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A Chevalley-Monk and Giambelli's Formula for Peterson Varieties of All Lie Types

Elizabeth Drellich*

University of Massachusetts, Amherst, USA

Abstract. A Peterson variety is a subvariety of the flag variety G/B defined by certain linear conditions. Peterson varieties appear in the construction of the quantum cohomology of partial flag varieties and in applications to the Toda flows. Each Peterson variety has a one-dimensional torus S^1 acting on it. We give a basis of Peterson Schubert classes for $H^*_{S^1}(Pet)$ and identify the ring generators. In type A Harada-Tymoczko gave a positive Monk formula, and Bayegan-Harada gave Giambelli's formula for multiplication in the cohomology ring. This paper gives a Chevalley-Monk rule and Giambelli's formula for all Lie types.

Résumé. Une variété de Peterson est une sous-variété de G/B qui provient de la construction de la cohomologie quantique des variétés de drapeaux partielles. Chacune a un tore d'une dimension, S^1 qui y agit. Nous présentons une base des classes de Peterson Schubert pour $H^*_{S^1}(Pet)$ et identifions les générateurs de l'anneau. Harada-Tymoczko ont fourni une formule de Monk positive pour le type A, et Bayegan-Harada ont donné une formule Giambelli pour multiplication dans l'anneau de la cohomologie. Cet article donne une règle Chevalley-Monk et une formule de Giambelli dans tous les types de Lie.

Keywords: Peterson variety, Chevalley-Monk formula, Giambelli's formula, equivariant cohomology

1 Introduction

Classical Schubert calculus computes the cohomology rings of Grassmannians and flag varieties. Each cohomology ring of a Grassmannian or flag variety has a basis of **Schubert classes** indexed by the elements of the corresponding Weyl group. This indexing set turns geometric and topological questions into combinatorial questions.

This paper will do "Schubert calculus" in the equivariant cohomology rings of Peterson varieties. Peterson varieties were introduced by D. Peterson in the 1990s. Peterson constructed the small quantum cohomology of partial flag varieties from what are now called Peterson varieties. Since then Kostant used Peterson varieties to describe the quantum cohomology of the flag manifold [13]. Rietsch described the totally non-negative part of type A Peterson varieties in 2006 using mirror symmetry constructions [16]. In 2012 Insko-Yong explicitly described the singular locus of type A Peterson varieties and intersected them with Schubert varieties [12].

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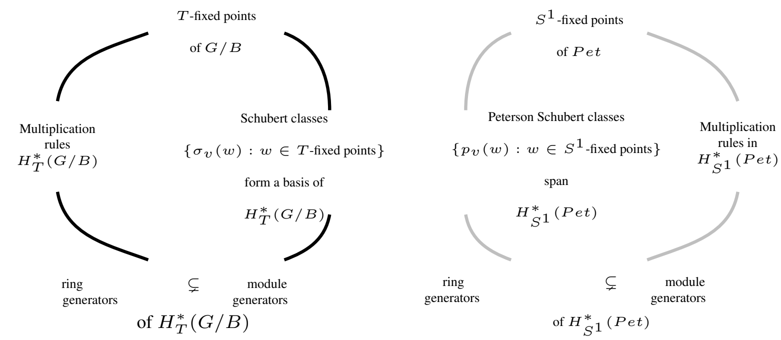


Fig. 1: Connections given by GKM theory for G/B, a GKM space, are in black. The corresponding connections we will make for the Peterson variety, which is not a GKM space, are in gray.

Named for Goresky, Kottwitz, and MacPherson, GKM theory expresses the T-equivariant cohomology of certain spaces in terms of polynomials corresponding to T-fixed points [7]. The flag variety with the action of a maximal torus $T \subseteq B \subseteq G$ is such a space. Peterson varieties are not GKM spaces; they lack many of the nice structures of G/B. However by using a one-dimensional torus S^1 several of those structures can be built for Peterson varieties anyway. Harada-Tymoczko gave the S^1-fixed points of the Peterson variety explicitly [9]. In this paper we expand the GKM-like properties of Peterson varieties as illustrated in Figure 1.

Using work by Harada-Tymoczko [10] and Precup [15], we construct a basis for the S^1-equivariant cohomology of Peterson varieties in *all* Lie types. This construction gives a basis of **Peterson Schubert classes**. A Peterson Schubert class is thus named because is the image of a Schubert classes under a certain ring homomorphism, not because the Peterson Schubert classes satisfy the standard properties of Schubert classes. In fact our first result is to give a subset of the Peterson Schubert classes which satisfy as many of the Schubert class properties as possible. Crucially this subset is a module basis for $H^*_{S^1}(Pet)$.

Once we have a basis of Peterson Schubert classes, we ask classical Schubert calculus questions about how to multiply them. We give a Chevalley-Monk formula for multiplying a ring generator and a module generator, and a Giambelli's formula for expressing any Peterson Schubert class in terms of the ring generators. In type A the equivariant cohomology of the Peterson variety was studied by Harada-Tymoczko who gave a basis and a Monk's rule for the equivariant cohomology ring [9]. A type A Giambelli's formula was given by Bayegan-Harada [1]. This paper extends those results to all Lie types.

1.1 Multiplication rules for Peterson Schubert classes in all Lie types

Where Schubert classes in the flag variety are indexed by permutations in the associated Weyl group, Peterson Schubert classes are indexed by subsets of the set of simple roots. To each subset K of simple roots we associate a Peterson Schubert class p_{v_K}.

In the flag variety case, the Schubert classes σ_{s_i} corresponding to words of length one generate the cohomology ring. For Grassmannians and partial flag varieties we need more complicated sets of Schubert classes to generate the cohomology ring. The Peterson case is analogous to the flag variety: classes indexed by words of length one generate the ring $H^*_{S^1}(Pet)$. For each simple root $\alpha_i \in \Delta$ we have a ring generator p_{s_i} of $H^*_{S^1}(Pet)$.

A Chevalley-Monk rule is an explicit formula for multiplying an arbitrary module basis element p_{v_K} by a ring generator class p_{s_i}. For the Peterson variety, a Chevalley-Monk formula gives a set of constants $c^J_{i,K} \in H^*_{S^1}(\mathrm{pt})$ such that

$$p_{s_i} p_{v_K} = \sum_{J \subseteq \Delta} c^J_{i,K} \cdot p_{v_J}.$$

Graham showed that flag varieties have a Chevalley-Monk formula with non-negative integer coefficients [8, Theorem 3.1]. Giving these coefficients explicitly has been the work of many including Buch [5], Bergeron-Sánchez-Ortega-Zabrocki [2], and Lam-Shimozono [14]. In the Peterson case, we give the following formula.

Theorem 1.1 *The Peterson Schubert classes satisfy*

$$p_{s_i} \cdot p_{v_K} = p_{s_i}(w_K) \cdot p_{v_K} + \sum_{\substack{K \subseteq J \subseteq \Delta \\ |J| = |K|+1}} c^J_{i,K} \cdot p_{v_J}$$

where $c^J_{i,K} = (p_{s_i}(w_J) - p_{s_i}(w_K)) \cdot \frac{p_{v_K}(w_J)}{p_{v_J}(w_J)}$.

Our Chevalley-Monk rule is uniform across Lie types. This formula is similar in complexity to the rule for G/B and has positive, although occasionally non-integral, coefficients. On the other hand Giambelli's formula provides explicit structure constants and is surprisingly simple and uniform across Lie types.

Theorem 1.2 *Giambelli's formula for Peterson Schubert classes:*

$$\begin{aligned}
|K|! \cdot p_{v_K} &= \prod_{\alpha_i \in K} p_{s_i} && \text{if } K \text{ is type } A_n, B_n, C_n, F_4, \text{ or } G_2 \\
\tfrac{|K|!}{2} \cdot p_{v_K} &= \prod_{\alpha_i \in K} p_{s_i} && \text{if } K \text{ is type } D_n \\
\tfrac{|K|!}{3} \cdot p_{v_K} &= \prod_{\alpha_i \in K} p_{s_i} && \text{if } K \text{ is type } E_n.
\end{aligned}$$

1.2 Proof Techniques

Multiplication rules in Schubert calculus frequently require new combinatorial machines to prove explicit formulas. Billey's formula evaluates an equivariant Schubert class at a fixed point [3]. This combinatorial formula gives a polynomial in the simple roots for each pair of a Schubert class with a fixed point. Ikeda-Naruse use excited Young diagrams to encode and compute Billey's formula [11]. We modify both Ikeda-Naruse's excited Young diagrams and Billey's formula in order to evaluate Peterson Schubert classes at S^1-fixed points of Peterson varieties.

While the uniform description of the result suggests that a geometric or topological proof exists, we proved the theorem in cases. The modified excited Young diagrams are used to prove Giambelli's formula in each of the classical Lie types. The exceptional Lie types do not have nice corresponding excited Young diagrams. In these cases we prove Giambelli's formula through direct computation. For types F_4 and G_2 the computation is relatively small and straightforward. Type E requires a computer assisted proof using Sage.

2 The S^1 Action on Peterson Varieties

Fix a complex reductive linear algebraic group G, a Borel subgroup B, and a maximal torus $T \subseteq B \subseteq G$. This choice gives

- a root system Φ
- positive roots $\Phi^+ \subset \Phi$
- simple roots $\Delta \subset \Phi^+$
- an associated Weyl group W
- associated Lie algebras $\mathfrak{t} \subseteq \mathfrak{b} \subseteq \mathfrak{g}$
- root spaces $\mathfrak{g}_\alpha \subset \mathfrak{g}$ for each root $\alpha \in \Phi$.

We also choose a basis element $E_\alpha \in \mathfrak{g}_\alpha$ for each of the root spaces. Some of our constructions rely on a specific ordering of the roots $\alpha_1, \alpha_2, \ldots, \alpha_{|\Delta|} \in \Delta$. This ordering is expressed in the Dynkin diagrams in Figure 2.

For any Lie type the Peterson subspace in $\mathfrak{g}$ is

$$H_{Pet} = \mathfrak{b} \oplus \bigoplus_{\alpha \in -\Delta} \mathfrak{g}_\alpha.$$

We fix a regular nilpotent operator $N_0 \in \mathfrak{g}$ is

$$N_0 = \sum_{\alpha \in \Delta} E_\alpha.$$

Definition 2.1 *The Peterson variety Pet is a subvariety of the flag variety defined by*

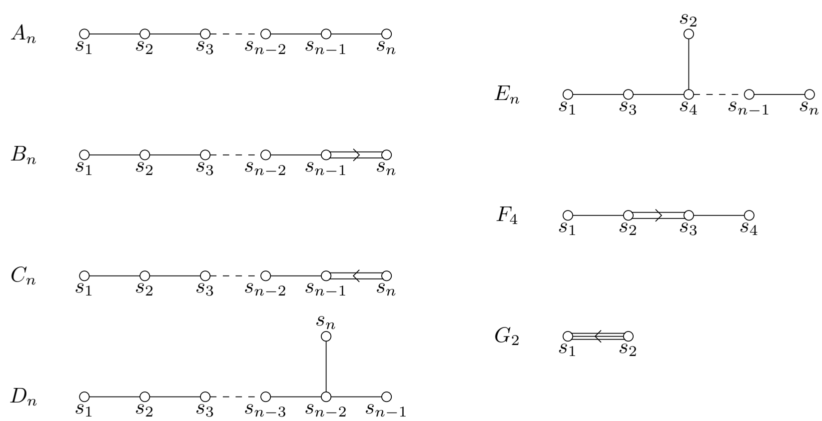
$$Pet = \{gB \in G/B : Ad(g^{-1})(N_0) \in H_{Pet}\}.$$

Peterson varieties are a type of regular nilpotent Hessenberg variety. They are irreducible and generally not smooth [12].

2.1 Billey's formula

With GKM theory the equivariant cohomology ring can be viewed as a subring of a product of polynomial rings: $H_T^*(G/B) \hookrightarrow \bigoplus_{w \in W} H_T^*(pt)$ where each ring $H_T^*(pt) \cong \mathbb{C}[\alpha_i : \alpha_i \in \Delta]$. From this perspective each Schubert class σ_v is actually a collection of polynomials $\sigma_v(w)$ for w in the Weyl group W.

Fig. 2: The Dynkin diagrams show the order on the simple reflections. The same order is imposed on the corresponding simple roots throughout this paper.



Billey gave an explicit combinatorial formula for the polynomial $\sigma_v(w)$ [3]. Fix a reduced word of w, $\tilde{w} = s_{b_1} s_{b_2} \cdots s_{b_{\ell(w)}}$ and let $\mathbf{r}(\mathbf{i}, w) = s_{b_1} s_{b_2} \cdots s_{b_{i-1}}(\alpha_{b_i})$. Then

$$\sigma_v(w) = \sum_{\substack{\text{reduced words} \\ v = s_{b_{j_1}} s_{b_{j_2}} \cdots s_{b_{j_{\ell(v)}}}}} \left(\prod_{i=1}^{\ell(v)} \mathbf{r}(\mathbf{j_i}, \tilde{w}) \right). \tag{1}$$

Proposition 2.2 (Billey [3]) *The polynomial $\sigma_v(w)$ has several useful properties:*

- *The polynomial $\sigma_v(w)$ is homogeneous of degree $\ell(v)$.*
- *The polynomial $\sigma_v(w)$ is non-zero if and only if $v \leq w$ in the Bruhat order.*
- *The coefficients of $\sigma_v(w)$ are non-negative integers.*
- *The polynomial $\sigma_v(w)$ does not depend on the choice of reduced word for w.*

When v and w are words of relatively short length it is simple to calculate $\sigma_v(w)$ by hand.

Example 2.3 *Let G/B have Weyl group $W = A_2$ and let $w = s_1 s_2 s_1$ and $v = s_1$. The word v is found as a subword of $s_1 s_2 s_1$ in two places, $\mathbf{s_1} s_2 s_1$ and $s_1 s_2 \mathbf{s_1}$.*

$$\sigma_v(w) = \mathbf{r}(\mathbf{1}, s_1 s_2 s_1) + \mathbf{r}(\mathbf{3}, s_1 s_2 s_1) = \alpha_1 + s_1 s_2(\alpha_1) = \alpha_1 + \alpha_2$$

2.2 GKM Theory and Peterson Varieties

The torus T does not preserve Pet but a one-dimensional subtorus $S^1 \subseteq T$ does. The equivariant cohomology of the Peterson variety can be defined with respect to S^1 and will still inject into the S^1-fixed points of the Peterson variety.

Definition 2.4 *[10, Lemma 5.1] The characters $\alpha_1, \dots \alpha_n \in \mathfrak{t}^*$ are a maximal $\mathbb{Z}$-linearly independent set. Let $\phi : T \to (\mathbb{C}^*)^n$ be the isomorphism of linear algebraic groups $t \mapsto (\alpha_1(t), \alpha_2(t), ..., \alpha_n(t))$. Then define a one-dimensional torus S^1 by*

$$S^1 = \phi^{-1}(\{(c, c, \dots, c) | c \in \mathbb{C}^*\}).$$

Proposition 2.5 *[10, Lemma 5.1] The torus S^1 acts on the Peterson variety.*

Any point in Pet fixed by T will also be fixed by S^1. In fact these are the only points in the Peterson variety fixed by S^1:

$$(Pet)^{S^1} = Pet \cap (G/B)^T.$$

Harada-Tymoczko gave the S^1-fixed points of Pet explicitly. Let $K \subseteq \Delta$ be a subset of the simple roots. Define $W_K \subseteq W$ as the parabolic subgroup generated by K and let w_K be the longest element of W_K.

Proposition 2.6 *[10, Proposition 5.8] An element $w \in W$ is an S^1-fixed point of Pet if and only if $w = w_K$ for some set $K \subseteq \Delta$.*

3 Peterson Schubert classes as a basis of $H^*_{S^1}(Pet)$

The first structure we want for $H^*_{S^1}(Pet)$ is a basis. We use a projection from $H^*_T(G/B)$ to $H_{S^1}(Pet)$.

$$\begin{array}{ccc} H^*_T(G/B) & \hookrightarrow & \bigoplus_{(G/B)^T} H^*_T(pt) \\ \downarrow & & \downarrow \pi_1 \\ H^*_{S^1}(G/B) & \hookrightarrow & \bigoplus_{(G/B)^{S^1}} H^*_{S^1}(pt) \\ \downarrow & & \downarrow \pi_2 \\ H^*_{S^1}(Pet) & \hookrightarrow & \bigoplus_{(Pet)^{S^1}} H^*_{S^1}(pt) \end{array} \tag{2}$$

A priori $H^*_T(G/B)$ is a module over $\mathbb{C}[\alpha_i : \alpha_i \in \Delta]$. The ring homomorphism π_1 takes simple roots $\alpha_i \in \Delta$ to the variable t. The map π_2 forgets the T-fixed points of G/B that are not in the Peterson variety.

3.1 Peterson Schubert classes

The image of a Schubert class $\sigma_v \in H^*_T(G/B)$ in $H^*_{S^1}(Pet)$ is denoted p_v and called a **Peterson Schubert class**. The class p_v has one polynomial for each S^1-fixed point of Pet so a Peterson Schubert class can be thought of as a $2^{|\Delta|}$-tuple of polynomials in $\mathbb{C}[t]$. Below is an example in type A_2.

$$\begin{array}{ll} & \sigma_{s_1} \\ \begin{array}{l} 1 \\ s_1 \\ s_2 \\ s_1s_2 \\ s_2s_1 \\ s_1s_2s_1 \end{array} & \begin{pmatrix} 0 \\ \alpha_1 \\ 0 \\ \alpha_1 \\ \alpha_1+\alpha_2 \\ \alpha_1+\alpha_2 \end{pmatrix} \end{array} \xmapsto{\pi_1} \begin{pmatrix} 0 \\ t \\ 0 \\ t \\ 2t \\ 2t \end{pmatrix} \xmapsto{\pi_2} \begin{array}{c} p_{s_1} \\ \begin{pmatrix} 0 \\ t \\ 0 \\ \\ \\ 2t \end{pmatrix} \end{array}.$$

Remark 3.1 *Because the longest word w_0 is a fixed point of the Peterson variety and the polynomial $\sigma_v(w_0)$ is non-zero by Proposition 2.2, the image p_v of Schubert class σ_v is always non-zero.*

Theorem 3.2 *The poset pinball machinery given by Harada-Tymoczko [10, Theorem 5.4] holds for Peterson varieties of all Lie types.*

As consequence of this theorem we can use the maps π_1 and π_2 to study $H^*_{S^1}(Pet)$.

3.2 A basis of Peterson Schubert classes

The set of Peterson Schubert classes $\{p_v : v \in W\}$ is the image of the set of Schubert classes. The set does not satisfy the standard properties of Schubert classes. We now identify a subset that satisfies many of those properties. This subset is linearly independent and spans $H^*_{S^1}(Pet)$.

Definition 3.3 *A subset of simple roots $K \subseteq \Delta$ is called* **connected** *if the induced Dynkin diagram of K is a connected subgraph of the Dynkin diagram of Δ.*

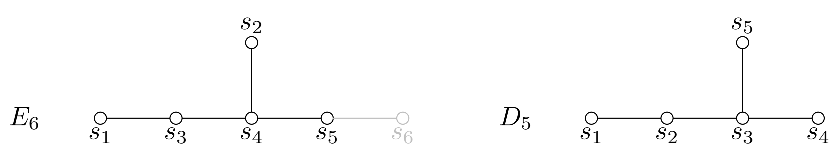
Any subset $K \subseteq \Delta$ can be written as $K = K_1 \times \cdots \times K_m$ where each K_i is a maximally connected subset. Each connected subset corresponds to its own Lie type.

Definition 3.4 *Let $K \subseteq \Delta$ be a connected subset. We define $v_K \in W_K$ to be*

$$v_K = \prod_{\mathrm{Root_K(i)=1}}^{|K|} s_i$$

where $\mathrm{Root}_K(i)$ is the index of the corresponding root in a root system of the same Lie type as K, ordered as in Figure 2. If $K = K_1 \times \cdots \times K_m$ we write $v_K = v_{K_1} v_{K_2} \cdots v_{K_m}$.

Example 3.5 *Let $\Delta = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6\}$ be a the set of simple roots of a type E_6 root system and let $K = \Delta \setminus \{\alpha_6\}$. The subset $K \subseteq \Delta$ is drawn as a marked set of vertices in the Dynkin diagram which is compared to the Dynkin diagram for D_5. The word v_K is $s_1 s_3 s_4 s_5 s_2$.*



Lemma 3.6 *For any set of simple roots Δ and any subsets $J, K \subseteq \Delta$ the Peterson Schubert class satisfies $p_{v_J}(w_K) = 0$ unless $J \subseteq K$. Since K contains itself, the polynomial $p_{v_K}(w_K)$ is non-zero.*

The following theorem is a version of Harada-Tymoczko's Theorem 5.9 [10] which, using Precup's work, we extend to all Peterson varieties.

Theorem 3.7 *The Peterson Schubert classes $\{p_{v_K} : K \subseteq \Delta\}$ are a basis of $H^*_{S^1}(Pet)$.*

Proof: Impose a partial order on the sets $\{K \subseteq \Delta\}$ by inclusion. Use that partial order to order the classes $\{p_{v_K}\}$ and the S^1-fixed points $w_K \in Pet$. Lemma 3.6 implies that the collection $\{p_{v_K}\}$ is lower-triangular and has full rank. Thus $\{p_{v_K}\}$ is a linearly independent set.

By the properties of Billey's formula, the polynomial degree of p_{v_K} is $|K|$ and its cohomology degree is $2|K|$. As there are $\binom{n}{|K|}$ subsets of Δ with size $|K|$, there are exactly $\binom{n}{|K|}$ Peterson Schubert varieties with cohomology degree $2|K|$. Precup's paving by affines reveals that the dimensions of the corresponding pavings are also $\binom{n}{|K|}$ [15, Corollary 4.13]. As a linearly independent set with the correct number of elements of each degree, the set $\{p_{v_K}\}$ is a module basis of $H^*_{S^1}(Pet)$ [9, Proposition A.1]. $\square$

Example 3.8 *Below we give the Peterson Schubert classes which form a basis of the S^1-equivariant cohomology of Pet in Lie type C_3. The classes and fixed points are indexed by the subsets $K \subseteq \Delta$.*

K	$p_{v_\emptyset}$	$p_{v_{\{\alpha_1\}}}$	$p_{v_{\{\alpha_2\}}}$	$p_{v_{\{\alpha_3\}}}$	$p_{v_{\{\alpha_1,\alpha_2\}}}$	$p_{v_{\{\alpha_1,\alpha_3\}}}$	$p_{v_{\{\alpha_2,\alpha_3\}}}$	$p_{v_{\{\alpha_1,\alpha_2,\alpha_3\}}}$
$\emptyset$	1	0	0	0	0	0	0	0
$\{\alpha_1\}$	1	t	0	0	0	0	0	0
$\{\alpha_2\}$	1	0	t	0	0	0	0	0
$\{\alpha_3\}$	1	0	0	t	0	0	0	0
$\{\alpha_1,\alpha_2\}$	1	$2t$	$2t$	0	$2t^2$	0	0	0
$\{\alpha_1,\alpha_3\}$	1	t	0	t	0	t^2	0	0
$\{\alpha_2,\alpha_3\}$	1	0	$3t$	$4t$	0	0	$6t^2$	0
$\{\alpha_1,\alpha_2,\alpha_3\}$	1	$5t$	$8t$	$9t$	$20t^2$	$45t^2$	$36t^2$	$60t^3$

4 Chevalley-Monk Formula

Now that we have a basis for $H^*_{S^1}(Pet)$ in terms of Peterson Schubert classes, we can examine the structure of $H^*_{S^1}(Pet)$ through its multiplication rules. Recall that the Peterson Schubert classes p_{s_i} generate the ring $H^*_{S^1}(Pet)$. A Chevalley-Monk rule is an explicit formula for multiplying an arbitrary module generator class p_{v_K} by a ring generator p_{s_i}. For the Peterson variety, it gives a set of constants $c^J_{i,K} \in H^*_{S^1}(pt)$ such that

$$p_{s_i} p_{v_K} = \sum_{J \subseteq \Delta} c^J_{i,K} \cdot p_{v_J}. \tag{3}$$

Lemma 4.1 *If $K \not\subseteq J$ then $c^J_{i,K} = 0$. Moreover if $|J| > |K| + 1$ then $c^J_{i,K} = 0$.*

Having determined which coefficients are always zero, we can give our Chevalley-Monk formula for Peterson varieties. Our coefficients are complex polynomials in t. We say such a polynomial is non-negative and rational if it is contained in $\mathbb{Q}_{\geq 0}[t]$.

Theorem 4.2 (Chevalley-Monk formula for Peterson varieties) *The Peterson Schubert classes satisfy*

$$p_{s_i} \cdot p_{v_K} = p_{s_i}(w_K) \cdot p_{v_K} + \sum_{J \text{ such that } K \subseteq J \subseteq \Delta \text{ and } |J| = |K|+1} c^J_{i,K} \cdot p_{v_J}$$

where the coefficients $c^J_{i,K}$ are non-negative rational numbers given by

$$c^J_{i,K} = (p_{s_i}(w_J) - p_{s_i}(w_K)) \cdot \frac{p_{v_K}(w_J)}{p_{v_J}(w_J)}.$$

Conjecture 4.3 *In classical Schubert calculus the structure constants are generally non-negative integers. Frequently they are in bijection with dimensions of irreducible representations. However, structure constants for the Peterson variety are* not *necessarily integers. For example in type D_5 if we let*

$K = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ *and* $J = \Delta$ *then*

$$c^J_{5,K} = \frac{5}{2}.$$

We conjecture that in this basis, non-integral structure constants only occur in Lie types D *and* E.

5 Giambelli's Formula

The last piece to understanding $H^*_{S^1}(Pet)$ is Giambelli's formula. It expresses an arbitrary module basis element in terms of the ring generators. For the basis of $H^*_T(G/B)$ consisting of Schubert classes it looks like

$$\sigma_\lambda = \det(\sigma_{\lambda_i+j-i})_{1\le i,j\le r}$$

where σ_λ is the Schubert class corresponding to the partition $\lambda = (\lambda_1, \ldots, \lambda_r)$ [6]. While easy to write down, this formula is hard to compute for a given Schubert class. Astonishingly, for Peterson varieties the Giambelli's formula for module basis elements of simplifies to a single product. A Giambelli's formula for the Peterson Schubert classes not in our basis is not yet known.

Lemma 5.1 *For a Peterson Schubert class* p_{v_K} *there is an integer constant* C *satisfying*

$$C \cdot p_{v_K} = \prod_{\alpha_i \in K} p_{s_i}. \tag{4}$$

Proof: If $|K| = m$ let $K = \{\alpha_{a_1}, \alpha_{a_2}, \cdots \alpha_{a_m}\}$. Define subsets $\emptyset = K_0 \subsetneq K_1 \subsetneq K_2 \subsetneq \cdots \subsetneq K_m = K$ by

$$K_i = \{\alpha_{a_1}, \alpha_{a_2}, \cdots \alpha_{a_i}\}.$$

From the Chevalley-Monk formula for Peterson varieties

$$p_{s_{a_{i+1}}} \cdot p_{v_{K_i}} = c^{K_{i+1}}_{a_{i+1},K_i} \cdot p_{v_{K_i}} + \sum_{\substack{K_i \subsetneq J \subseteq \Delta \\ |J| = |K_i|+1}} c^J_{a_{i+1},K_i} \cdot p_{v_J}.$$

Theorem 4.2 says $c^{K_{i+1}}_{a_{i+1},K_i} = p_{s_{a_{i+1}}}(w_{K_i})$. Because $\alpha_{a_{i+1}} \notin K_i$ the polynomial $p_{s_{a_{i+1}}}(w_{K_i})$ is zero. If $\alpha_{a_{i+1}} \notin J$ the term $p_{s_{a_{i+1}}}(w_J)$ is also zero. Thus if $J \neq K_{i+1}$ then $c^J_{a_{i+1},K_i} = 0$. Now the Chevalley-Monk rule reduces to

$$p_{s_{a_{i+1}}} \cdot p_{v_{K_i}} = c^{K_{i+1}}_{a_{i+1},K_i} \cdot p_{v_{K+1}}.$$

Solving for $p_{v_{K+1}}$ gives

$$\frac{p_{s_{a_{i+1}}} \cdot p_{v_{K_i}}}{c^{K_{i+1}}_{a_{i+1},K_i}} = p_{v_{K i+1}}.$$

By induction on i we see

$$p_{v_K} = \frac{\prod_{i=1}^{|K|} p_{s_{a_i}}}{\prod_{i=1}^{|K|} c^{K_i}_{a_i,K_{i-1}}}.$$

This gives that

$$C = \prod_{i=1}^{|K|} c_{a_{i+1},K_i}^{K_{i+1}}.$$

□

Remark 5.2 *Like the determinantal formula for flag varieties, this formula holds in both the equivariant and ordinary cohomology of the Peterson variety.*

Knowing that Giambelli's formula is a single product rather than a determinantal formula, we want to give the constant C explicitly. To find this C we consider the simplest non-trivial Peterson Schubert classes, those that are connected.

Theorem 5.3 *If $J, K \subset \Delta$ are disjoint subsets such that no root in J is adjacent to any root of K, then*

$$p_{v_{J\cup K}} = p_{v_J} \cdot p_{v_K}. \tag{5}$$

It follows that if $K = K_1 \times \cdots \times K_m$ then $p_{v_K} = p_{v_{K_1}} \times \cdots \times p_{v_{K_m}}$.

Proof: We show that equality holds when Equation (5) is evaluated at any S^1-fixed point w_L. If L does not contain $J \cup K$ we can suppose without loss of generality that $J \not\subseteq L$. Then both $p_{v_{J\cup K}}(w_L)$ and $p_{v_J}(w_L)$ are zero.

If L contains $J \cup K$ then a subword b of a fixed reduced word $\tilde{w}_L$ is a reduced word for $v_{J\cup K}$ if and only if $b = b_J \cup b_K$ for subwords $b_J, b_K \prec \tilde{w}_L$ reduced words for v_J and v_K. Billey's formula in Equation (1) is a sum over such subwords. We use it to rewrite the left- and right-hand sides of Equation (5). The left hand side becomes:

$$p_{v_{J\cup K}}(w_L) = \sum_{\substack{b\prec\tilde{w}_L \text{ and } b=v_{J\cup K} \\ b=s_{a_{j_1}} s_{a_{j_2}} \cdots s_{a_{j_{|J\cup K|}}}}} \left(\prod_{1\le i\le |J\cup K|} \mathbf{r}(\mathbf{j}_i, \tilde{w}_L) \right). \tag{6}$$

Similarly the right-hand side becomes $p_{v_J}(w_L) \cdot p_{v_K}(w_L) =$

$$\left[\sum_{\substack{b\prec\tilde{w}_L \text{ and } b=v_J \\ b=s_{a_{j_1}} s_{a_{j_2}} \cdots s_{a_{j_{|J|}}}}} \left(\prod_{1\le i\le |J|} \mathbf{r}(\mathbf{j}_i, \tilde{w}_L) \right) \right] \cdot \left[\sum_{\substack{b\prec\tilde{w}_L \text{ and } b=v_K \\ b=s_{a_{j_1}} s_{a_{j_2}} \cdots s_{a_{j_{|K|}}}}} \left(\prod_{1\le i\le |K|} \mathbf{r}(\mathbf{j}_i, \tilde{w}_L) \right) \right]. \tag{7}$$

Both Equations (6) and (7) expand out to the same expression. □

Theorem 5.4 *If $K \subseteq \Delta$ is a connected root subsystem of type A_n, B_n, C_n, F_4, or G_2 then*

$$|K|! \cdot p_{v_K} = \prod_{\alpha_i \in K} p_{s_i}.$$

If K is a connected root subsystem of type D_n then

$$\frac{|K|!}{2} \cdot p_{v_K} = \prod_{\alpha_i \in K} p_{s_i}.$$

If K is a connected root subsystem of type E_n then

$$\frac{|K|!}{3} \cdot p_{v_K} = \prod_{\alpha_i \in K} p_{s_i}.$$

Our proof of this theorem is combinatorial and treats each Lie type as its own case. The uniformity across Lie types suggests that a uniform proof exists. Such a proof might shed light on the topology of these varieties. In fact, the theorem can be stated in an even more uniform manner.

Theorem 5.5 *If $K \subseteq \Delta$ is a connected root subsystem of any Lie type and $|\mathcal{R}(v_K)|$ is the number of reduced words for v_K then*

$$\frac{|K|!}{|\mathcal{R}(v_K)|} \cdot p_{v_K} = \prod_{\alpha_i \in K} p_{s_i}.$$

Proof: Given Theorem 5.4 it is sufficient to show that $|\mathcal{R}(v_K)| = 1$ if K is type A, B, C, F, or G, that $|\mathcal{R}(v_K)| = 2$ for type D and that $|\mathcal{R}(v_K)| = 3$ for type E. Given one reduced word any other reduced word can be obtained by a series of braid moves and commutations [4]. If K is type A, B, C, F, or G then s_i and s_{i+1} do not commute for any i. Therefore $s_1 s_2 \cdots s_{n-1} s_n$ is the only reduced word for v_K.

If K is of type D then s_i and s_{i+1} commute if and only if $i = n-1$. Also s_{n-2} and s_n do not commute. The only two reduced words for v_K are $s_1 s_2 \cdots s_{n-2} s_{n-1} s_n$ and $s_1 s_2 \cdots s_{n-2} s_n s_{n-1}$ so $|\mathcal{R}(v_K)| = 2$.

If K is type E_n then we start with the word $v_K = s_1 s_2 s_3 s_4 \cdots s_n$ with the labels given as in Figure 2. The reflection s_2 commutes with s_1 and s_3 but not s_4. The reflection s_3 does not commute with s_1. When $i > 2, s_i$ and s_{i+1} do not commute. Thus v_K has exactly 3 reduced words: $s_1 s_2 s_3 s_4 \cdots s_n$ and $s_1 s_3 s_2 s_4 \cdots s_n$ and $s_2 s_1 s_3 s_4 \cdots s_n$. □

We can now give Giambelli's formula explicitly for all Peterson Schubert classes.

Corollary 5.6 *If $K \subseteq \Delta$ and $K = K_1 \times \cdots \times K_m$ where each K_ℓ is a maximally connected subset then*

$$C_K \cdot p_{v_K} = \prod_{i \in K} p_{s_i} \tag{8}$$

where $C_K = \prod_{\ell=1}^{j} \frac{|K_\ell|!}{|\mathcal{R}(v_{K_\ell})|}$.

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