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The Bruhat order on conjugation-invariant sets of involutions in the symmetric group

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Abstract. Let I_n be the set of involutions in the symmetric group S_n, and for $A \subseteq \{0, 1, \ldots, n\}$, let

$$F_n^A = \{\sigma \in I_n \mid \sigma \text{ has } a \text{ fixed points for some } a \in A\}.$$

We give a complete characterisation of the sets A for which F_n^A, with the order induced by the Bruhat order on S_n, is a graded poset. In particular, we prove that $F_n^{\{1\}}$ (i.e., the set of involutions with exactly one fixed point) is graded, which settles a conjecture of Hultman in the affirmative. When F_n^A is graded, we give its rank function. We also give a short new proof of the EL-shellability of $F_n^{\{0\}}$ (i.e., the set of fixed point-free involutions), which was recently proved by Can, Cherniavsky, and Twelbeck.

Résumé. Soit I_n l'ensemble d'involutions dans le groupe symétrique S_n, et pour $A \subseteq \{0, 1, \ldots, n\}$, soit

$$F_n^A = \{\sigma \in I_n \mid \sigma \text{ a } a \text{ points fixes pour quelque } a \in A\}.$$

Nous caractérisons tous les ensembles A dont les F_n^A, avec l'ordre induit par l'ordre de Bruhat sur S_n, est un poset gradué. En particulier, nous démontrons que $F_n^{\{1\}}$ (c'est-à-dire, l'ensemble d'involutions avec précis en point fixe) est gradué, ce qui résout une conjecture d'Hultman à l'affirmative. Lorsque F_n^A est gradué, nous donnons sa fonction de rang. En plus, nous donnons une nouvelle démonstration courte l'EL-shellability de $F_n^{\{0\}}$ (c'est-à-dire, l'ensemble d'involutions sans points fixes), établie récemment par Can, Cherniavsky et Twelbeck.

Keywords: Bruhat order, symmetric group, involution, conjugacy class, graded poset, EL-shellability

1 Introduction

Partially ordered by the Bruhat order, the symmetric group S_n is a graded poset whose rank function is given by the number of inversions, and Edelman [4] proved that it is EL-shellable. Richardson and Springer [10] proved that the set I_n of involutions in S_n and the set F_n^0 of fixed point-free involutions are graded. Incitti [9] proved that the rank function of I_n can be expressed as the average of the number of inversions and the number of exceedances, and that I_n is EL-shellable. Hultman [8] studied (in a more general setting, which we shall describe shortly) F_n^0 and F_n^1, the set of involutions with exactly one

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fixed point. It follows that F_n^0 is graded and Hultman conjectured that the same is true for F_n^1. Can, Cherniavsky, and Twelbeck [3] recently proved that F_n^0 is EL-shellable.

We consider the following generalisation. For $a \in \{0, 1, \ldots, n\}$, let F_n^a be the conjugacy class in S_n consisting of the involutions with a fixed points, and for $A \subseteq \{0, 1, \ldots, n\}$, let

$$F_n^A = \bigcup_{a \in A} F_n^a.$$

Both I_n and F_n^A are regarded as posets with the order induced by the Bruhat order on S_n. Note that

$$F_n^A = \{\sigma \in I_n \mid \sigma \text{ has } a \text{ fixed points for some } a \in A\}.$$

Also note that for all elements in I_n, the number of fixed points is congruent to n modulo 2. Hence, we may assume that all members of A have the same parity as n.

Depicted in Figures 1 and 2, are the Hasse diagrams of I_4, F_4^0, and F_4^2.

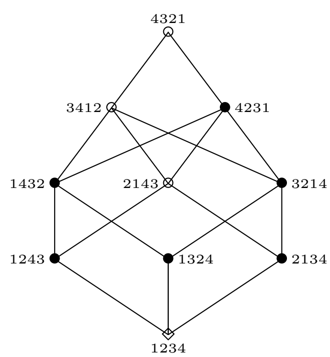

Figure 1: Hasse diagram of I_4 with the involutions with zero ($\circ$), two ($\bullet$), and four ($\diamond$) fixed points indicated.

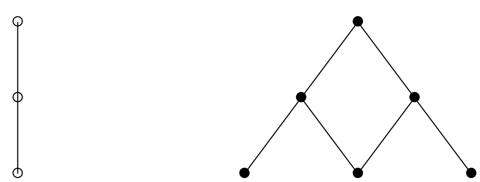

Figure 2: Hasse diagrams of F_4^0 (left) and F_4^2 (right).

Our main result is a complete characterisation of the sets A for which F_n^A is graded. In particular, we prove that F_n^1 is graded.

Informally, F_n^A is graded precisely when $A - \{n\}$ is empty or an "interval," which may consist of a single element if it is 0, 1, or $n - 2$. The following theorem, which is our main result, makes the above precise. It also gives the rank function of F_n^A when it exists.

Theorem 1 *The poset F_n^A is graded if and only if $A - \{n\} = \emptyset$ or $A - \{n\} = \{a_1, a_1 + 2, \ldots, a_2\}$ with $a_1 \in \{0, 1\}$, $a_2 = n - 2$, or $a_2 - a_1 \geq 2$. Furthermore, when F_n^A is graded, its rank function ρ is given by*

$$\rho(\sigma) = \frac{\mathrm{inv}(\sigma) + \mathrm{exc}(\sigma) - n + \tilde{a}}{2} + \begin{cases} 1 & \text{if } n \in A \\ 0 & \text{otherwise,} \end{cases}$$

where $\mathrm{inv}(\sigma)$ and $\mathrm{exc}(\sigma)$ denote the number of inversions and exceedances, respectively, of σ, and $\tilde{a} = \max(A - \{n\})$. In particular, F_n^A has rank

$$\rho(F_n^A) = \frac{n^2 - a^2 - 2n + 2\tilde{a}}{4} + \begin{cases} 1 & \text{if } n \in A \\ 0 & \text{otherwise,} \end{cases}$$

where $a = \min A$.

The following result is direct consequence of Theorem 1.

Corollary 2 *The posets F_n^0, F_n^1, F_n^{n-2}, and F_n^n are the only graded conjugacy classes of involutions in S_n. Furthermore, the rank function ρ of F_n^0 and F_n^1 is given by*

$$\rho(\sigma) = \frac{\mathrm{inv}(\sigma) - \lfloor n/2 \rfloor}{2},$$

and the rank function ρ of F_n^{n-2} is given by

$$\rho(\sigma) = \frac{\mathrm{inv}(\sigma) - 1}{2}.$$

It is well known that F_n^{n-2} is graded (in fact, it coincides with the root poset of the Weyl group $A_{n-1} \cong S_n$). As was mentioned above, the gradedness of F_n^0 was proved by Richardson and Springer, and that of F_n^1 was conjectured by Hultman. These two posets are special cases of a more general construction from Hultman's paper [8], which we now describe.[(i)]

Given a finitely generated Coxeter system (W, S) and an involutive automorphism θ of (W, S) (i.e., a group automorphism θ of W such that $\theta(S) = S$ and $\theta^2 = \mathrm{id}$), let

$$\iota(\theta) = \{\theta(w^{-1})w \mid w \in W\}$$

and

$$\mathfrak{I}(\theta) = \{w \in W \mid \theta(w) = w^{-1}\}$$

be the sets of *twisted identities* and *twisted involutions*, respectively. Clearly, $\iota(\theta) \subseteq \mathfrak{I}(\theta) \subseteq W$. Note that when $\theta = \mathrm{id}$, $\iota(\theta)$ and $\mathfrak{I}(\theta)$ reduce to the sets of the (ordinary) identity and (ordinary) involutions in W. Each subset of W is regarded as a poset with the order induced by the Bruhat order on W. When W is the symmetric group S_n, there is a unique non-trivial automorphism of (W, S), mapping $s_i = (i, i+1)$ to s_{n-i}.

(i) The results below are taken from [6, 7, 8]. In general, we do not indicate which results are from which paper. For general Coxeter group terminology and results, see [2].

We say that θ has the *no odd flip property* if the order of $s\theta(s)$ is even or infinite for all $s \in S$ with $s \neq \theta(s)$. If W is finite and irreducible, then θ has the no odd flip property, unless W is of type $A_{2n} \cong S_{2n+1}$ or $I_2(2n+1)$ for some $n \geq 1$, and θ is the unique non-trivial automorphism. The poset $\mathfrak{I}(\theta)$ is always graded. Furthermore, we have the following result, from which it follows that F_n^0 is graded, as we shall see.

Theorem A ([8, Theorem 4.6 and Proposition 6.7]) *If θ has the no odd flip property, then $\iota(\theta)$ is graded with the same rank function as $\mathfrak{I}(\theta)$.*

If W is finite, it contains a greatest element w_0, and $\theta(w) = w_0ww_0$ defines an involutive automorphism of (W, S). Since $\iota(\theta) = \{w_0w^{-1}w_0w \mid w \in W\}$ and $\mathfrak{I}(\theta) = \{w \in W \mid w_0ww_0 = w^{-1}\}$,

$$w_0 \cdot \iota(\theta) = \{w^{-1}w_0w \mid w \in W\}$$

and

$$w_0 \cdot \mathfrak{I}(\theta) = \{w_0w \mid w_0ww_0 = w^{-1}\} = \{w_0w \mid (w_0w)^2 = e\}.$$

Since left (as well as right) multiplication by w_0 is a poset anti-automorphism (i.e., an order-reversing bijection whose inverse is order-reversing), $\iota(\theta)$ is isomorphic to the dual of $[w_0]$, where $[w_0]$ is the conjugacy class of w_0, and $\mathfrak{I}(\theta)$ is isomorphic to the dual of $I(W)$, where $I(W)$ is the set of involutions in W.

When W is the symmetric group S_n, this θ is the unique non-trivial automorphism of (W, S), and $I(W) = I_n$. For n even, $[w_0] = F_n^0$, and for n odd, $[w_0] = F_n^1$. Thus, it follows from Theorem A that F_n^0 is graded.

It was conjectured by Hultman [8, Conjecture 6.1] that $\iota(\theta)$ is graded when $W = A_{2n}$. As we have seen, this is equivalent to F_n^1 being graded, which is the case (see Corollary 2). Since $\iota(\theta)$ is graded whenever W is dihedral, as is easily seen, it therefore follows that $\iota(\theta)$ is graded whenever W is finite and irreducible. From this, we get the following (we omit the proof):

Theorem 3 *If W is finite, then $\iota(\theta)$ is graded.*

Let us also mention a connection to work by Richardson and Springer [10, 11], who studied a graded poset V of orbits of certain symmetric varieties (depending on, inter alia, a group G). They did so by defining an order-preserving function $\varphi : V \to \mathfrak{I}(\theta) \subseteq W$ (where the Weyl group W depends on, inter alia, G).

When $W = S_n$, $\mathfrak{I}(\theta)$ is the image of an injective φ (for details, see [10, Example 10.2]). When n is even, the same is true for $\iota(\theta)$ (see [10, Example 10.4] or [8, Example 3.1]). Thus, I_n and F_n^0 are isomorphic to the duals of such posets V. Hence, I_n and F_n^0 are graded.

However, these are not the only F_n^A that occur as the image of a φ. To describe these sets, and for later purposes, define

$$F_n^{\leq a} = \bigcup_{i \geq 0} F_n^{a-2i} \quad \text{and} \quad F_n^{\geq a} = \bigcup_{i \geq 0} F_n^{a+2i},$$

and for $a_2 = a_1 + 2m$, where m is a positive integer, let

$$F_n^{a_1:a_2} = F_n^{\geq a_1} \cap F_n^{\leq a_2}.$$

As described in [10], the image of φ can be read off from the corresponding Satake diagram. It follows from Satake diagrams A III and A IV in Helgason [5, Table VI] that for each $a \le n-2$, $F_n^{\ge a}$ is the image of a φ. (From Satake diagrams A I and A II, it follows that $\mathfrak{I}(\theta)$ and $\iota(\theta)$, respectively, are the images of such functions).

The remainder of this extended abstract is organised as follows. In Section 2, we agree on notation and gather the necessary definitions and previous results. Then, in Section 3, we sketch the proof of our main result (Theorem 1). Finally, in Section 4, we give a short new proof of the following result, which was recently proved by Can, Cherniavsky, and Twelbeck.

Theorem B ([3, Theorem 1]) *The poset F_n^0 is EL-shellable.*

2 Notation and preliminaries

Poset notation and terminology will follow [12]. In particular, if P is a poset and $x \le y$ in P, then $[x,y] = \{z \in P \mid x \le z \le y\}$ and $(x,y) = \{z \in P \mid x < z < y\}$. Furthermore, in a finite poset P, $\lhd$ denotes the covering relation, a chain $x_0 < x_1 < \cdots < x_k$ is *saturated* if $x_{i-1} \lhd x_i$ for all $i \in [k]$, P is *bounded* if it has a minimum (denoted by $\hat{0}$) and a maximum (denoted by $\hat{1}$), and P is *graded of rank n* if every maximal chain has length n. In this case, there is a unique *rank function* $\rho : P \to \{0,1,\dots,n\}$ such that $\rho(x) = 0$ if x is a minimal element of P, and $\rho(y) = \rho(x) + 1$ if $x \lhd y$ in P; x has *rank i* if $\rho(x) = i$. An *x-y-chain* is a *saturated* chain from x to y.

Let P be a finite, bounded, and graded poset. An *edge-labelling* of P is a function $\lambda : \{(x,y) \in P^2 \mid x \lhd y\} \to Q$, where Q is a totally ordered set. If λ is an edge-labelling of P and $x_0 \lhd x_1 \lhd \cdots \lhd x_k$ is a saturated chain, let $\lambda(x_0, x_1, \dots, x_k) = (\lambda(x_0,x_1), \lambda(x_1,x_2), \dots, \lambda(x_{k-1},x_k))$. The chain is said to be *increasing* if $\lambda(x_{i-1},x_i) \le \lambda(x_i,x_{i+1})$ for all $i \in [k-1]$, and *decreasing* if $\lambda(x_{i-1},x_i) > \lambda(x_i,x_{i+1})$ for all $i \in [k-1]$. An edge-labelling λ of P is an *EL-labelling* if, for all $x < y$ in P, there is exactly one increasing x-y-chain, say $x_0 \lhd x_1 \lhd \cdots \lhd x_k$, and this chain is *lexicographically minimal*, or *lex-minimal*, among the x-y-chains in P (i.e., if $y_0 \lhd y_1 \lhd \cdots \lhd y_k$ is any other x-y-chain, then $\lambda(x_{j-1},x_j) < \lambda(y_{j-1},y_j)$, where $j = \min\{i \in [k] \mid \lambda(x_{i-1},x_i) \ne \lambda(y_{i-1},y_i)\}$; this is known as the *lexicographic order*). If P has an EL-labelling, P is said to be *EL-shellable*. The reason for this is the following result, due to Björner.

Theorem C ([1, Theorem 2.3]) *Let P be a finite, bounded, and graded poset. If P is EL-shellable, then its order complex $\Delta(P)$ is shellable.*

For $\sigma \in S_n$ and $(k,l) \in [n]^2$, let $\sigma[k,l] = |\{i \le k \mid \sigma(i) \ge l\}|$. The Bruhat order on S_n may be defined as follows (see, e.g., [2, Theorem 2.1.5]):

Definition 4 *Let $\sigma, \tau \in S_n$. Then $\sigma \le \tau$ if and only if $\sigma[k,l] \le \tau[k,l]$ for all $(k,l) \in [n]^2$.*

Let us turn to involutions in the symmetric group. Here, notation will follow [9].

Let $\sigma \in S_n$. A *rise* of σ is a pair $(i,j) \in [n]^2$ such that $i < j$ and $\sigma(i) < \sigma(j)$. A rise (i,j) is called *free* if there is no $i < k < j$ such that $\sigma(i) < \sigma(k) < \sigma(j)$. An *inversion* is a pair $(i,j) \in [n]^2$ such that $i < j$ and $\sigma(i) > \sigma(j)$. An element $i \in [n]$ is a *fixed point* of σ if $\sigma(i) = i$, an *exceedance* if $\sigma(i) > i$, and a *deficiency* if $\sigma(i) < i$. Let $\mathrm{inv}(\sigma)$ and $\mathrm{exc}(\sigma)$ denote the number of inversions and exceedances, respectively, of σ.

Let $\sigma \in I_n$. A free rise (i,j) of σ is *suitable* if it is an ff-rise (Type 1), an fe-rise (Type 2), an ef-rise (Type 3), a non-crossing ee-rise (Type 4), a crossing ee-rise (Type 5), or an ed-rise (Type 6). Here fe, e.g., means that i is a fixed point of σ while j is an exceedance, and an ee-rise is *crossing* if $\sigma(i) < j$ and *non-crossing* otherwise.

The following definition is very important.

Definition 5 *Let $\sigma \in I_n$ and let (i,j) be a suitable rise of σ. We define a new involution $\mathrm{ct}_{(i,j)}(\sigma)$ as follows:*

If (i,j) is of Type 1, then $\mathrm{ct}_{(i,j)}(\sigma) = \sigma(i,j)$.
If (i,j) is of Type 2, then $\mathrm{ct}_{(i,j)}(\sigma) = \sigma(i,j,\sigma(j))$.
If (i,j) is of Type 3, then $\mathrm{ct}_{(i,j)}(\sigma) = \sigma(i,j,\sigma(i))$.
If (i,j) is of Type 4, then $\mathrm{ct}_{(i,j)}(\sigma) = \sigma(i,j)(\sigma(i),\sigma(j))$.
If (i,j) is of Type 5, then $\mathrm{ct}_{(i,j)}(\sigma) = \sigma(i,j,\sigma(j),\sigma(i))$.
If (i,j) is of Type 6, then $\mathrm{ct}_{(i,j)}(\sigma) = \sigma(i,j)(\sigma(i),\sigma(j))$.

See [9, Table 1] for pictures describing the action of $\mathrm{ct}_{(i,j)}$ on the diagram of σ.

Incitti characterised the covering relation in I_n as follows.

Lemma 6 ([9, Theorem 5.1]) *Let $\sigma,\tau \in I_n$. Then $\sigma \lhd \tau$ in I_n if and only if $\tau = \mathrm{ct}_{(i,j)}(\sigma)$ for some (necessarily unique) suitable rise (i,j) of σ.*

If $\tau = \mathrm{ct}_{(i,j)}(\sigma)$ for some suitable rise (i,j) of σ, let $\lambda(\sigma,\tau) = (i,j)$. By Lemma 6, this defines an edge-labelling of I_n (with $\{(i,j) \in [n]^2 \mid i < j\}$ totally ordered by the lexicographic order, i.e., $(i_1,j_1) < (i_2,j_2)$ if and only if $i_1 < i_2$, or $i_1 = i_2$ and $j_1 < j_2$). Whenever we consider an edge-labelling of I_n, it is this one. If $\lambda(\sigma,\tau) = (i,j)$, then (i,j) is the *label* on the cover $\sigma \lhd \tau$; (i,j) is a label on a chain if it is the label on some cover of the chain.

Let $\tau \in I_n$ and let (i,j) be an inversion of τ. If (i,j) is a suitable rise of some $\sigma \in I_n$ and $\mathrm{ct}_{(i,j)}(\sigma) = \tau$, then σ is unique, and we write $\sigma = \mathrm{ict}_{(i,j)}(\tau)$.

For $\sigma < \tau$ in I_n, let $\mathrm{di}(\sigma,\tau) = \min\{i \in [n] \mid \sigma(i) \neq \tau(i)\}$.

We shall need the following results, due to Incitti:

Lemma 7 ([9, Theorem 5.2]) *The poset I_n is graded with rank function ρ given by*

$$\rho(\sigma) = \frac{\mathrm{inv}(\sigma) + \mathrm{exc}(\sigma)}{2}.$$

Lemma 8 ([9, Theorem 6.2]) *Let $\sigma < \tau$ in I_n. Then there is exactly one increasing σ-τ-chain, and it is lex-minimal.*

Lemma 9 ([9, Theorem 7.3]) *Let $\sigma < \tau$ in I_n. Then there is exactly one decreasing σ-τ-chain.*

Remark 1 *Since $\mathrm{ct}_{(i,j)}(\sigma)(i) > \mathrm{ct}_{(i,j)}(\sigma)(j)$, there is also exactly one "weakly" decreasing σ-τ-chain. This fact is used in Section 4.*

3 Sketch of the proof of the main result

In this section, we prove, sketch the proofs of, or simply state, a number of lemmas and propositions, from which Theorem 1 easily follows.

The strategy for proving that a poset F_n^A is graded is as follows. We first prove that F_n^A has a maximum and that all its minimal elements have the same rank in I_n (see Propositions 11 and 12). We then prove that if $\sigma, \tau \in F_n^A$, then $\sigma \lhd \tau$ in F_n^A if and only if $\sigma \lhd \tau$ in I_n (one implication is obvious). This is done in Lemmas 15, 16, and 17. Since I_n is graded, it thus follows that F_n^A is graded.

In particular, when $F_n^A \in \{F_n^{\leq a}, F_n^{\geq a}\}$, to prove that $\sigma \lhd \tau$ in I_n if $\sigma \lhd \tau$ in F_n^A, we assume that $\sigma \not\lhd \tau$ in I_n, and consider the increasing and the decreasing σ-τ-chains in I_n. We then prove that either the element in the increasing chain that covers σ, or the element in the decreasing chain that is covered by τ, has to belong to F_n^A. This contradicts the fact that $\sigma \lhd \tau$ in F_n^A.

To prove that a poset F_n^A is not graded, we consider an interval $[\sigma, \tau]$, and then construct two σ-τ-chains in F_n^A of different lengths (see Propositions 19 and 20).

Let us first note the following fact:

Lemma 10 *For all n and all A, F_n^A is graded if and only if $F_n^{A-\{n\}}$ is graded.*

Proof: This is obvious if $n \notin A$. Otherwise, deleting the identity permutation gives a bijection between maximal chains in F_n^A of length k and maximal chains in $F_n^{A-\{n\}}$ of length $k-1$. □

In the next two results, we describe the maximal and minimal elements of F_n^A. The proofs are not given here.

Proposition 11 *For all n and all A, F_n^A has a $\hat{1}$. Furthermore,* $\mathrm{inv}(\hat{1}) = \frac{n-a}{2}(n+a-1)$ *and* $\mathrm{exc}(\hat{1}) = \frac{n-a}{2}$*, where $a = \min A$.*

Proposition 12 *For all n and all A, all minimal elements of F_n^A have rank $(n - \max A)/2$ in I_n.*

Recall that

$$F_n^{\leq a} = \bigcup_{i\geq 0} F_n^{a-2i}, \quad F_n^{\geq a} = \bigcup_{i \geq 0} F_n^{a+2i}, \quad \text{and} \quad F_n^{a_1:a_2} = F_n^{\geq a_1} \cap F_n^{\leq a_2},$$

where $a_2 = a_1 + 2m$ for some positive integer m. Note that $F_n^{a_1:a_2}$ is not defined for $a_1 = a_2$.

The following lemma will eventually allow us to conclude that $F_n^{\leq a}$, $F_n^{\geq a}$, and $F_n^{a_1:a_2}$ are graded.

Lemma 13 *If every cover in F_n^A is a cover in I_n, then F_n^A is graded.*

Proof: This follows from Lemma 7 and Propositions 11 and 12. □

The next lemma is used in the proofs of Lemmas 15, 16, and 17, which, together with Lemma 13, show that $F_n^{\leq a}$, $F_n^{\geq a}$, and $F_n^{a_1:a_2}$ are graded.

Lemma 14 *Let $\sigma < \tau$ in I_n. Then the label (i, j) on any cover in $[\sigma, \tau]$ satisfies $i \geq \mathrm{di}(\sigma, \tau)$.*

Proof: Suppose $i < \mathrm{di}(\sigma, \tau)$ for the label (i, j) on $\sigma \lhd \pi \leq \tau$. Then $\pi(k) = \tau(k)$ for $k < i$ and $\sigma(i) = \tau(i)$. However, it follows from Definition 5 that $\pi(i) > \sigma(i)$. Hence, $\pi[i, \sigma(i)+1] > \tau[i, \sigma(i)+1]$. By Definition 4, this contradicts the fact that $\pi \leq \tau$. Thus $i \geq \mathrm{di}(\sigma, \tau)$. The result follows by induction. □

Lemma 15 *Let $\sigma \lessdot \tau$ in $F_n^{\leq a}$. Then $\sigma \lessdot \tau$ in I_n.*

Proof: Assume that $\sigma \not\lessdot \tau$ in I_n, and let $C_I = \sigma \lessdot \sigma_1 \lessdot \cdots \lessdot \sigma_k \lessdot \tau$ be the increasing σ-τ-chain in I_n and $C_D = \sigma \lessdot \tau_k \lessdot \cdots \lessdot \tau_1 \lessdot \tau$ the decreasing σ-τ-chain in I_n. Since $\sigma \lessdot \tau$ in $F_n^{\leq a}$, neither σ_1 nor τ_1 belongs to $F_n^{\leq a}$.

Let $h = \mathrm{di}(\sigma, \tau)$, and let (i_σ, j_σ) and (i_τ, j_τ) be the labels on $\sigma \lessdot \sigma_1$ and $\tau_1 \lessdot \tau$, respectively. Since $\sigma(h) \neq \tau(h)$, it follows from Lemma 14 that h is in some label on C_I and some label on C_D. Since C_I is increasing, $i_\sigma = h$, and since $\sigma_1 \notin F_n^{\leq a}$, h is an exceedance of σ (Type 5). Since C_D is decreasing, $i_\tau = h$, and since $\tau_1 \notin F_n^{\leq a}$, h is a fixed point of τ_1 (Type 1). Hence, $\sigma[h, h+1] > \tau_1[h, h+1]$. By Definition 4, this contradicts the fact that $\sigma \leq \tau_1$. □

Lemma 16 *Let $\sigma \lessdot \tau$ in $F_n^{\geq a}$. Then $\sigma \lessdot \tau$ in I_n.*

Proof: Assume that $\sigma \not\lessdot \tau$ in I_n, and let $C_I = \sigma \lessdot \sigma_1 \lessdot \cdots \lessdot \sigma_k \lessdot \tau$ be the increasing σ-τ-chain in I_n and $C_D = \sigma \lessdot \tau_k \lessdot \cdots \lessdot \tau_1 \lessdot \tau$ the decreasing σ-τ-chain in I_n. Since $\sigma \lessdot \tau$ in $F_n^{\geq a}$, neither σ_1 nor τ_1 belongs to $F_n^{\geq a}$.

Let $h = \mathrm{di}(\sigma, \tau)$, and let (i_σ, j_σ) and (i_τ, j_τ) be the labels on $\sigma \lessdot \sigma_1$ and $\tau_1 \lessdot \tau$, respectively. Since $\sigma(h) \neq \tau(h)$, it follows from Lemma 14 that h is in some label on C_I and some label on C_D. Since C_I is increasing, $i_\sigma = h$, and since $\sigma_1 \notin F_n^{\geq a}$, h is a fixed point of σ (Type 1). Since C_D is decreasing, $i_\tau = h$, and since $\tau_1 \notin F_n^{\geq a}$, h is an exceedance of τ_1 (Type 5).

Let m be such that h is an exceedance of $\tau_1, \ldots, \tau_{m-1}$ and a fixed point of τ_m (with $\tau_{k+1} = \sigma$). Then the labels on $\tau \gtrdot \tau_1 \gtrdot \cdots \gtrdot \tau_m$ are $(h, j_1), \ldots, (h, j_m)$, where $j_1 < j_2 < \cdots < j_m$. Since $\tau_1 > \tau_2 > \cdots > \tau_{m-1}$, $\tau_1(h) > \tau_2(h) > \cdots > \tau_{m-1}(h)$. Since h is a fixed point of τ_m but an exceedance of τ_{m-1}, the cover $\tau_m \lessdot \tau_{m-1}$ is of Type 1 or 2, whence $\tau_{m-1}(h) = j_m$ or $\tau_{m-1}(h) = \tau_m(j_m) > j_m$, respectively; hence, $\tau_{m-1}(h) \geq j_m$. Therefore, $j_1 < j_m \leq \tau_{m-1}(h) \leq \tau_1(h)$. However, since the cover $\tau_1 \lessdot \tau$ is of Type 5, $\tau_1(h) < j_1$, which is a contradiction. □

Lemma 17 *Let $\sigma \lessdot \tau$ in $F_n^{a_1:a_2}$. Then $\sigma \lessdot \tau$ in I_n.*

The proof, which is omitted here, is largely a combination of the proofs of Lemmas 15 and 16.

Proposition 18 *The posets $F_n^{\leq a}$, $F_n^{\geq a}$, and $F_n^{a_1:a_2}$ are graded.*

Proof: This follows from Lemmas 13, 15, 16, and 17. □

In the following two results, we consider the sets A for which F_n^A is not graded.

Proposition 19 *If there is an $i \in [2, n-4]$ such that $i \in A$ but $i-2, i+2 \notin A$, then F_n^A is not graded.*

The proof, which is not given here, is similar to, but easier than, the proof of Proposition 20.

Proposition 20 *If there is an $i \notin A$ and a positive integer m such that $i-2, i+2m \in A - \{n\}$, then F_n^A is not graded.*

Proof sketch: We first prove that $F_k^{\{0,k-2\}}$, where $k \geq 6$ is even, is not graded. Let $\sigma = 12\cdots(k-2)k(k-1)$ and $\tau = k23\cdots(k-1)1$, and consider the interval $[\sigma, \tau]$. We obtain a σ-τ-chain C in I_k by $k-2$ fe-rises with labels $(k-2, k-1), (k-3, k-2), \ldots, (1, 2)$ (from σ to τ). We also obtain a

σ-τ-chain in I_n by $(k-2)/2$ ff-rises with labels $(1,2),(3,4),\dots,(k-3,k-2)$, followed by $(k-2)/2$ crossing ee-rises with labels $(k-3,k-1),(k-5,k-3),\dots,(1,3)$.

Let π be the fixed point-free involution obtained after the ff-rises. Since each ff-rise decreases the number of fixed points and I_k is graded, $(\sigma,\pi)\cap F_k^{\{0,k-2\}}=\emptyset$, and since each crossing ee-rise increases the number of fixed points and I_k is graded, $(\pi,\tau)\cap F_k^{\{0,k-2\}}=\emptyset$. Hence, C is a σ-τ-chain in $F_k^{\{0,k-2\}}$ of length $k-2$, while $\sigma\lhd\pi\lhd\tau$ is a σ-τ-chain in $F_k^{\{0,k-2\}}$ of length 2. Thus $F_k^{\{0,k-2\}}$ is not graded. Figure 3 illustrates the situation when $k=6$.

Now we have to obtain the right number of fixed points. The details are not given here. □

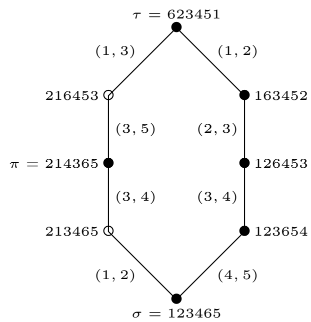

Figure 3: Two σ-τ-chains in I_6 of length 4, and two σ-τ-chains in $F_6^{\{0,4\}}$ of length 4 (right) and length 2 (left); the involutions marked by a • belong to $F_6^{\{0,4\}}$, and the involutions marked by a ○ belong to $I_6-F_6^{\{0,4\}}$. On the edges (covers in I_6) are the labels (i,j).

We are now ready to prove our main result:

Proof of Theorem 1: The first claim follows from Lemma 10 and Propositions 18, 19, and 20. (It is readily checked that if $F_n^{A-\{n\}}$ does not belong to $\{\emptyset,F_n^{\le a},F_n^{\ge a},F_n^{a_1:a_2}\}$, then either there is an $i\in[2,n-4]$ such that $i\in A$ but $i-2,i+2\notin A$, or there are an $i\notin A$ and a positive integer m such that $i-2,i+2m\in A-\{n\}$.) The second claim follows from Lemma 7, Proposition 12, and Lemmas 15, 16, and 17. The third claim follows from the second claim and Proposition 11. □

4 EL-shellability of F_n^0

In this section, we give a new proof of Theorem B, due to Can, Cherniavsky, and Twelbeck [3]. Our proof is largely based on the same main idea as their proof, together with the technique used in the proof of Lemma 15. The proof in [3] goes as follows:

Let $\sigma<\tau$ in F_n^0. It follows from, e.g., Theorem A and the paragraphs following it, that there exists a σ-τ-chain in I_n that is contained in F_n^0. Let C be the lex-maximal such chain. The idea of the proof is to show that C is decreasing. Then, by reversing the lexicographic order on the set $\{(i,j)\in[n]^2\mid i<j\}$ (i.e., by letting $(i_1,j_1)<(i_2,j_2)$ if and only if $i_1>i_2$, or $i_1=i_2$ and $j_1>j_2$), one obtains an edge-labelling of F_n^0 such that in each interval, there is an increasing σ-τ-chain which is lex-minimal. By Lemma 9 and the remark following it, this is an EL-labelling of F_n^0.

We use the same main idea, namely, to show that the decreasing σ-τ-chain in I_n is contained in F_n^0, and then reverse the lexicographic order. However, we give a direct proof of this fact. By using the same technique as in the proof of Lemma 15, we get a very short argument.

Lemma 21 *Let $\sigma < \tau$ in F_n^0 and let $C_D = \sigma \lhd \tau_k \lhd \cdots \lhd \tau_1 \lhd \tau$ be the decreasing σ-τ-chain in I_n, where $k \geq 1$. Then $\tau_1, \ldots, \tau_k \in F_n^0$.*

Proof: Since the decreasing σ-τ_1-chain in I_n is $\sigma \lhd \tau_k \lhd \cdots \lhd \tau_2 \lhd \tau_1$, it suffices to prove that $\tau_1 \in F_n^0$.

Let $h = \mathrm{di}(\sigma, \tau)$, let $C_I = \sigma \lhd \sigma_1 \lhd \cdots \lhd \sigma_k \lhd \tau$ be the increasing σ-τ-chain in I_n, and let (i_σ, j_σ) and (i_τ, j_τ) be the labels on $\sigma \lhd \sigma_1$ and $\tau_1 \lhd \tau$, respectively. Since $\sigma(h) \neq \tau(h)$, it follows from Lemma 14 that h is in some label on C_I and some label on C_D. Since C_I is increasing, $i_\sigma = h$, and since σ has no fixed points, h is an exceedance of σ (Type 4, 5, or 6). Since C_D is decreasing, $i_\tau = h$, and were $\tau_1 \notin F_n^0$, h would be a fixed point of τ_1 (Type 1). Hence, by Definition 4, $\tau_1 \in F_n^0$. □

We can now complete the proof of Theorem B:

Proof of Theorem B: Let $\sigma < \tau$ in F_n^0. By Lemma 21, the decreasing σ-τ-chain in I_n is contained in F_n^0. If we can show that this chain is lex-maximal, then by reversing the lexicographic order and invoking Lemma 9, we are done.

In order to obtain a contradiction, let $C = \sigma_1 \lhd \cdots \lhd \sigma_k$ be the lex-maximal σ-τ-chain in I_n, and assume that it is not decreasing; say that $\lambda(\sigma_1, \sigma_2) \leq \lambda(\sigma_2, \sigma_3)$. By Lemma 8, $\sigma_1 \lhd \sigma_2 \lhd \sigma_3$ is lex-minimal among the σ_1-σ_3-chains in I_n. Hence, $\sigma_1 \lhd \sigma_2' \lhd \sigma_3 \lhd \cdots \lhd \sigma_k$, where $\sigma_1 \lhd \sigma_2' \lhd \sigma_3$ is the decreasing σ_1-σ_3-chain, is lex-larger than C, which is a contradiction. □

Is it possible to use the same idea to prove that every interval in $F_n^A \subset I_n$ is EL-shellable for some $A \neq \{0\}$? Unfortunately, the answer is no, since for all $A \neq \{0\}$ (except the trivial cases $A = \emptyset$ and $A = \{n\}$), it is possible to find $\sigma_1 < \tau_1$ and $\sigma_2 < \tau_2$ in F_n^A, such that the increasing σ_1-τ_1-chain and the decreasing σ_2-τ_2-chain in I_n, are of length 2 and are not contained in F_n^A.

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