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A Categorification of One-Variable Polynomials

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Abstract. We develop a diagrammatic categorification of the polynomial ring $\mathbb{Z}[x]$, based on a geometrically-defined graded algebra and show how to lift various operations on polynomials to the categorified setting. Our categorification satisfies a version of the Bernstein-Gelfand-Gelfand reciprocity property, with indecomposable projective modules corresponding to x^n and standard modules to $(x-1)^n$ in the Grothendieck ring. This construction generalizes to categorification of various orthogonal polynomials.

Résumé. Catégorification de l'anneau des polynômes $\mathbb{Z}[x]$

Nous développons une catégorification diagrammatique de l'anneau des polynômes $\mathbb{Z}[x]$, s'appuyant sur une algèbre graduée définie de manière géométrique, et nous décrivons comment on peut relever certaines opérations sur les polynômes dans cette catégorification.

Notre catégorification vérifie une version de la réciprocité de Bernstein-Gelfand-Gelfand, avec les modules projectifs indécomposables correspondants à x^n et les modules standards correspondants à $(x-1)^n$ dans l'anneau de Grothendieck. Cette construction se généralise à certains polynômes orthogonaux.

Keywords: categorification, diagrammatic algebra, Grothendieck ring, Bernstein-Gelfand-Gelfand reciprocity, crossing less matchings,

1 Introduction

Inspired by the general idea of categorification, introduced by L. Crane and I. Frenkel, we construct a categorification of the polynomial ring $\mathbb{Z}[x]$, more precisely of polynomials $(x-1)^n$ that can be generalized to orthogonal one-variable polynomials, including Chebyshev polynomials of the second kind and the Hermite polynomials. In this paper, we interpret the ring $\mathbb{Z}[x]$ as the Grothendieck ring of a suitable additive monoidal category $A-\mathrm{pmod}$ of (finitely generated) projective modules over an idempotented geometrically defined ring A. Monomials x^n become indecomposable projective modules P_n,

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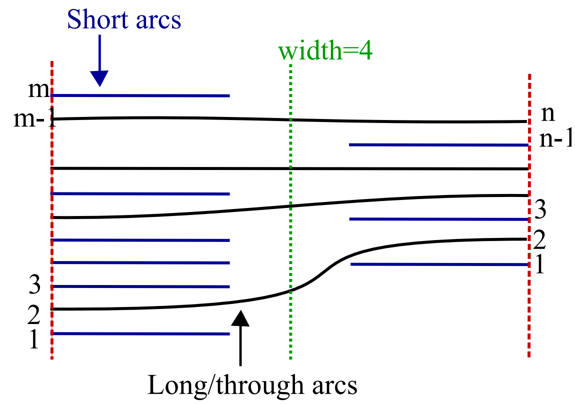

Fig. 1: A diagram in ${}_mB_n$.

while polynomials $(x-1)^m$ turn into so-called standard modules M_m. Ring A has one more distinguished family of modules - simple modules L_n. A remarkable feature of these three collections of modules is the Bernstein–Gelfand–Gelfand (or BGG) reciprocity property [BG76]. Projective modules P_n have a filtration by standard modules M_m, for $m \leq n$, and the multiplicities satisfy the relation:

$$[P_n : M_m] = [M_m : L_n].$$

Original examples of algebras and modules with this property are due to J. Bernstein, I. Gelfand, and S. Gelfand and come up in infinite-dimensional representation theory of simple Lie algebras. The algebra A has a purely topological-geometric definition, yet satisfies the BGG property. Moreover, the standard modules M_n have a clear geometric interpretation. An additional sophistication appears due to non-unitality of algebras A. Instead, they contain an infinite collection of idempotents 1_n, $n \geq 0$, serving as a substitute for the unit element 1. Projectives P_n and standard modules M_n are infinite-dimensional, and the multiplicity $[M_m : L_n]$ should be understood in the generalized sense, as $\dim(1_n M_m)$. We hope that our approach will lead to geometric interpretation of the BGG reciprocity in many other cases, including the ones considered by J. Bernstein, I. Gelfand, and S. Gelfand. In the sequel we will generalize this constructions to categorify the Hermite and Chebyshev polynomials.

2 Diagrammatic algebra

Denote by ${}_mB_n$ the set of isotopy classes of planar diagrams (see Fig. 1) which connect k out of m points on the line $x = 0$ to k out of n points on the line $x = 1$ by k arcs called larcs (long arcs), $k \leq \min(n, m)$. The remaining $m - k$ left and $n - k$ right points extend to *short arcs* or *sarcs*, with one endpoint on either line $x = 0$ or $x = 1$ and the other in the interior of the strip $0 < x < 1$. We require that the projection of the resulting 1-manifold onto the x-axis has no critical points. The number of larcs k is called the *width* of the diagram. Let ${}_mB_n(k)$ and ${}_mB_n(\leq k)$ denote the subsets of diagrams in ${}_mB_n$ of width k and less than or equal to k, respectively.

The set ${}_mB_n$ has cardinality $\binom{n+m}{n}$. Let $B_n = \coprod_{n\geq 0} {}_mB_n$ and $B = \coprod_{n,m\geq 0} {}_mB_n$. Given a field $\mathbf{k}$,

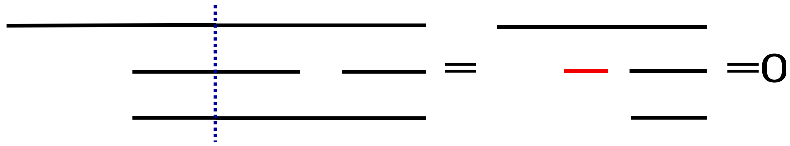

Fig. 2: Concatenation of these two diagrams equals zero since the resulting diagram contains a floating arc.

form $\mathbf{k}$-algebra A as a vector space with the basis B and the multiplication generated by the concatenation of elements of B. The product is zero if the resulting diagram has an arc which is not attached to the lines $x = 0$ or $x = 1$, called *floating* arc, Fig. 2. Also, if $y \in {}_mB_n$, $z \in {}_kB_l$ and $n \neq k$, then the concatenation is not defined and we set $yz = 0$. Thus, for any two elements y, z of B the product yz is either 0 or an element of B.

The composition induces an associative $\mathbf{k}$-algebra structure on A. For each n there exists a unique diagram in ${}_nB_n$ without sarcs. We denote this diagram and its image in A by 1_n. These elements are minimal idempotents in A.

We have $A = \bigoplus_{n,m \geq 0} {}_nA_m$, where ${}_nA_m$ is the vector space with the basis ${}_nB_m$. A is a non-unital associative algebra with a system of mutually orthogonal idempotents $\{1_n\}_{n \geq 0}$. We consider left modules M over A with the property

$$M = \oplus_{n \geq 0} 1_n M.$$

This property is analogous to the unitality condition $1M = M$ for modules over a unital algebra. For a module M, we write M^m for the direct sum of m copies of M.

Let $P_n = A1_n$ be the projective A-module P_n with a basis consisting of all diagrams in B_n. Define M_n, called the *standard module*, as the quotient of P_n by the submodule spanned by all diagrams which have right sarcs. Therefore, a basis of M_n is the set of diagrams in B_n with no right sarcs. In particular, if $1_m M_n \neq 0$ then $m \geq n$. Notice that $b \cdot a = 0$ for any $a \in M_n$ and a diagram $b \in B$ with at least one right sarc, Fig. 3.

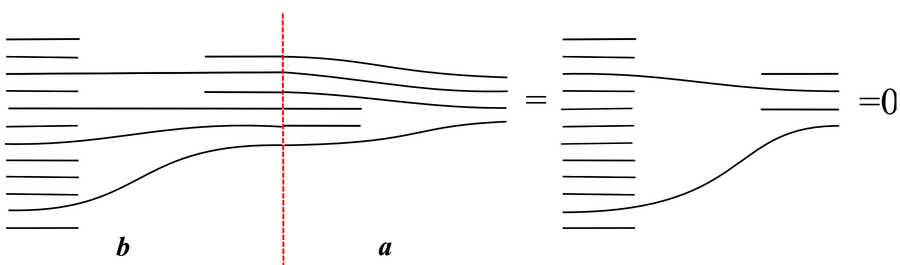

Fig. 3: For any diagram a representing an element of a standard module and a diagram $b \in B$ with right sarcs the product $b \cdot a = 0$.

A left A–module M is called finitely–generated if for some finite subset $\{m_1, m_2, \ldots, m_k\}$ of M we have $M = Am_1 + \cdots + Am_k$. M is finitely generated if and only if it is a quotient of $\bigoplus_{n=0}^{N} P_n^{a_n}$ for some $a_n \geq 0$, $N \in \mathbb{N}$.

Let $A-\mathrm{mod}$ be the category of finitely-generated left A-modules and $A-\mathrm{pmod}$ the category of finitely-generated projective left A-modules.

Let $L_n = \mathbf{k}1_n$ be the one-dimensional module over A on which any element of B other than 1_n acts by zero.

Lemma 2.1 *Any simple A-module is isomorphic to L_n, for some $n \geq 0$.*

Theorem 2.2 *Any finitely–generated projective left A-module P is isomorphic to a finite direct sum of indecomposable projective modules P_n,*

$$P \cong \bigoplus_{n=0}^{N} P_n^{a_n}.$$

The multiplicities $a_n \in \mathbb{Z}_+$ are invariants of P.

The projective module P_n has a filtration by standard modules M_m, over $m \leq n$. Specifically, consider the filtration

$$P_n = P_n(\leq n) \supset P_n(\leq n-1) \supset \cdots \supset P_n(\leq 0) = 0, \tag{1}$$

where $P_n(\leq m)$ is spanned by the diagrams in B_n of width at most m (equivalently, with at least $n-m$ right sarcs). Left multiplication by a basis vector cannot increase the width, hence $P_n(\leq m)$ is a submodule of P_n. The quotient $P_n(\leq m)/P_n(\leq m-1)$ has a basis of diagrams of width exactly m. These diagrams can be partitioned into $\binom{n}{m}$ classes enumerated by positions of the $n-m$ right sarcs. The quotient $P_n(\leq m)/P_n(\leq m-1)$ is isomorphic to the direct sum of $\binom{n}{m}$ copies of the standard module M_m. Consequently, we have an equality in the Grothendieck group of the additive category $A-\mathrm{mod}$:

$$[P_n] = \sum_{m=0}^{n} \binom{n}{m} [M_m]. \tag{2}$$

Next, we prove that the non-unital algebra A is Noetherian, hence the category $A-\mathrm{mod}$ is abelian.

Proposition 2.3 *A submodule of a finitely-generated left A-module is finitely-generated.*

The involution of the set B which reflects a diagram about a vertical axis takes ${}_nB_m$ to ${}_mB_n$ and induces an anti-involution of A. Hence the ring A is right Noetherian as well.

Definition 2.4 *Grothendieck group $K_0(A)$ of finitely generated projective A-modules is an abelian group generated by symbols $[P]$ of finitely-generated projective left A modules P, with defining relations $[P] = [P'] + [P'']$ if $P \cong P' \oplus P''$.*

Proposition 2.5 $K_0(A)$ *is a free abelian group with basis $\{[P_n]\}_{n\geq 0}$.*

Proposition 2.5 follows from Theorem 2.2.

Observe that the existence of the filtration (1) of projective modules P_n by standard modules M_m implies that M_m has a finite projective resolution $P(M_m)$ by P_n's, for $n \leq m$. Consequently, we can view M_m as an object of the category $\mathcal{C}(A{-}\mathrm{pmod})$ of bounded complexes of finitely-generated projective A-modules. Morphisms in this category are homomorphisms of complexes modulo zero-homotopic homomorphisms. Grothendieck groups of categories $A{-}\mathrm{pmod}$ and $\mathcal{C}(A{-}\mathrm{pmod})$ are canonically isomorphic:

$$K_0(\mathcal{C}(A{-}\mathrm{pmod})) \cong K_0(A{-}\mathrm{pmod})$$

via the isomorphism taking the symbol of

$$Q = (\ldots \to P^i \to P^{i+1} \to \ldots) \in \mathcal{C}(A{-}\mathrm{pmod}) \text{ to } [Q] = \sum_{i\in\mathbb{Z}} (-1)^i [P^i] \in K_0(A).$$

Hence, the equality (2) can be interpreted within $K_0(A)$.

The transformation matrix from the basis of the symbols $[P_n]$ of indecomposable projective modules to the basis of symbols $[M_m]$ of standard modules is upper-triangular, with ones on the diagonal and nonzero coefficients being the binomials $\binom{n}{m}$. The entries of the inverse matrix are $(-1)^{n+m}\binom{n}{m}$. Thus we have the following equation in $K_0(A)$:

$$[M_n] = \sum_{m=0}^{n} (-1)^{n+m} \binom{n}{m} [P_m]. \tag{3}$$

We identify the projective Grothendieck group $K_0(A)$ with $\mathbb{Z}[x]$ by sending the symbols of projective modules $[P_n]$ to monomials x^n, and define an inner product on the basis $\{x^n\}_{n\geq 0}$ by

$$(x^n, x^m) = \dim \mathrm{Hom}(P_n, P_m) = |{}_nB_m| = \binom{n+m}{m} \tag{4}$$

This identification can be justified by introducing a monoidal structure on $A{-}\mathrm{pmod}$ under which $P_n \otimes P_m \cong P_{n+m}$, [KS10].

Under this identification, equation (3) gives

$$[M_n] = \sum_{m\leq n} (-1)^{n+m} \binom{n}{m} x^m = (x-1)^n, \tag{5}$$

so the symbols of standard modules $[M_n]$ correspond to $(x-1)^n$.

Equation (3) hints at the existence of a projective resolution of M_n which starts with P_n and has $\binom{n}{m}$ copies of P_m in the $(n-m)$-th position:

$$0 \to P_0 \to \ldots \to P_{n-m}^{\binom{n}{m}} \to \ldots \to P_{n-2}^{\binom{n}{2}} \to P_{n-1}^{\binom{n}{1}} \to P_n \to M_n \to 0 \tag{6}$$

Let us construct this resolution.

Denote the diagram with n larcs and one left sarc at the i-th position by ${}^ib_{n-1} \in {}_nB_{n-1}$. The diagram obtained from ${}^ib_n$ by a reflection along the vertical axis is denoted by $b_n^i \in {}_{n-1}B_n$, Fig. 4. The product of ${}^ib_{n-1}$ or b_n^i with an arbitrary diagram $a \in B$, when defined and non-zero, differs from the diagram a in the following way:

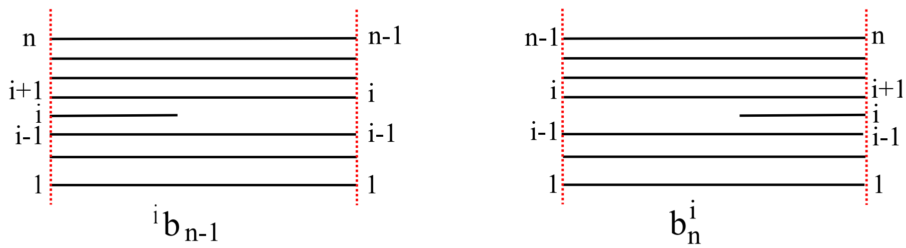

Fig. 4: Diagrams ${}^{i}b_{n-1}$ and b_n^i used in defining differentials in projective resolution of standard modules and resolution of simple by standard modules.

1. $a \cdot {}^{i_j}b_n$ turns i_jth larc in a diagram a into left sarc,
2. ${}^{i_j}b_n \cdot a$ adds left sarc between ith and $i+1$-st larc in a,
3. $a \cdot b_n^{i_j}$ adds right sarc between ith and $i+1$-st larc in a,
4. $b_n^{i_j} \cdot a$ turns i_jth larc in a diagram a into right sarc.

Let $I_m = \{i_1, \dots, i_m\} \subseteq \{1, \dots, n\}$, $i_1 < \cdots < i_m$ be a subset of cardinality $m \leq n$. Label the summands of the m-th term $P_{n-m}^{\binom{n}{m}}$ by these subsets I_m, denoting the summand by $P_{n-m}^{I_m}$. Let $I_{m,l} := I_m \setminus \{i_l\}$. Removing an element i_l of the set I_m can be interpreted as composing a diagram in B_{n-m} on the right with a diagram b_{n-m+1}^p, obtained in the following way. Take a diagram $b_n^{i_l}$ and delete all long arcs at positions labeled by elements in $I_{m,l}$, resulting in a diagram b_{n-m+1}^p, where p denotes the position of i_l in the ordered set $\{1, 2, \dots, n\} \setminus I_m \cup \{i_l\}$, Figures 4 and 5.

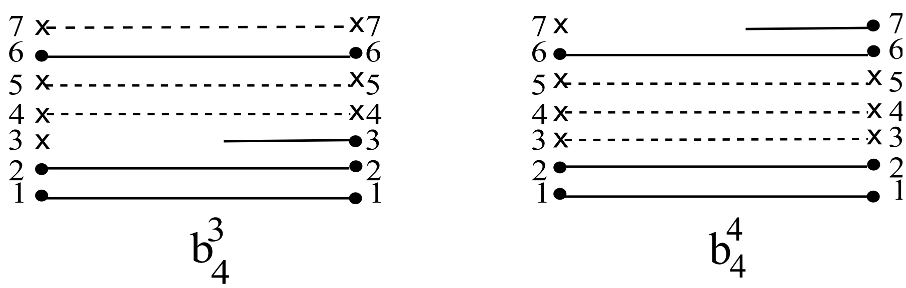

Fig. 5: Differentials $d_{\{3,4,5,7\}}^{+1}$ and $d_{\{3,4,5,7\}}^{+4}$ in the projective resolution of standard module M_7 sending $P_3^{\{3,4,5,7\}}$ into $P_4^{\{4,5,7\}}$ and $P_4^{\{3,4,5\}}$, respectively. They are determined by composing on the right by diagrams b_4^3 and b_4^4 obtained from diagrams b_7^3 and b_7^7 by deleting dashed larcs corresponding to the label sets of $P_4^{\{4,5,7\}}$ and $P_4^{\{3,4,5\}}$.

Next, define the differential

$$d : P_{n-m}^{\binom{n}{m}} \longrightarrow P_{n-(m-1)}^{\binom{n}{m-1}}$$

as the sum

$$d = \sum_{I_m} \sum_{l=1}^{m} d_{I_m,+l}.$$

of maps $d_{I_m,+l} : P^{I_m}_{n-m} \to P^{I_{m,l}}_{n-(m-1)}$ sending $a \in P^{I_m}_{n-m}$ into $d_{I_m,+l}(a) = (-1)^{l-1} a \cdot b^p_{n-m+1}$, For example, Fig.5 shows how to define the differentials $d_{\{1,3,4,5\},+5}$ and $d_{\{1,3,4,5\},+1}$ in the resolution of M_7 sending $P_3^{\{1,3,4,5\}}$ into $P_4^{\{1,3,4\}}$ and $P_4^{\{3,4,5\}}$, respectively.

Proposition 2.6 *The complex (6) with the differential defined above is exact.*

A finite–dimensional A–module M has a finite filtration with simple modules L_n as subquotients. Due to one-dimensionality of L_n the multiplicity of L_n in M, denoted by $[M : L_n]$, equals $\dim 1_n M$. A finitely-generated A–module M is not necessarily finite dimensional but it satisfies the following property $\dim(1_n M) < \infty$, for $n \geq 0$, which we call a locally finite–dimensional property.

For locally finite–dimensional module M we define the multiplicity of L_n in M as:

$$[M : L_n] = \dim(1_n M). \tag{7}$$

This definition is compatible with the usual notion of multiplicity of L_n in M as the number of times L_n appears in the composition series of M when M is finite–dimensional.

Let us now specialize to standard modules M_m. We have

$$[M_m : L_n] = \dim(1_n M_m) = \begin{cases} \binom{n}{m}, & \text{for } n \geq m; \\ 0, & \text{if } n < m. \end{cases} \tag{8}$$

Recall that $[P_n : M_m] = \binom{n}{m}$, hence

$$[P_n : M_m] = [M_m : L_n]. \tag{9}$$

Thus, our diagrammatically defined algebra possesses the Bernstein–Gelfand–Gelfand (BGG) reciprocity property. Indecomposable projective modules P_n have filtration by standard modules M_m, with $m \leq n$ and $[P_n : M_n] = 1$. The multiplicity in the RHS in the equality (9) is understood in the generalized sense, as explained above.

Proposition 2.7 *Homological dimension of slarc algebra standard module M_n is n.*

For detailed construction of a resolution of a simple module L_k by standard modules M_m for $m \geq k$:

$$\xrightarrow{d} M_{k+m}^{\binom{k+m}{m}} \xrightarrow{d} \cdots \xrightarrow{d} M_{k+2}^{\binom{k+2}{2}} \xrightarrow{d} M_{k+1}^{\binom{k+1}{1}} \xrightarrow{d} M_k \xrightarrow{d} L_k \longrightarrow 0. \tag{10}$$

Notice that the m-th term of the resolution is a direct sum $M_{k+m}^{\binom{k+m}{m}}$ of standard modules M_{k+m}. On the level of diagrams, multiplicity $\binom{k+m}{m}$ represents the number of ways to add m right sarcs to a diagram in M_k to obtain a diagram in M_{k+m}. Let $I_m = \{i_1, i_2, \ldots, i_m\} \subseteq \{1, 2, \ldots, k+m\}$ be the set describing positions of added larcs. Each summand $M_{k+m}^{I_m}$ is labeled by one of these subsets, and the differential will take summand labeled by I_m into summands labeled by $I_{m,-l}$, for $0 < l \leq m$, by composing on the right with diagrams containing a single short right arc and no left sarcs, see Fig. 4.

Informally, on the level of Grothendieck groups we have the following relation:

$$\begin{aligned} [L_n] &= \sum_{k=0}^{\infty} (-1)^k \binom{n+k}{k} [M_{n+k}] \\ &= \sum_{k=0}^{\infty} (-1)^k \binom{n+k}{k} (x-1)^{n+k} = \frac{(x-1)^n}{x^{n+1}}. \end{aligned}$$

This infinite sum does not play a direct role in our categorification, but we can obtain projective resolution of a simple module L_n via constructing a bicomplex, with a projective resolution (6) of M_{n+k}, $k \geq 0$ lying above each copy of a standard module in the resolution (10) of simple modules L_n by standard modules M_m, $m \geq n$.

This construction can be summarized by saying that suitably defined Grothendieck ring of category of bounded complexes of projective modules is isomorphic to the one-variable polynomial ring $\mathbb{Z}[x]$ with the following correspondence:

$$[P_n] = \sum_{m=0}^{n} \binom{n}{m} [M_m] \quad \leftrightarrow \quad x^n = \sum_{m=0}^{n} \binom{n}{m} (x-1)^m$$

$$[M_n] = \sum_{m \leq n} (-1)^{n+m} \binom{n}{m} [P_m] \quad \leftrightarrow \quad (x-1)^n = \sum_{m \leq n} (-1)^{n+m} \binom{n}{m} x^m$$

for more details and the background see [Ben91], [Wei94], [GM96], [Mil], and [KS10]. This construction can be generalized to the categorification of various orthogonal polynomials by varying the underlying diagrammatic algebra.

3 Approximations of the identity

Various operations on polynomials can be lifted to the categorified setting. In this section we define a functor which corresponds to the truncation of polynomials on the level of Grothendieck group. Recall that $B(\leq k) = \bigsqcup_{i=0}^{k} B(i)$ denotes diagrams in B of width less than or equal to k. Let $A(\leq k)$, $k \geq 0$ denote the subspace of A spanned by diagrams in $B(\leq k)$. This subspace is an A–subbimodule of A. Let $A(k)$ be the quotient subbimodule $A(\leq k)/A(\leq k-1)$. Let ${}_nP$ denote a right projective module ${}_nP = 1_n A$ and, analogously to the standard modules M_n, let ${}_nM$ be the quotient of ${}_nP$ by the submodule spanned by all diagrams with a left sarcs. One can think of diagrams of ${}_nM$ as reflections along vertical axis of diagrams in M_n.

Proposition 3.1 *$A(\leq k)/A(\leq k-1) \cong M_k \otimes_{\mathbf{k}} {}_kM$ as A-bimodules (Fig. 6).*

For a given $k \geq 0$, define a right exact functor $F_k : A\!-\!\mathrm{mod} \to A\!-\!\mathrm{mod}$ by

$$F_k(M) = A(\leq k) \otimes_A M, \tag{11}$$

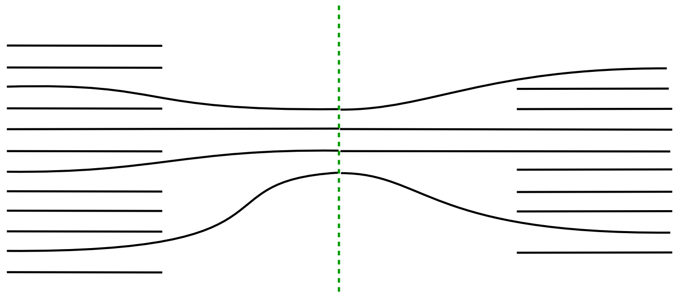

Fig. 6: Diagram in $B(4)$ viewed as a product of elements in M_4 and ${}_4M$.

for an A-module M. The image of the standard module M_m under functor F_k is:

$$A(\leq k) \otimes_A M_m = \begin{cases} M_m, & \text{if } k \geq m; \\ 0, & \text{otherwise.} \end{cases} \tag{12}$$

By definition $P_m = A1_m$, hence $A(\leq k) \otimes_A P_m = A(\leq k) \otimes_A A1_m = A(\leq k)1_m$, and this is a submodule of P_m spanned by diagrams of width less than or equal to k:

$$F_k(P_m) = A(\leq k) \otimes_A P_m = \begin{cases} P_m, & \text{if } k \geq m; \\ P_m(\leq k), & \text{if } k < m. \end{cases} \tag{13}$$

In the Grothendieck group, projective modules P_n correspond to x^n and standard modules M_n to $(x-1)^n$. Modules $P_n(\leq k)$ have finite homological dimension, since they admit finite filtrations with successive quotients isomorphic to standard modules. Therefore, functor F_k descends to an operator on the Grothendieck group $K_0(A)$, denoted by $[F_k]$. The action of $[F_k]$ on $[P_n] = \sum_{m=0}^{n} \binom{n}{m} [M_m]$ is equal to:

$$[F_k][P_n] = \begin{cases} [P_n] = x^n, & \text{if } k \geq n; \\ \sum_{m=0}^{k} \binom{n}{m} [M_m] = \sum_{m=0}^{k} \binom{n}{m} (x-1)^m, & \text{if } k < n. \end{cases} \tag{14}$$

Since all higher derived functors of F_k are zero on standard modules, for $k \geq n$ operator $[F_k]$ acts via identity on $[P_n]$, and for $k < n$ it approximates identity and can be viewed as taking the first $k+1$ terms $\sum_{m=0}^{k} \binom{n}{m} [M_m]$ in the expansion of $[P_n]$ in the basis $\{(x-1)^m\}_{m \geq 0}$.

4 Cabling functors

For every A–module M and a positive integer k construct the corresponding cabled module ${}^{[k]}M$ in the following way:

$$1_n {}^{[k]}M = 1_{nk}M, \text{ hence } {}^{[k]}M = \bigoplus_{n \geq 0} 1_{nk}M. \tag{15}$$

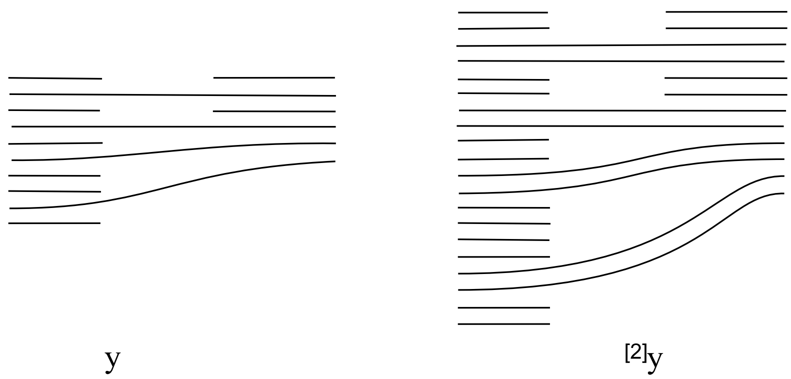

Fig. 7: A diagram $y \in {}_{11}B_6$ and 2-cable ${}^{[2]}y \in {}_{22}B_{12}$.

Given a diagram $y \in {}_sB_l$, construct a diagram ${}^{[k]}y \in {}_{sk}B_{lk}$, called the k-cabling of y, by taking k parallel copies of each arc (Fig. 7). For example, ${}^{[k]}1_n = 1_{nk}$. By definition, the action of an element $\alpha \in A$ on ${}^{[k]}M_n$ is the regular action of its k-cabling α^k.

What is the result of k-cabling simple, standard and projective modules? It is easy to see that, if k divides n, the k-cabling of the simple module L_n is the module $L_{n/k}$:

$$1_m{}^{[k]}L_n = 1_{km}L_n = \begin{cases} \mathbf{k}, & \text{if } km = n; \\ 0, & \text{otherwise.} \end{cases} \tag{16}$$

If k does not divide n the result is zero, ${}^{[k]}L_n = 0$.

Recall that basis elements of standard A modules M_n correspond to diagrams in B_n with n through arcs and an arbitrary number of left sarcs. Let $S(n,k,i)$ denote the number of ways to select n numbers between 1 and ki such that each of the sets $\{kj+1,\ldots,k(j+1)\}_{0\le j<i}$ contains at least one of the selected numbers.

Proposition 4.1

$$^{[k]}M_n \cong \bigoplus_{i=\lceil \frac{n}{k}\rceil}^{n} M_i^{S(n,k,i)} \tag{17}$$

Proof: The proof is left to the reader following examples shown on Fig. 8. $S(n,k,i)$ is the sum of products $\prod_{j=1}^{i}\binom{k}{\lambda_j}$, over all possible partitions $\lambda = (\lambda_1,\ldots,\lambda_i)$ of n into i blocks of length at most k. □

Example 4.2 *We compute cabling modules of M_n for small values of n:*

- ${}^{[k]}M_0 = M_0$, ${}^{[k]}M_1 = M_1^k$,
- ${}^{[k]}M_2 = M_2^{k^2} \oplus M_1^{\binom{k}{2}}$,

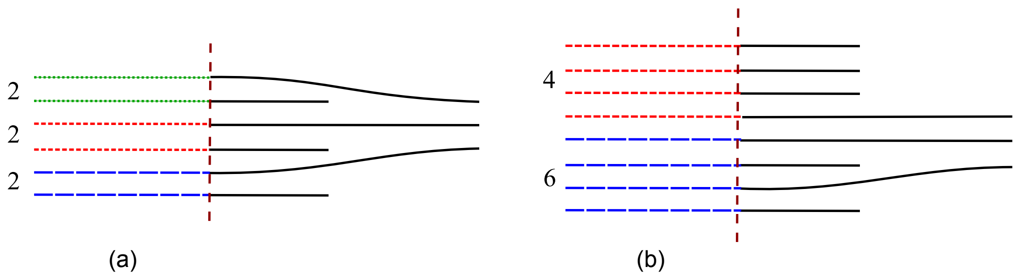

Fig. 8: (a) 2-cabling of M_3; (b) 4-cabling of M_3 corresponding to the partition $(2,1)$: 2 arcs in the same part contribute 6 hence, the total contribution is 24.

- ${}^{[k]}M_3 = M_3^{k^3} \oplus M_2^{2\binom{k}{1}\binom{k}{2}} \oplus M_1^{\binom{k}{3}}$.

Studying cablings of projective modules reduces to the case of standard modules: ${}^{[k]}P_n$ has a filtration with the i-th term consisting of $\binom{n}{i}\,{}^{[k]}M_i$, based on the filtration (1) of P_n by $P_n(i)$, $i \leq n$.

Cabling functor ${}^{[k]}$ is exact, sending an A-module M to its k-cabled module ${}^{[k]}M$, and categorifies the following operator on the Grothendieck group:

$$[M_n] = (x-1)^n \mapsto [{}^{[k]}M_n] = \sum_{i=\lceil \frac{n}{k} \rceil}^{n} S(n,k,i)(x-1)^i. \tag{18}$$

In the following paper we will further explore the structure and functors this categorification admits. For example, notice that ${}^{[s][k]}M \cong {}^{[ks]}M$ functorially in M. This identity indicates that the cabling operator may correspond to a certain plethysm on the polynomial ring. Moreover, we will generalize this construction using a slightly different diagrammatics to category certain classes of orthogonal polynomials, such as Chebyshev and Hermite.

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