety to translate this problem into a purely combinatorial one in terms of a subword complex. These simplicial complexes, defined by Knutson and Miller, encode a lot of information about reduced words in a Coxeter system. Pilaud and Stump realized certain subword complexes as the dual to the boundary of a polytope which generalizes the brick polytope defined by Pilaud and Santos. For a nice family of words, the brick polytope is the generalized associahedron realized by Hohlweg and Lange. These stories connect in a nice way: the moment polytope of a fiber of the Bott-Samelson map is the Brick polytope. In particular, we give a nice description of the toric variety of the associahedron.

Keywords: Bott-Samelson variety, Subword complex, Brick polytope, Moment polytope, Toric variety, Associahedra

1 Introduction

The Bott-Samelson varieties were first defined in Bott and Samelson (1955). In Bott and Samelson (1961) the authors gave a desingularization of Schubert varieties in $G/B$ using these varieties. Demazure used...
them to analyze $H^0(G/B, C)$ and the projective coordinate ring $\mathbb{C}[G/B]$. Intuitively, Bott-Samelson varieties factor $G/B$ into a product of $\mathbb{CP}^1$'s via a map into $G/B$. These varieties are mostly studied in the case in which the map into $G/B$ is birational.

In this paper we will study the fibers of this map when it is not birational. We will show that in some cases this fiber is a toric variety. In order to do so we will translate this problem into a purely combinatorial one in terms of subword complexes. These simplicial complexes $\Delta(Q, w)$ are parametrized by a word $Q$ in the generators of the Weyl group $W$ of $G$ and an element $w \in W$. They were defined by Allen Knutson and Ezra Miller in [Knutson and Miller 2004] to describe the geometry of determinantal ideals and Schubert polynomials. In Pilaud and Stump (2011), the authors define the brick polytope and realize certain subword complexes as the boundary of a polytope dual to the brick polytope. In Ceballos et al. (2011) Ceballos, Labbè and Stump show that for a nice family of words, the brick polytope is the cluster polytope and for the Weyl group of type A it is a associahedron. In Theorem 4.4 we prove that for the words Pilaud and Stump call realizable, a fiber of the Bott-Samelson map is the toric variety of the Brick polytope. We then get a great description of the toric variety of the associahedron in terms of flags arranged in a poset.

Actually the toric case is just a shadow of a more general situation. We will see in Theorem 4.3 that for any word $Q$ and element $w \in W$ the brick polytope is the moment polytope of a fiber of the Bott-Samelson variety. This motivates us to define the Brick manifold as the fiber we study in this paper. This paper shows a very nice connection between three combinatorial objects and solves the toric variety case for Brick manifolds. However, there is much structure here that can be discovered.

2 Some Definitions

2.1 Subword complexes

Let $W$ be the Weyl group of a complex Lie group, i.e., $W$ is a crystallographic Coxeter group and let $S = \{s_i : i \in I\}$ denote its generators.

Notation: Let $Q = (q_1, \ldots, q_k)$ be a word in $S$ and let $P = (q_{i_1}, \ldots, q_{i_k})$ be a subword of $Q$, we denote $\prod^b P := \prod_{j=1}^b q_{i_j}$ and $\prod P := \prod_{j=1}^k q_{i_j}$.

**Definition 2.1** Let $Q = (q_1, \ldots, q_k)$ be a word in $S$ and $w \in W$. The subword complex $\Delta(Q, w)$ is the simplicial complex whose facets are the subwords $F$ of $Q$ such that the product $\prod Q \setminus F$ is a reduced expression for $w$. The cofacets of $\Delta(Q, w)$ are the complements of the facets.

**Definition 2.2** We define the Demazure product of a word $Q$ inductively as follows:

- $\text{Dem}(\text{empty word}) = \text{id}$
- $\text{Dem}(Q \cup s) = \begin{cases} \text{Dem}(Q) \cdot s & \text{if } \ell(\text{Dem}(Q)s) > \ell(\text{Dem}(Q)) \\ \text{Dem}(Q) & \text{if } \ell(\text{Dem}(Q)s) < \ell(\text{Dem}(Q)) \end{cases}$

In Knutson and Miller (2004) the authors prove that $\Delta(Q, w)$ is a sphere if and only if $\text{Dem}(Q) = w$; in this paper we only consider such pairs.
2.2 Brick polytopes

Let $\Delta(W) := \{\alpha_s : s \in S\}$ be the simple roots of $W$ and let $\nabla(W) := \{\lambda_s : s \in S\}$ be its fundamental weights. Given a subword complex $\Delta(Q, w)$ with $|Q| = m$ define the root function $r(I, \cdot) : \{\text{subwords of } Q\} \to \Delta(W)$ and the weight function $w(I, \cdot) : \{\text{subwords of } Q\} \to \nabla(W)$ by

$$r(I, k) := (\prod_{Q \setminus I} (\alpha_{q_k}))$$
$$w(I, k) := (\prod_{Q \setminus I} (\lambda_{q_k})).$$

(1)

**Definition 2.3** The brick vector of a face $F$ of $\Delta(Q, w)$ is defined by

$$B(F) = \sum_{k \in [m]} w(F, k),$$

and the brick polytope is the convex hull of the brick vectors of some faces of $\Delta(Q, w)$

$$B(Q, w) := \text{conv}(B(F) : F \in \Delta(Q, w) \text{ and } \prod_{Q \setminus F} w = w).$$

The following definition is referred to as realizing or root independent in [Pilaud and Stump (2011)]. We will call it toric due to one of the main theorems of this paper which states that the brick manifold of a word $Q$ is toric if and only if $Q$ is root independent.

**Definition 2.4** A word $Q$ is toric if for some vertex $B(F)$ of $B(Q, w)$ (or all vertices) we have that the multiset $r(F) := \{|r(F, i) : i \in F\}$ is linearly independent.

3 Brick Manifolds for $SL_n(\mathbb{C})$

We start with the case $G = SL_n(\mathbb{C})$ both because it’s nice combinatorial pictures and as a motivation to the general complex semi simple Lie group case.

3.1 **Definition of Bott-Samelson varieties for $SL_n(\mathbb{C})$**

Let $G = SL_n(\mathbb{C})$ and fix an ordered basis for $\mathbb{C}^n$. Let $B$ be the subgroup $SL_n(\mathbb{C})$ consisting of upper triangular matrices with respect to this basis. We then get an ascending flag of vector spaces

$$\langle e_1 \rangle \subset \cdots \subset \langle e_1, \ldots, e_n \rangle.$$

Let $T$ be the maximal torus contained in $B$, then $T$ consists of all diagonal matrices. The matrices in the minimal parabolic subgroups $P_i$ are almost upper triangular, except they have a possibly nonzero entry at the position $(i + 1, i)$. The quotient $G/B$ is the flag variety, that is, the space of flags $\{0\} \subset V_1 \subset \cdots \subset V_n = \mathbb{C}^n$ where each $V_i$ is an $i$-dimensional vector space. Moreover, the Weyl group of $G$ is $W = A_{n-1}$.

We begin the definition of $BS^Q$ by an example

**Example 3.1** Let $G = SL_3(\mathbb{C})$ and $Q = (s_1, s_2, s_1, s_2, s_1)$, then the Bott-Samelson variety $BS^Q$ is constructed iteratively by reading the word from left to right: if $k$-th letter of $Q$ is $s_i$, we have an $i$-th dimensional vector space $V_k$ such that $V_{k-1} \subset V_k \subset V_{k+1}$. In this example we have that

$$BS^Q = \{(V_1, V_2, V_3, V_4, V_5) : \text{each ascending chain in the poset below is a complete flag}\}$$
More generally, if \( Q = (q_1, \ldots, q_m) \) then \( BS^Q \) is a list of \( m + 1 \) flags where the first one is the base flag and such that each one agrees with the previous one except possibly in one subspace. We can give a point in \( BS^Q \) by giving subspaces (\( V_1, \ldots, V_m \)) such that the incidence relations given by the flags hold. The map \( BS^Q \rightarrow G/B \) maps the list to the last flag.

**Example 3.2** Continuing with the previous example, we have that \( m_Q : BS^{(s_1, s_2, s_1, s_2, s_1)} \rightarrow G/B \) is the map

\[
\begin{array}{c}
\mathbb{C}^3 \\
\langle e_1, e_2 \rangle \\
\langle e_1 \rangle \\
\{0\}
\end{array} \quad \xrightarrow{\mu} \quad
\begin{array}{c}
\mathbb{C}^3 \\
V_1 \\
V_2 \\
V_3 \\
V_4 \\
V_5 \\
0
\end{array}
\]

Note that \( T \) acts on \( \mathbb{C}^n \) by multiplication, and this action can be extended to \( T \) acting on \( BS^Q \). In Section 4.1, we will show that Bott-Samelson varieties are Hamiltonian symplectic manifolds with respect to this torus action. Therefore, they have moment maps with respect to the torus action

\[
\mu : BS^Q \rightarrow \mathbb{R}\langle \nabla(W) \rangle,
\]

where \( \mathbb{R}\langle \nabla(W) \rangle \) is real span of the fundamental weights of \( W \). We now describe this moment map very concretely. Given \( p = (V_1, \ldots, V_m) \in BS^Q \) and \( i \in [m] \) define

\[
\mu(p, i) = (\dim_{e_1}(V_i), \ldots, \dim_{e_m}(V_i)),
\]

where \( \dim_{e_j}(V) \) denotes the dimension of \( V \) on the \( e_j \)-th coordinate, then the moment map is

\[
BS^Q \xrightarrow{\mu} \mathbb{R}^n
\]

\[
(V_1, \ldots, V_m) \xrightarrow{\mu} \left( \sum_{i=1}^m \dim_{e_1}(V_i), \ldots, \sum_{i=1}^m \dim_{e_n}(V_i) \right).
\]

The image of this map is the *moment polytope* and it equals the convex hull of the images of the \( T \)-fixed points. There is a 1-1 correspondence between \( T \)-fixed points on \( BS^Q \) and subwords \( R \) of \( Q \): if \( p(R) \) is
the $T$-fixed point corresponding to $R$ then $m_Q(p(R)) = (\prod Q \setminus R) B \in G/B$. In more combinatorial terms, the point $p(R)$ corresponding to the subword $S$ is determined by deciding between $=$ and $\neq$ in each diamond

$$V_b = V_a \oplus (e_x, e_y)$$

$$V_i = \neq V_j$$

For $Q = (q_1, \ldots, q_m)$, we pick $=$ if $q_j \in R$ and $\neq$ if $q_j \notin R$. We illustrate the correspondence by an example.

**Example 3.3** The subword $J = (-, s_2, -, -, s_1)$ of $Q = (s_1, s_2, s_1, s_2, s_1)$ corresponds to the flags below and the image of $m_Q : BS^Q \to G/B$ is $(s_1s_2)B = (s_2)B$.

This correspondence motivates the relation between fibers of the map $m_Q : BS^Q \to G/B$ and subword complexes.

### 3.2 Brick polytopes in the $SL_n(\mathbb{C})$ case

The *sorting network* $N_Q$ of a word $Q = (q_1, \ldots, q_m)$ consists of $n$ horizontal lines (called the *levels*) and $m$ vertical segments (called the *commutators*) drawn from left to right so that each commutator joins consecutive levels, no two commutators share a common endpoint, and if $q_k = s_i$ then the $k$-th commutator connects levels $i$ and $i + 1$. A *brick* of $N_Q$ is a connected component of its complement, bounded on the right by a commutator.

A *pseudoline* supported by $N_Q$ is a path on $N_Q$ traveling monotonically from left to right. A commutator of $N_Q$ is called a *crossing* between two pseudolines if it is crossed by the two pseudolines and it is called a *contact* if its endpoints are one on each pseudoline. A *pseudoline arrangement* on $N_Q$ is a collection of $n$ pseudolines such that each two have at most one crossing and no other intersection.

**Example 3.4** Let $Q = (s_1, s_2, s_1, s_2, s_1)$ then $w_0 = Dem(Q) = s_1s_2s_1 = s_2s_1s_2$ and we have that the sorting network $N_Q$ is
and the pseudoline arrangement corresponding to the subword \( J = (s_1, -, -, -, s_1) \) is

![Diagram of a pseudoline arrangement]

Given a pseudoline arrangement supported by \( \mathcal{N}_Q \) with contacts \( F = (q_i, \ldots, q_k) \) the product \( w = \prod Q \setminus F \) is an element of \( W \) and the pseudoline starting at level \( i \) will end at level \( w(i) \). We call such an arrangement a \( w \)-pseudoline arrangement. There is a one-to-one correspondence between facets \( F \) of \( \Delta(Q, w) \) and \( \prod Q \setminus F \)-pseudoline arrangements supported on \( \mathcal{N}_Q \). In this setup, we have that \( w(F, j) \) is the characteristic vector of the pseudolines passing below the \( j \)-th brick of \( \mathcal{N}_Q \). Moreover, the \( i \)-th coordinate of the brick vector \( B(F) \) is the number of bricks in \( \mathcal{N}_Q \) that lie above the \( i \)-th pseudoline with contacts \( F \), and the brick polytope \( B(Q, w) \) is the following convex hull:

\[
B(Q, w) := \text{conv}\{B(F) : F \in \Delta(Q, w) \text{ and } \prod Q \setminus F = w\}.
\]

**Example 3.5** Let \( Q = (s_1, s_2, s_1, s_2, s_1) \) and \( w = s_1 s_2 s_1 \), then the pseudoline arrangement corresponding to the subword \( J = (s_1, -, -, -, s_1) \) gives the vector \( B(J) = (2, 0, 2) \) obtained by counting bricks above each line. The brick polytope \( B(Q, w) \) is pictured below.

3.3 Toric varieties for brick polytopes in the \( SL_n(\mathbb{C}) \) case

We have already seen that subwords \( R \) of \( Q \) correspond to \( T \)-fixed points of \( BS^Q \) and that if \( p(R) \) is the point corresponding to \( R \) then \( m_Q(p(R)) = (\prod Q \setminus R) B \in G/B \). This means that the rightmost flag of the point \( p(R) \) is the flag corresponding to \( \prod Q \setminus R \in W \) and so the pseudoline arrangement corresponding to \( R \) is an \( (\prod Q \setminus R) \)-arrangement. The following example shows the correspondence.

**Example 3.6** The pseudoline arrangement corresponding to the word \( J = (s_1, -, -, -, s_1) \) gives a \( T \)-fixed point of \( BS^{(s_1, s_2, s_1, s_2, s_1)} \) as the following diagrams show.
**Theorem 3.7** Suppose $Q$ is toric and $w = \text{Dem}(Q)$. There is a bijective correspondence between $w$-pseudoline arrangements supported by $N_Q$ and $T$-fixed points of $m_Q^{-1}(wB)$. Moreover, this correspondence makes the moment map

$$\mu : m_Q^{-1}(wB) \rightarrow \mathbb{R}^n$$

be equivalent to the mapping

$$B : \{w\text{-pseudoline arrangements supported by } N_Q\} \rightarrow \mathbb{R}^n$$

given in [Pilaud and Santos 2012].

**Proof:** The first part of the proposition is proven in the paragraph preceding the example above. The second part of this theorem follows from using induction on $|Q| = m$ to prove that $\mu(I, j) = w(I, j)$ for all subwords $I$ and all $j \in [m]$. \qed

**Theorem 3.8** The fiber $m_Q^{-1}(wB)$ is a toric variety if and only if $Q$ is toric, $\ell(w) \leq |Q| \leq \ell(w) + n$ and $\text{Dem}(Q) = w$. Moreover, $m_Q^{-1}(wB)$ is the toric variety associated to the polytope $B(Q)$.

We have proved the if part of this theorem, however the only if part will be follow from Theorem 4.4. The following corollary follows from the work of [Pilaud and Santos 2012]. For the statement we define a Coxeter element $c$ to be the product of all simple reflections in some order. Define also the $c$-sorting word of $w$ to be the lexicographically first subword of $c^\infty$ that is a reduced expression for $w$.

**Corollary 3.9** If $Q$ is the word starting with a word $c$ representing a Coxeter element $c$ and then having the $c$-sorting word for $w_0$, then $m_Q^{-1}(w_0B)$ is the toric variety of the associahedron as realized in [Hohlweg and Lange 2007] and in [Pilaud and Santos 2012].

**Example 3.10** The toric variety of the pentagon from example 3.5, i.e. the associahedron corresponding to the Coxeter element $c = (s_1, s_2)$, is

$$m_Q^{-1}(wB) = \{(V_1, V_2, V_3) : \text{each ascending chain in the poset below is a complete flag}\}$$
4 Brick manifolds in the general case

Let $G$ be a complex semisimple Lie group, let $B$ be a Borel subgroup of $G$, i.e., a maximal solvable subgroup, and $T$ be a maximal torus contained in $B$. Let $W$ be the Weyl group of $G$ with generators $S = \{s_1, \ldots, s_n\}$, which correspond to the simple roots $\Delta(W) = \{\alpha_1, \ldots, \alpha_n\}$. Let $P$ be a parabolic subgroup of $G$, i.e., a subgroup of $G$ for which the quotient $B/P$ is a projective algebraic variety; this condition is equivalent to $P$ contains $B$. We denote by $P_i$ the minimal parabolic subgroup corresponding to $s_i$, we then have that $P_i/B \cong \mathbb{CP}^1$. The torus $T$ acts on this quotient and this action has exactly two $T$-fixed points: one corresponding to the identity element and one corresponding to the generator $s_i$.

**Definition 4.1** Let $Q = (s_{i_1}, \ldots, s_{i_m})$ be a word in the generators of $W$. Then the product $P_{i_1} \times \cdots \times P_{i_m}$ has an action of $B^m$ given by:

$$(b_1, \ldots, b_m) \cdot (p_1, \ldots, p_m) = (p_1 b_1 b_1^{-1} p_2 b_2, \ldots, b_{m-1}^{-1} p_m b_m)$$

The Bott-Samelson variety of $Q$ is the quotient of the product of the $P_i$'s by this action

$$BS^Q := (P_{i_1} \times \cdots \times P_{i_m})/B^m$$

Bott-Samelson varieties are smooth, irreducible and $|Q|$-dimensional algebraic varieties. These varieties come equipped with a natural map

$$BS^Q \xrightarrow{\text{m}_{Q^{-1}}} G/B$$

$$(p_1, \ldots, p_m) \mapsto (p_1 \cdots p_m)B.$$  

The image of this map is the opposite Schubert cell $X_w := BwB$, where $w = \text{Dem}(Q)$. In the case in which $Q$ is reduced, this map is a resolution of singularities for non-smooth $X_w$, however in this paper we will study cases in which $Q$ is not reduced.

**Definition 4.2** Let $Q = (s_{i_1}, \ldots, s_{i_m})$ be a word in the generators of $W$ and $w = \text{Dem}(Q)$, then the Brick manifold is the fiber $m_Q^{-1}((B)w)$.

4.1 Symplectic Structure on Bott-Samelson Varieties

Let $P_i$ be the maximal parabolic subgroup of $G$ corresponding to $S_i := \{s_1, \ldots, \hat{s}_i, \ldots, s_n\}$. Note that for $G = SL_n(\mathbb{C})$ each quotient $G/P_i$ is a Grassmannian. Let $K$ be the maximal compact subgroup of $G$. Then we can view $G/P_i$ as a coadjoint orbit, i.e., a $K$-orbit trough the fundamental weight $\lambda_{s_i} \in \mathfrak{t}^*$. 

\[
\begin{array}{c}
\mathbb{C}^3 \\
\langle e_1, e_2 \rangle & V_2 & \langle e_2, e_3 \rangle \\
\langle e_1 \rangle & V_1 & \langle e_3 \rangle \\
0
\end{array}
\]
where $\mathfrak{t}$ is the Lie algebra of $K$. This interpretation gives us a symplectic structure on $G/P_1$ with respect to the action of $K$ such that the inclusion

$$G/P_1 \hookrightarrow \mathfrak{t}^*$$

is a moment map for the $K$-action. Then the composition

$$G/P_1 \hookrightarrow \mathfrak{t}^* \rightarrow \mathfrak{t}^*$$

is the moment map of $G/P_1$ with respect to the Torus action, where $\mathfrak{t}$ is the Lie algebra of the torus.

Moreover, the moment map for the diagonal $T$-action on a product $\prod G/P_i$ is the sum of each moment map $G/P_i \to \mathfrak{t}^*$.

Let $T$ act on $BS^Q$ by

$$t \cdot (p_1, p_2, \ldots, p_m) = (t \cdot p_1, p_2, \ldots, p_m).$$

Given $Q = (q_1, \ldots, q_m)$ we have a $T$-equivariant map

$$BS^Q \xrightarrow{\varphi} \prod_{i : s_i \in Q} G/P_i,$$

where $\varphi = (\varphi_1, \ldots, \varphi_m)$ and the $k$-th component is

$$BS^Q \xrightarrow{\varphi_k} G/P_k$$

$$(p_1, \ldots, p_m) \mapsto (\prod_{i < j} p_i) P_k.$$  

The composition

$$BS^Q \xrightarrow{\varphi} \prod_{i : s_i \in Q} G/P_i \rightarrow \mathfrak{t}^*$$

gives us a moment map for this Bott-Samelson variety with respect to the $T$-action. Thus Bott-Samelson varieties are Hamiltonian symplectic manifolds with respect to this torus action. The image of this map is the moment polytope, the convex hull of the images of the $T$-fixed points. This result is a theorem proved independently in Atiyah (1982) and in Guillemin and Sternberg (1982). There is a correspondence between $T$-fixed points on $BS^Q$ and subwords $R$ of $Q$ and if $p(R)$ is the $T$-fixed point corresponding to $S$ then

$$m_Q(p(R)) = \left(\prod Q \setminus R \right) B \in G/B.$$  

This correspondence motivates the relation between fibers of the map $m_Q : BS^Q \to G/B$ and subword complexes.

We now describe the image of the $T$-fixed points under the moment map. For each $k$ we have the moment map

$$\mu_k : G/P_k \rightarrow \mathfrak{t}^*,$$

where $\mu_k(P_k) = \lambda_{s_k}$, the fundamental weight corresponding to $s_k$, and it maps a general element to a Weyl conjugate of this fundamental weight. Before we finish describing the maps $\mu_k$, we note that the moment map of $BS^Q$ is then

$$\sum_{k=1}^m \varphi_k \circ \mu_k.$$
Consider the fixed point \((p_1, \ldots, p_m)\) in \(BS^Q\) corresponding to the subword \(R\) of \(Q\) then under the moment map \(\mu_k\) each \(p_j\) corresponds to either the reflection \(s_{i_j}\) if \(q_j \in S\) or to the identity in \(W\). In other words, \(p_j\) corresponds to \(s_{i_j}\) if \(p_j \notin B\) and to the identity in \(W\) otherwise. In conclusion we have that for \(R\) subword of \(Q\) and

\[
p_R = \text{the fixed point corresponding to } R
\]

\[
BS^Q \xrightarrow{\mu_k} t^* \quad p_R \mapsto (\prod_{i<k,q_i \in S} q_i)(\lambda_{s_{i_k}}).
\]

It then follows that

\[
BS^Q \xrightarrow{\mu} t^* \quad (2)
\]

\[
p_R \mapsto \sum_{k=1}^m (\prod_{i<k,q_i \in S} q_i)(\lambda_{s_{i_k}}) \quad (3)
\]

### 4.2 Moment polytopes of Brick Manifolds

We now state and prove the main results of the paper.

**Theorem 4.3** The image of \(m_Q^{-1}(B)\) under the moment map is the brick polytope \(B(Q,w)\).

**Proof:** \(T\)-fixed points of \(BS^Q\) are in 1-1 correspondence with subwords \(R\) of \(Q\). This induces a 1-1 correspondence between \(T\)-fixed points of \(m_Q^{-1}(B)\) and the subwords \(R\) of \(Q\) with \(\prod Q \setminus R = w\). If the subword \(R\) is not a facet of the subword complex \(\Delta(Q,w)\) then it gives a non reduced product \(\prod Q \setminus R\). This implies that the root configuration \(r(R) = \{|r(R,i) : i \in F\}\) has a smaller dimension than the root configuration of a facet and thus it cannot be a vertex. Therefore, moment polytope is the convex hull of the points corresponding to facets of \(\Delta(Q,w)\) and by Equation 3 the image of each fixed point is precisely the one defined in Equation 1 in Section 2.2.

**Theorem 4.4** The fiber \(m_Q^{-1}(wB)\) is a toric variety if and only if \(Q\) is toric, \(\ell(w) \leq |Q| \leq \ell(w) + \dim(W)\) and \(Dem(Q) = w\). Moreover, \(m_Q^{-1}(wB)\) is the toric variety associated to the polytope \(B(Q,w)\).

**Proof:** We sketch the proof for the case \(|Q| = \ell(w) + \dim(W)\). Note that \(\dim(T) = \dim(m_Q^{-1}(wB))\).

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