

# Maximal increasing sequences in fillings of almost-moon polyominoes

Svetlana Poznanovic<sup>1†</sup> and Catherine H. Yan<sup>2‡</sup>

<sup>1</sup>*Department of Mathematical Sciences, Clemson University, Clemson, SC 29634, USA*

<sup>2</sup>*Department of Mathematics, Texas A&M University, College Station, TX 77843, USA*

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**Abstract.** It was proved by Rubey that the number of fillings with zeros and ones of a given moon polyomino that do not contain a northeast chain of a fixed size depends only on the set of column lengths of the polyomino. Rubey's proof uses an adaptation of jeu de taquin and promotion for arbitrary fillings of moon polyominoes and deduces the result for 01-fillings via a variation of the pigeonhole principle. In this paper we present the first completely bijective proof of this result by considering fillings of almost-moon polyominoes, which are moon polyominoes after removing one of the rows. More precisely, we construct a simple bijection which preserves the size of the largest northeast chain of the fillings when two adjacent rows of the polyomino are exchanged. This bijection also preserves the column sum of the fillings. In addition, we also present a simple bijection that preserves the size of the largest northeast chains, the row sum and the column sum if every row of the filling has at most one 1. Thereby, we not only provide a bijective proof of Rubey's result but also two refinements of it.

**Résumé.** Rubey a montré que le nombre de remplissages d'un polyomino lunaire donné par des zéros et des uns qui ne contiennent pas de chaîne nord-est d'une taille fixée ne dépend que de l'ensemble des longueurs des colonnes du polyomino. La preuve de Rubey utilise une adaptation du jeu de taquin et de la promotion sur des remplissages arbitraires de polyominos lunaires et déduit le résultat pour les remplissages 0/1 par inclusion-exclusion. Dans cet article, nous présentons la première preuve bijective de ce résultat en considérant des remplissages de polyominos presque lunaires, qui sont des polyominos lunaires dont on a supprimé une ligne. Plus précisément, nous construisons une bijection simple qui préserve la taille de la plus longue chaîne nord-est des remplissages lorsque deux lignes adjacentes du polyomino sont échangées. Cette bijection préserve aussi la somme des colonnes des remplissages. En outre, nous présentons aussi une bijection simple qui préserve la taille de la plus longue chaîne nord-est, la somme des lignes et la somme des colonnes si chaque ligne du remplissage contient au plus un 1. Nous fournissons donc non seulement une preuve bijective du résultat de Rubey, mais aussi deux raffinements de celui-ci.

**Keywords:** moon polyominoes, maximal chains, 01-fillings

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<sup>†</sup>Email: [spoznan@clemson.edu](mailto:spoznan@clemson.edu). The first author is partially supported by NSF grant DMS-1312817.

<sup>‡</sup>Email: [cyan@math.tamu.edu](mailto:cyan@math.tamu.edu). The second author is partially supported by NSF grant DMS-1161414.

## 1 Introduction

In [4], Chen, Deng, Du, Stanley, and Yan used Robinson-Schensted-like insertion/deletion processes to show the symmetry between the sizes of the largest crossings and the largest nestings in set partitions. These results have been generalized and put in a larger context of enumeration of fillings of boards where one imposes restrictions on the increasing and decreasing chains of the fillings [15]. Moon polyominoes are polyominoes that are convex and intersection-free. In a moon polyomino  $\mathcal{M}$  the lengths of rows (and columns) are arranged in a unimodal order. Let  $\mathcal{F}_{\mathbb{N}}(\mathcal{M}, n)$  be the set of assignments of nonnegative integers to the cells of  $\mathcal{M}$  such that the sum of the integers equals  $n$  and let  $\mathcal{F}_{01}(\mathcal{M}, n)$  be the subset of  $\mathcal{F}_{\mathbb{N}}(\mathcal{M}, n)$  where the cells may only be filled with zeros and ones. Let  $\text{ne}(M)$  be the length of the longest strict northeast chain in a filling  $M$ . The motivation for this paper is the very important result of Rubey [18] that the cardinalities of the sets  $\{M : M \in \mathcal{F}_{\mathbb{N}}(\mathcal{M}, n) \text{ and } \text{ne}(M) = k\}$  and  $\{M : M \in \mathcal{F}_{01}(\mathcal{M}, n) \text{ and } \text{ne}(M) = k\}$  only depend on the multiset of column lengths of the moon polyomino  $\mathcal{M}$ . This is a very interesting property for fillings of moon polyominoes: many combinatorial statistics are invariant under permutations of rows (or columns). For example, a similar statement is also known for the major index introduced by Chen, Poznanović, Yan and Yang [5], for the numbers of northeast and southeast chains of length 2 studied by Kasraoui [14], and for various analogs and generalizations of 2-chains [6, 21].

The aforementioned result of Rubey is a culmination of the work of several people who proved it for various subclasses of moon polyominoes. The case for Ferrers and reverse Ferrers shapes was proved bijectively by Krattenthaler [15] using Fomin's growth diagrams [9, 10, 11]. Jonsson, motivated by the problem of counting generalized triangulations with a given size of the maximal crossings, first proved the result for maximal 01-fillings of stack polyominoes using an involved induction [12], and later with Welker [13] for all 01-fillings with a fixed number of 1's using the machinery of simplicial complexes and commutative algebra. A stack polyomino is a convex polyomino in which the rows are arranged in a descending order from top to bottom. A bijective proof for the general case of  $\mathbb{N}$ -fillings of moon polyominoes was obtained by Rubey [18] using an adaptation of *jeu de taquin* and promotion. Rubey additionally deduced the statement for  $\mathcal{F}_{01}(\mathcal{M}, n)$  using inclusion-exclusion. Later, a bijective proof for  $\mathcal{F}_{01}(\mathcal{M}, n)$  with the additional assumption that  $n$  is maximal was provided by Serrano and Stump [20] for the special case of Ferrers shapes and extended to stack polyominoes by Rubey [19]. Other papers that are part of this whole picture but phrase the results in terms of avoiding patterns are [1, 2, 3, 8].

In this paper we give a bijective proof of Rubey's result for  $\mathcal{F}_{01}(\mathcal{M}, n)$  [18]. In fact, we construct two simple bijections for 01-fillings demonstrating two different refinements of this result. For a moon polyomino  $\mathcal{M}$ , let  $\sigma\mathcal{M}$  be another moon polyomino obtained by permuting the rows of  $\mathcal{M}$ . Inspired by the work on layer polyominoes [16], our idea is to extend the class of moon polyominoes to a more general family that would allow us to transform the moon polyomino  $\mathcal{M}$  to any  $\sigma\mathcal{M}$  by a sequence of steps that interchange two adjacent rows at each time. For this purpose we introduce the notion of almost-moon polyominoes, which become moon polyominoes after removing one of its rows (see Section 2 for the exact definition). Let  $\mathcal{M}$  and  $\mathcal{N}$  be two almost-moon polyominoes that are related by an interchange of two adjacent rows. We present two bijections. The first is a map  $\phi_{\mathcal{M}, \mathcal{N}}$  from 01-fillings of  $\mathcal{M}$  to those of  $\mathcal{N}$  which preserves the number of non-zero entries, the size of the longest northeast chain and the column sums. The second map  $\psi_{\mathcal{M}, \mathcal{N}}$  is restricted to fillings in which every row has at most one 1 and preserves the size of the longest northeast chains, the row sum, and the column sum. Both our maps are simple to describe but the proofs that they have the desired properties are technical; we omit them here and refer the

reader to the full version of the paper [17].

The paper is organized as follows. Section 2 contains the necessary notation and the statements of the main results. In Section 3 we construct the bijection  $\phi_{\mathcal{M},\mathcal{N}}$  for fillings of almost-moon polyominoes which preserves the size of maximal northeast chains and the column sums. In Section 4 we restrict to fillings that have at most one 1 in each row and describe the bijection  $\psi_{\mathcal{M},\mathcal{N}}$ . We conclude the paper with some comments and counterexamples to a few seemingly natural generalizations in Section 5.

## 2 Notation and statements of the main results

A *polyomino* is a finite subset of  $\mathbb{Z}^2$ , where we represent every element  $(i, j)$  of  $\mathbb{Z}^2$  by a square cell. The polyomino is *row-convex* (*column-convex*) if its every row (column) is connected. If the polyomino is both row- and column-convex, we say that it is *convex*. It is *intersection-free* if every two columns are *comparable*, i.e., the row-coordinates of one column form a subset of those of the other column. Equivalently, it is intersection-free if every two rows are comparable. A *moon polyomino* is a convex intersection-free polyomino (e.g. Figure 1a). The length of a row (or a column) is the number of cells in it. Note that in a moon polyomino the lengths of rows from top to bottom form a unimodal sequence. We will say that a row  $\mathcal{R}$  is an *exceptional row* of a polyomino  $\mathcal{M}$  if there are rows above  $\mathcal{R}$  and below  $\mathcal{R}$  with larger lengths in  $\mathcal{M}$ . An *almost-moon polyomino* is a polyomino with comparable convex rows and at most one exceptional row (e.g. Figure 1b). Therefore, every moon polyomino is also an almost-moon polyomino and an almost-moon polyomino is not necessarily column-convex.



Fig. 1: A moon polyomino and an almost-moon polyomino with an exceptional row  $\mathcal{R}$  that differ by an interchange of adjacent rows.

In this paper we will consider polyominoes whose cells are filled with zeros and ones. A *northeast chain*, or shortly *ne-chain*, of size  $k$  in such a filling is a set of  $k$  cells  $\{(i_1, j_1), (i_2, j_2), \dots, (i_k, j_k)\}$  with  $i_1 < \dots < i_k, j_1 < \dots < j_k$  filled with 1's such that the  $k \times k$  submatrix

$$\mathcal{G} = \{(i_r, j_s) : 1 \leq r \leq k, 1 \leq s \leq k\}$$

is contained in the polyomino (with no restriction on the filling of the other cells). See Figure 2 for an illustration. We will call the ne-chains of size  $k$  shortly  $k$ -chains. Note that in a moon polyomino  $\mathcal{M}$ ,  $k$  1-cells in a northeast direction satisfy the submatrix condition if and only if the corners  $(i_1, j_k)$  and  $(i_k, j_1)$  are contained in  $\mathcal{M}$ , which is equivalent to the whole rectangle determined by these corners being contained in  $\mathcal{M}$ . In an almost-moon polyomino, the submatrix condition is satisfied if and only if the vertices  $(i_1, j_k)$  and  $(i_k, j_1)$  either determine a rectangle which is completely contained in  $\mathcal{M}$  or an almost-rectangle with one exceptional row contained in  $\mathcal{M}$ . In the latter case, the exceptional row does not contain any elements from the ne-chain.

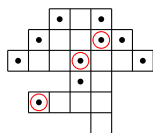


Fig. 2: A 01-filling  $M$  of an almost-moon polyomino with  $\text{ne}(M) = 3$ . The 1's are represented by dots and the 0-cells are drawn empty. The circled dots form the only 3-chain in  $M$ .

For a 01-filling  $M$  of an almost-moon polyomino, we denote by  $\text{ne}(M)$  the size of its largest  $\text{ne}$ -chains. Suppose that  $\mathcal{M}$  has  $k$  rows and  $\ell$  columns and let  $\mathbf{r} \in \mathbb{N}^k$  and  $\mathbf{c} \in \mathbb{N}^\ell$ . We will denote by  $\mathcal{F}(\mathcal{M})$  the set of all 01-fillings of  $\mathcal{M}$ , by  $\mathcal{F}(\mathcal{M}, n)$  those fillings with exactly  $n$  1's, and by  $\mathcal{F}(\mathcal{M}, \mathbf{r}, \mathbf{c})$  the set of all 01-fillings with row sums given by  $\mathbf{r}$  and column sums given by  $\mathbf{c}$ . Our first main result states that if  $\mathcal{M}$  and  $\mathcal{N}$  are two almost-moon polyominoes related by an interchange of adjacent rows (e.g. Figure 1), then the statistic  $\text{ne}$  is equidistributed over the sets  $\mathcal{F}(\mathcal{M}, *, \mathbf{c})$  and  $\mathcal{F}(\mathcal{N}, *, \mathbf{c})$  of fillings of  $\mathcal{M}$  and  $\mathcal{N}$ , respectively, with fixed column sums but arbitrary row sums:

$$\sum_{M \in \mathcal{F}(\mathcal{M}, *, \mathbf{c})} q^{\text{ne}(M)} = \sum_{M \in \mathcal{F}(\mathcal{N}, *, \mathbf{c})} q^{\text{ne}(M)}.$$

More precisely, we have the following theorem.

**Theorem 1** *Let  $\mathcal{M}$  and  $\mathcal{N}$  be two almost-moon polyominoes such that  $\mathcal{N}$  can be obtained from  $\mathcal{M}$  by an interchange of two adjacent rows. In addition assume that  $\mathcal{M}$  and  $\mathcal{N}$  have no exceptional rows other than the swapped ones. Then there is a bijection*

$$\phi_{\mathcal{M}, \mathcal{N}} : \mathcal{F}(\mathcal{M}) \longrightarrow \mathcal{F}(\mathcal{N})$$

*that preserves the column sums of the fillings and such that  $\text{ne}(\phi_{\mathcal{M}, \mathcal{N}}(M)) = \text{ne}(M)$  for  $M \in \mathcal{F}(\mathcal{M})$ . Moreover,  $\phi_{\mathcal{N}, \mathcal{M}} \circ \phi_{\mathcal{M}, \mathcal{N}} = 1_{\mathcal{F}(\mathcal{M})}$ .*

Let  $\mathcal{M}$  be an almost-moon polyomino with  $k$  rows,  $\sigma \in S_k$ , and suppose the polyomino  $\sigma\mathcal{M}$  obtained by permuting the rows of  $\mathcal{M}$  according to  $\sigma$  is also an almost-moon polyomino. Note that  $\sigma\mathcal{M}$  can also be obtained by a sequence of steps in which only two adjacent rows are interchanged. Moreover, the order of steps can be chosen so that the intermediate polyominoes are all almost-moon polyominoes with no exceptional rows other than the swapped ones. In other words, the set of all  $\sigma\mathcal{M}$  which are almost-moon polyominoes is connected by transposition of adjacent rows. One way to see this is to note that one can reach the polyomino in which the row lengths are descending from top to bottom by starting from  $\mathcal{M}$  and first moving its exceptional row down until there is no longer rows below it, then moving the shortest row of  $\mathcal{M}$  to the bottom, the second shortest row to the second position from below, etc. Consequently, by composing the maps from Theorem 1 we get the following corollary.

**Corollary 2** *Let  $\mathcal{M}$  be an almost-moon polyomino with  $k$  rows and  $\ell$  columns,  $\sigma \in S_k$  be a permutation of the row indices such that  $\sigma\mathcal{M}$  is also an almost-moon polyomino. Let  $\mathbf{c} \in \mathbb{N}^\ell$ . Then there is a bijection  $\phi : \mathcal{F}(\mathcal{M}, *, \mathbf{c}) \longrightarrow \mathcal{F}(\sigma\mathcal{M}, *, \mathbf{c})$  such that  $\text{ne}(\phi(M)) = \text{ne}(M)$  for  $M \in \mathcal{F}(\mathcal{M}, *, \mathbf{c})$ . Moreover, the size of  $\{M : M \in \mathcal{F}(\mathcal{M}, n), \text{ne}(M) = k\}$  depends only on the multiset of column lengths of  $\mathcal{M}$ .*

**Proof:** To prove the second part of the statement, suppose  $\mathcal{M}_1$  is the moon polyomino with descending row lengths that can be obtained by reordering the rows of  $\mathcal{M}$  as in the discussion above. Suppose that the same procedure applied to the transpose  $\mathcal{M}_1^t$  of  $\mathcal{M}_1$  yields the polyomino  $\mathcal{M}_2$  with descending row lengths. Then  $\mathcal{M}_2$  is a Ferrers shape and it depends only on the set of column lengths of  $\mathcal{M}$ . Since the transpose of ne-chains are also ne-chains, it follows from the first part that the size of  $\{M : M \in \mathcal{F}(\mathcal{M}, n), \text{ne}(M) = k\}$  also depends only on the set of lengths of the columns of  $\mathcal{M}$ . ■

The fact that  $|\{M : M \in \mathcal{F}(\mathcal{M}, *, \mathbf{c}), \text{ne}(M) = k\}| = |\{M : M \in \mathcal{F}(\sigma\mathcal{M}, *, \mathbf{c}), \text{ne}(M) = k\}|$  if  $\mathcal{M}$  and  $\sigma\mathcal{M}$  are moon polyominoes as well as the second part of Corollary 2 was proved using a variation of the pigeonhole principle by Rubey [18]. Therefore, our results provide a bijective proof of these facts and extend them to a larger set of polyominoes in which these properties hold. In Section 5 we discuss why this extension is in a sense the best possible.

As discussed in [18], one cannot hope to simultaneously preserve both  $\mathbf{r}$  and  $\mathbf{c}$ , i.e., the natural generalization  $|\{M : M \in \mathcal{F}(\mathcal{M}, \mathbf{r}, \mathbf{c}), \text{ne}(M) = k\}| = |\{M : M \in \mathcal{F}(\sigma\mathcal{M}, \sigma\mathbf{r}, \mathbf{c}), \text{ne}(M) = k\}|$  does not hold. However, our second main result implies that ne can be preserved together with both the row and the column sums if the fillings are restricted to have at most one 1 in each row.

**Theorem 3** *Let  $\mathcal{M}$  and  $\mathcal{N}$  be two almost-moon polyominoes such that  $\mathcal{N}$  can be obtained from  $\mathcal{M}$  by an interchange of two adjacent rows. In addition assume that  $\mathcal{M}$  and  $\mathcal{N}$  have no exceptional rows other than the swapped ones. If  $\mathbf{r} \in \{0, 1\}^*$  and  $\mathbf{c} \in \mathbb{N}^*$ , then there is a bijection*

$$\psi_{\mathcal{M}, \mathcal{N}} : \mathcal{F}(\mathcal{M}, \mathbf{r}, \mathbf{c}) \longrightarrow \mathcal{F}(\mathcal{N}, \mathbf{r}', \mathbf{c})$$

such that  $\text{ne}(\psi_{\mathcal{M}, \mathcal{N}}(M)) = \text{ne}(M)$  for  $M \in \mathcal{F}(\mathcal{M}, \mathbf{r}, \mathbf{c})$ , where  $\mathbf{r}'$  is obtained from  $\mathbf{r}$  by exchanging the entries corresponding to the two swapped rows.

This theorem was proved for moon polyominoes by Rubey [18]. By the same discussion after Theorem 1 and a combination of the maps  $\psi_{\mathcal{M}, \mathcal{N}}$  and  $\phi_{\mathcal{M}, \mathcal{N}}$ , we get the following corollary.

**Corollary 4** *Let  $\mathcal{M}$  be an almost-moon polyomino with  $k$  rows and  $\ell$  columns,  $\sigma \in S_k$  be a permutation of the row indices such that  $\sigma\mathcal{M}$  is also an almost-moon polyomino. Let  $\mathbf{r} \in \{0, 1\}^k$  and  $\mathbf{c} \in \mathbb{N}^\ell$ . Then there is a bijection  $\psi : \mathcal{F}(\mathcal{M}, \mathbf{r}, \mathbf{c}) \longrightarrow \mathcal{F}(\sigma\mathcal{M}, \sigma\mathbf{r}, \mathbf{c})$  such that  $\text{ne}(\psi(M)) = \text{ne}(M)$  for  $M \in \mathcal{F}(\mathcal{M}, \mathbf{r}, \mathbf{c})$ . Moreover, the number of fillings  $M \in \mathcal{F}(\mathcal{M}, \mathbf{r})$  with at most one 1 per row and fixed column sums depends only on the multiset of column lengths of  $\mathcal{M}$ .*

### 3 Maximal increasing sequences in 01-fillings with fixed total sum

In this section we will describe the maps  $\phi_{\mathcal{M}, \mathcal{N}}$  and prove Theorem 1. To this end, let  $\mathcal{M}$  and  $\mathcal{N}$  be two almost-moon polyominoes related by an interchange of two adjacent rows (e.g. Figure 1). Assume that  $\mathcal{M}$  and  $\mathcal{N}$  have no exceptional rows other than the swapped ones. If the swapped rows are of equal length then  $\mathcal{M} = \mathcal{N}$  and we define  $\phi_{\mathcal{M}, \mathcal{N}}$  to be the identity map on  $\mathcal{F}(\mathcal{M})$ .

Otherwise, suppose the lengths of the swapped rows are not equal and let  $\mathcal{R}_s$  and  $\mathcal{R}_l$ , respectively, be the shorter and longer rows. Note that  $\mathcal{M} \setminus \mathcal{R}_s$  and  $\mathcal{N} \setminus \mathcal{R}_s$  have no exceptional rows. Let  $\alpha, \beta, \gamma, \delta$  be the fillings of the regions of  $\mathcal{R}_s$  and  $\mathcal{R}_l$  in  $\mathcal{M}$  as depicted on the left side of Figure 3. Precisely,  $\alpha$  is the filling of the shorter row  $\mathcal{R}_s$ ,  $\beta$  is the filling of the part of the longer row  $\mathcal{R}_l$  which has the same column support as row  $\mathcal{R}_s$ , and  $\gamma$  and  $\delta$  are the fillings of the two ends of row  $\mathcal{R}_l$  so that the whole filling of row

$\mathcal{R}_l$  viewed as a binary string is a concatenation of  $\gamma$ ,  $\beta$ , and  $\delta$ . Note that one of  $\gamma$  and  $\delta$  may be the empty string. First we define a map  $f_{\mathcal{M},\mathcal{N}} : \mathcal{F}(\mathcal{M}) \rightarrow \mathcal{F}(\mathcal{N})$  as follows.

**Definition 5** Using the notation above,  $f_{\mathcal{M},\mathcal{N}}(M)$  is the filling of  $\mathcal{N}$  in which (1) the filling of the rows in  $\mathcal{N}$  other than  $\mathcal{R}_s$  and  $\mathcal{R}_l$  is the same as in  $M$  (2) the filling of the shorter row  $\mathcal{R}_s$  of  $\mathcal{N}$  is  $\beta$  (3) the filling of the longer row  $\mathcal{R}_l$  of  $\mathcal{N}$  is the concatenation of  $\gamma$ ,  $\alpha$ , and  $\delta$ .

See Figure 3 for an illustration.

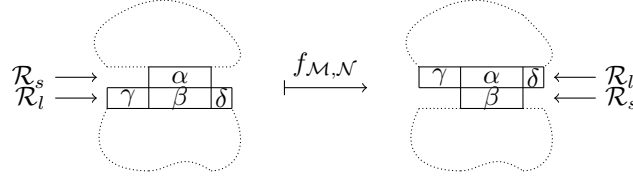


Fig. 3: The fillings  $M$  and  $f_{\mathcal{M},\mathcal{N}}(M)$  differ only in the rows  $\mathcal{R}_s$  and  $\mathcal{R}_l$ .

**Lemma 6**  $f_{\mathcal{N},\mathcal{M}} \circ f_{\mathcal{M},\mathcal{N}} = 1_{\mathcal{F}(\mathcal{M})}$ .

If it is clear what the polyominoes  $\mathcal{M}$  and  $\mathcal{N}$  are, we will leave out the subscripts and write only  $f(M)$ .

**Lemma 7** If the almost-moon polyominoes  $\mathcal{M}$  and  $\mathcal{N}$  are related by an interchange of the rows  $\mathcal{R}_s$  and  $\mathcal{R}_l$  as above, then for every  $M \in \mathcal{F}(\mathcal{M})$ ,

$$|\text{ne}(M) - \text{ne}(f_{\mathcal{M},\mathcal{N}}(M))| \leq 1.$$

Therefore, every filling  $M \in \mathcal{F}(\mathcal{M})$  satisfies exactly one of the following 3 conditions:

- (I)  $\text{ne}(f(M)) = \text{ne}(M)$
- (II)  $\text{ne}(f(M)) = \text{ne}(M) + 1$
- (III)  $\text{ne}(f(M)) = \text{ne}(M) - 1$ .

Let  $\mathcal{F}^I(\mathcal{M})$ ,  $\mathcal{F}^{II}(\mathcal{M})$ , and  $\mathcal{F}^{III}(\mathcal{M})$ , be the fillings of  $\mathcal{M}$  that satisfy the conditions (I), (II), and (III), respectively. Below we describe how  $\phi_{\mathcal{M},\mathcal{N}}(M)$  is defined on each of these three sets.

**Case I.** For  $M \in \mathcal{F}^I(\mathcal{M})$  we define  $\phi_{\mathcal{M},\mathcal{N}}(M) = f_{\mathcal{M},\mathcal{N}}(M)$ . It is clear that this defines a bijection from  $\mathcal{F}^I(\mathcal{M})$  onto  $\mathcal{F}^I(\mathcal{N})$ .

**Case II.** For  $M \in \mathcal{F}^{II}(\mathcal{M})$  with  $\text{ne}(M) = k$ , we have  $\text{ne}(N') = k + 1$  where  $N' = f(M)$ . Reasoning as in the proof of Lemma 7, one can see that all  $(k + 1)$ -chains in  $N'$  contain exactly one cell from  $\mathcal{R}_s \cup \mathcal{R}_l$  and that that cell must be in the part  $\alpha$  of the longer row  $\mathcal{R}_l$  of  $\mathcal{N}$  (see the right filling in Figure 3).

**Definition 8** Suppose  $M \in \mathcal{F}^{II}(\mathcal{M})$  and  $\text{ne}(f(M)) = \text{ne}(M) + 1 = k + 1$ . The 1-cells in row  $\mathcal{R}_l$  of  $f(M)$  which are part of a  $(k + 1)$ -chain are called problem cells.

Let  $\alpha_0$  be the set of problem cells in  $N' = f(M)$  and let  $\beta_0$  be the set of cells in row  $\mathcal{R}_s$  of  $N'$  that share a horizontal edge with the problem cells. We define  $\phi_{\mathcal{M},\mathcal{N}}(M)$  to be the filling  $N''$  of  $\mathcal{N}$  obtained by replacing the problem cells in  $\alpha_0$  by zeros and the cells in  $\beta_0$  by ones. In other words,  $\phi_{\mathcal{M},\mathcal{N}}(M) = N''$  is obtained by a vertical shift of the problem cells in  $N'$  from row  $\mathcal{R}_l$  to row  $\mathcal{R}_s$ .

**Case III.** For  $M \in \mathcal{F}^{III}(\mathcal{M})$ , we have  $f_{\mathcal{M},\mathcal{N}}(M) \in \mathcal{F}^{II}(\mathcal{N})$ . In this case we set  $\phi_{\mathcal{M},\mathcal{N}}(M) = f_{\mathcal{M},\mathcal{N}}(\phi_{\mathcal{N},\mathcal{M}}(f_{\mathcal{M},\mathcal{N}}(M)))$ .

Although the description of the map  $\phi_{\mathcal{M},\mathcal{N}}$  is simple, the proof that it is a well-defined bijection from  $\mathcal{F}(\mathcal{M}, *, \mathbf{c})$  to  $\mathcal{F}(\mathcal{N}, *, \mathbf{c})$  preserving the statistic  $\text{ne}$  is rather technical and we omit it here. It would be interesting to see whether there is a relationship between the Edelman-Greene bijection as used in [20, 19] and the map on moon polyominoes induced by the  $\phi_{\mathcal{M},\mathcal{N}}$ 's. The analogous question for the Robinson-Schensted map as used in [15] and the BWX map was answered by Bloom and Saracino [2].

### 4 Maximal increasing sequences in fillings with restricted row sum

In this section we restrict to fillings with at most one 1 in each row and prove Theorem 3. Explicitly, let  $\mathcal{M}$  and  $\mathcal{N}$  be two almost-moon polyominoes that can be obtained from each other by an interchange of two adjacent rows. Assume that  $\mathcal{M}$  and  $\mathcal{N}$  have no exceptional rows other than the swapped ones. Let  $\mathbf{r} \in \{0, 1\}^*$  and  $\mathbf{c} \in \mathbb{N}^*$ , we shall construct a bijection  $\psi_{\mathcal{M},\mathcal{N}}$  from  $\mathcal{F}(\mathcal{M}, \mathbf{r}, \mathbf{c})$  to  $\mathcal{F}(\mathcal{N}, \mathbf{r}', \mathbf{c})$  that preserves the size of the largest ne-chains, where  $\mathbf{r}'$  is obtained from  $\mathbf{r}$  by exchanging the entries corresponding to the two swapped rows.

If the two swapped rows are of equal length, then  $\mathcal{M} = \mathcal{N}$  and we can simply take  $\psi_{\mathcal{M},\mathcal{N}}$  to be the identity map. In the following, we assume that the two swapped rows are  $\mathcal{R}_s$  and  $\mathcal{R}_l$ , where the length of  $\mathcal{R}_s$  is smaller than that of  $\mathcal{R}_l$ . We keep the notations as in the previous section. For any filling  $M$  let  $\alpha, \beta, \gamma, \delta$  be as defined in Figure 3. Note that for  $M \in \mathcal{F}(\mathcal{M}, \mathbf{r}, \mathbf{c})$ , there is at most one 1 in  $\alpha$ , as well as in the union of  $\beta, \gamma$  and  $\delta$ . Define the *filling coupled with  $M$*  to be the filling  $M'$  of  $\mathcal{M}$  which is obtained from  $M$  by exchanging the fillings  $\alpha$  and  $\beta$ . Let  $N$  (resp.  $N'$ ) be obtained from  $M$  (resp.  $M'$ ) by swapping the rows  $\mathcal{R}_s$  and  $\mathcal{R}_l$  together with their fillings. In other words,  $N = f_{\mathcal{M},\mathcal{N}}(M')$  and  $N' = f_{\mathcal{M},\mathcal{N}}(M)$ . Clearly  $N$  and  $N'$  are fillings coupled with each other.

We need the following lemma, which is the crucial observation for the construction of  $\psi_{\mathcal{M},\mathcal{N}}$ .

**Lemma 9** *Let fillings  $(M, M')$  be a pair of fillings in  $\mathcal{F}(\mathcal{M}, \mathbf{r}, \mathbf{c})$  coupled with each other and let  $(N, N') = (f_{\mathcal{M},\mathcal{N}}(M'), f_{\mathcal{M},\mathcal{N}}(M))$  be fillings of  $\mathcal{N}$ . Then  $\text{ne}(M) = \text{ne}(N)$  or  $\text{ne}(M) = \text{ne}(N')$ .*

Lemma 9 implies that

$$\{\text{ne}(M), \text{ne}(M')\} = \{\text{ne}(N), \text{ne}(N')\} \tag{1}$$

as multisets. To see this, note that if  $\text{ne}(N) = \text{ne}(N') = k$ , then applying Lemma 9 to both  $M$  and  $M'$  yields  $\text{ne}(M) = \text{ne}(M') = k$ . Otherwise, if  $\text{ne}(N) \neq \text{ne}(N')$ , then applying Lemma 9 to  $N, N'$  yields that one of  $\text{ne}(M), \text{ne}(M')$  equals  $\text{ne}(N)$ , and the other equals  $\text{ne}(N')$ . Equation (1) allows us to construct an ne-preserving bijection between the pairs of coupled fillings  $\{M, M'\}$  and  $\{N, N'\}$ . Combining the bijections of all the coupled fillings we get the desired map  $\psi_{\mathcal{M},\mathcal{N}}$ . Explicitly, we can describe the map  $\psi_{\mathcal{M},\mathcal{N}}$  as follows:

Let  $M$  be a filling in  $\mathcal{F}(\mathcal{M}, \mathbf{r}, \mathbf{c})$ .

1. If either  $\alpha$  or  $\beta$  has no 1, then let  $\psi_{\mathcal{M},\mathcal{N}}(M) = N$ , the filling of  $\mathcal{N}$  obtained by swapping the two rows together with their fillings.
2. If both  $\alpha$  and  $\beta$  contain a 1 in the same column, then again let  $\psi_{\mathcal{M},\mathcal{N}}(M) = N$ .
3. If each of  $\alpha$  and  $\beta$  contain a unique 1 in a distinct column,

$$\psi_{\mathcal{M},\mathcal{N}}(M) = \begin{cases} N & \text{if } \text{ne}(M) = \text{ne}(N) \\ N' & \text{if } \text{ne}(M) \neq \text{ne}(N). \end{cases}$$

It is clear that the map  $\psi_{\mathcal{M},\mathcal{N}}$  is well-defined and preserves the statistic  $ne$ . In addition,  $\psi_{\mathcal{M},\mathcal{N}}$  and  $\psi_{\mathcal{N},\mathcal{M}}$  are inverse to each other.

We remark that the map  $\psi_{\mathcal{M},\mathcal{N}}$  is different from the restriction of  $\phi_{\mathcal{M},\mathcal{N}}$  on the set  $\mathcal{F}(\mathcal{M}, \mathbf{r}, \mathbf{c})$ . Namely,  $\phi_{\mathcal{M},\mathcal{N}}$  does not always preserve the row sum, hence not necessarily maps fillings of  $\mathcal{F}(\mathcal{M}, \mathbf{r}, \mathbf{c})$  to  $\mathcal{F}(\mathcal{N}, \mathbf{r}', \mathbf{c})$  when  $\mathbf{r} \in \{0, 1\}^*$ .

### 5 Concluding remarks

We conclude this paper with some comments and counterexamples to a few seemingly natural generalizations of Theorem 1 and 3.

**Symmetry of  $(ne, se)$  for fillings with  $\mathbf{r} \in \{0, 1\}^*$ .** A *southeast chain*, or shortly *se-chain*, of size  $k$  in a 01-filling  $M$  is a set of  $k$  cells  $\{(i_1, j_k), (i_2, j_{k-1}), \dots, (i_k, j_1)\}$  with  $i_1 < \dots < i_k, j_1 < \dots < j_k$  filled with 1's such that the  $k \times k$  submatrix  $\mathcal{G} = \{(i_r, j_s) : 1 \leq r \leq k, 1 \leq s \leq k\}$  is contained in the polyomino. We denote by  $se(M)$  the size of the largest se-chains of  $M$ . By the symmetry of  $ne$  and  $se$  we have that Lemma 9 also holds for  $se$ .

It is known that in 01-fillings of a Ferrers shape with fixed row sum and column sum in  $\mathbf{N}$ , the pair  $(ne, se)$  may not be symmetrically distributed (e.g. see [18, 7]). On the other hand, when both  $\mathbf{r}, \mathbf{c} \in \{0, 1\}^*$ ,  $(ne, se)$  does have a symmetric joint distribution. This was proved for Ferrers shapes by Krattenthaler [15] and for moon polyominoes by Rubey [18].

The above results raise the question whether for almost-moon polyominoes the pair of statistics  $(ne, se)$  has a symmetric joint distribution when one or both of  $\mathbf{r}, \mathbf{c}$  are in  $\{0, 1\}$ , and whether the distribution of  $(ne, se)$  is unchanged when one swaps two adjacent rows.

Surprisingly, the answers are all negative. In the following we give two sets of counterexamples. The first one is for the case that  $\mathbf{r} \in \{0, 1\}^*$  but  $\mathbf{c} \in \mathbf{N}$ . The involved polyominoes are of small sizes, and we can list all the fillings explicitly. The second one is for the case when both  $\mathbf{r}$  and  $\mathbf{c}$  are in  $\{0, 1\}$ . We found the counterexample by running a computer program, and we will just describe the results without listing all the details.

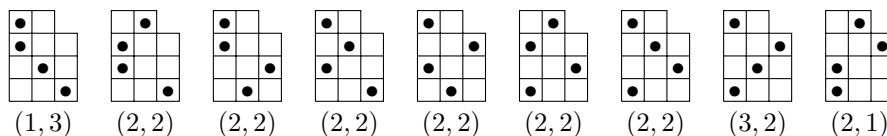


Fig. 4: Fillings of a moon polyomino with  $\mathbf{r} = (1, 1, 1, 1)$  and  $\mathbf{c} = (2, 1, 1)$ .

**Example 10** Figures 4 and 5 list all the 01-fillings of three polyominoes, where the polyominoes in Figure 5 are obtained from the moon polyomino in Figure 4 by moving down the first row. In all the fillings we require that  $\mathbf{r} = (1, 1, 1, 1)$  and  $\mathbf{c} = (2, 1, 1)$ . The data  $(ne, se)$  is given under each filling.

Figure 4 shows that even for moon polyomino with  $\mathbf{r} \in \{0, 1\}^*$  but  $\mathbf{c} \in \mathbf{N}^*$ , the distribution of  $(ne, se)$  is not necessarily symmetric. Figure 5 gives an example that the distribution of the pair  $(ne, se)$  is not preserved when two adjacent rows are swapped in an almost-moon polyomino. Note that the first two fillings  $(M, M')$  in Figure 5 are coupled fillings, whose corresponding coupled fillings are the first two



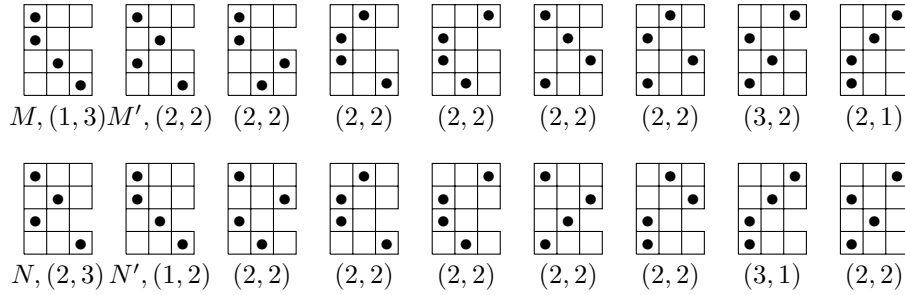


Fig. 5: Fillings of almost-moon polyominoes. The first row is for an almost-moon polyomino  $\mathcal{M}$ , and the second row is for  $\mathcal{N}$  which differ from  $\mathcal{M}$  by swapping the second and the third rows.

fillings  $(N, N')$  in the second row of Figure 5. For these two pairs Lemma 9 does not hold for  $(ne, se)$ , i.e.,

$$\{(ne(M), se(M)), (ne(M'), se(M'))\} \neq \{(ne(N), se(N)), (ne(N'), se(N'))\}.$$

**Example 11** Our second example is restricted to 01-fillings where each row as well as each column has exactly one 1. We say that such fillings are restricted. Let  $\mathcal{M}_1, \mathcal{M}_2$  and  $\mathcal{M}_3$  be the polyominoes whose cells are given by

$$\begin{aligned} \mathcal{M}_1 &= \{(i, j) : 1 \leq i \leq 6, 1 \leq j \leq 6\} - \{(1, 6)\}. \\ \mathcal{M}_2 &= \{(i, j) : 1 \leq i \leq 6, 1 \leq j \leq 6\} - \{(1, 5)\}. \\ \mathcal{M}_3 &= \{(i, j) : 1 \leq i \leq 6, 1 \leq j \leq 6\} - \{(1, 4)\}. \end{aligned}$$

Each polyomino has 600 restricted fillings. Let  $G_1(x, y) = \sum_M x^{ne(M)} y^{se(M)}$  be the joint distribution of  $(ne, se)$  over restricted fillings of  $\mathcal{M}_1$ . Similarly define  $G_2(x, y)$  and  $G_3(x, y)$  for restricted fillings in  $\mathcal{M}_2$  and  $\mathcal{M}_3$ . With the help of a computer program we obtained

$$\begin{aligned} G_1(x, y) = G_2(x, y) &= xy^5 + x^5y + 72(x^2y^4 + x^4y^2) + 48(x^3y^4 + x^4y^3) \\ &\quad + 50(x^2y^3 + x^3y^2) + 8(x^2y^5 + x^5y^2) + 242x^3y^3 \end{aligned} \tag{2}$$

and

$$\begin{aligned} G_3(x, y) &= xy^5 + x^5y + 72x^2y^4 + 73x^4y^2 + 48x^3y^4 + 47x^4y^3 \\ &\quad + 50x^2y^3 + 49x^3y^2 + 8x^2y^5 + 8x^5y^2 + 243x^3y^3. \end{aligned} \tag{3}$$

Equation (2) is symmetric with respect to  $x, y$ , which is expected for  $\mathcal{M}_1$  since it is a moon polyomino. Equation (3) shows that the joint distribution of  $(ne, se)$  over almost-moon polyominoes is not necessarily symmetric, even if we require that every row and every column has exactly one 1. The difference between the two equations implies that the distribution of  $(ne, se)$  may not be preserved when two adjacent rows are swapped.

**Coupled fillings with arbitrary row sums.** Another natural question is whether we can extend the idea of *coupling* in Theorem 3 to construct a bijection for Theorem 1. The following example shows that the direct application does not work.

**Example 12** Consider fillings in  $\mathcal{F}(\mathcal{M}, *, \mathbf{c})$  where a row may have multiple 1s. For a filling  $M$  of  $\mathcal{M}$ , we couple with it the filling  $M'$  obtained from  $M$  by swapping the fillings  $\alpha$  and  $\beta$ , just as what we did in Section 4. Again let  $N = f_{\mathcal{M}, \mathcal{N}}(M')$  and  $N' = f_{\mathcal{M}, \mathcal{N}}(M)$ . In the fillings shown in Figure 6,  $\text{ne}(M) = 3$ ,  $\text{ne}(M') = 4$  while  $\text{ne}(N) = \text{ne}(N') = 4$ . Lemma 9 does not hold for this case.

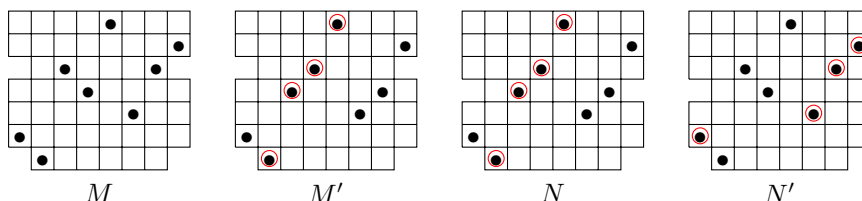


Fig. 6: Coupled fillings in  $\mathcal{F}(\mathcal{M}, \mathbf{r}, \mathbf{c})$  with  $\mathbf{r} \in \mathbb{N}^*$  and  $\mathbf{r} \in \{0, 1\}^*$ . The circled dots form 4-chains in the polyominoes.

It is still open whether Theorem 3 holds for the family of fillings for which  $\mathbf{r} \in \mathbb{N}^*$  but  $\mathbf{c} \in \{0, 1\}^*$ . We point out that Lemma 9 does not hold in this case either. Given a filling  $M$  with multiple 1-cells in a row, the natural way to define the coupling with the same row sum is to let  $M'$  be obtained from  $M$  by keeping the empty columns of  $\alpha \cup \beta$  and reversing the fillings in the remaining columns of  $\alpha \cup \beta$ . Then  $(N, N')$  are obtained from  $(M, M')$  by exchanging the two rows  $\mathcal{R}_s$  and  $R_l$  with their fillings.

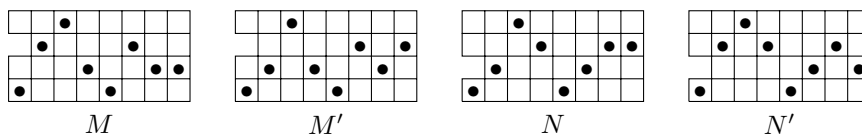


Fig. 7: Coupled fillings in  $\mathcal{F}(\mathcal{M}, \mathbf{r}, \mathbf{c})$  with  $\mathbf{r} \in \mathbb{N}^*$  and  $\mathbf{c} \in \{0, 1\}^*$ .

**Example 13** The fillings in Figure 7 gives an example where  $\text{ne}(M) = 2$ ,  $\text{ne}(M') = 3$  while  $\text{ne}(N) = \text{ne}(N') = 3$ .

**Northeast chains in fillings of general polyominoes.** It is natural to ask whether Theorem 1 or Theorem 3 can be extended to a more general family of polyominoes. The following example shows that if there are several exceptional rows in the polyomino, then the distribution of  $\text{ne}(M)$  may not be the same after a swap of two adjacent rows.

**Example 14** This example was first given in [16] for layer polyominoes, which are polyominoes that are row-convex and row-intersection-free. The right polyomino in Figure 8 is almost-moon but the left one is not. Let  $G(x, \mathcal{F}) = \sum_{M \in \mathcal{F}} x^{\text{ne}(M)}$  be the generating function for the statistic  $\text{ne}$  over the fillings in a set  $\mathcal{F}$ . First consider 01-fillings where every row and every column has exactly one 1. We obtained the following generating functions. For the left polyomino  $\mathcal{M}_1$ ,

$$G(x, \mathcal{F}(\mathcal{M}_1, \mathbf{r}, \mathbf{c})) = x + 37x^2 + 31x^3 + 3x^4,$$

while for the right polyomino  $\mathcal{M}_2$ ,

$$G(x, \mathcal{F}(\mathcal{M}_2, \mathbf{r}, \mathbf{c})) = p + 36x^2 + 32x^3 + 3x^4,$$

where  $\mathbf{r} = \mathbf{c} = (1, 1, 1, 1, 1)$ . This provides a counterexample of Theorem 3 for general polyominoes.

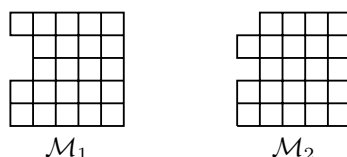


Fig. 8: Two general polyominoes related by an interchange of two adjacent rows.

The above example also implies that Theorem 1 can not hold for general polyominoes. Namely, take the same two polyominoes and consider all the fillings in which each column has exactly one 1, but there is no constraint on the row. Note that (1) empty rows do not affect the statistic  $ne$  and can be ignored and (2) any sub-polyomino of  $\mathcal{M}_1$  or  $\mathcal{M}_2$  containing three rows is an almost-moon polyomino. By Theorem 1 rearranging rows for such three-row polyominoes does not change the distribution of  $ne(M)$  over  $\mathcal{F}(\mathcal{M}, *, \mathbf{c})$ .

Now let  $\mathbf{c} = (1, 1, 1, 1, 1)$ . We have

1. Either  $G(ne, \mathcal{F}(\mathcal{M}_1, *, \mathbf{c})) \neq G(ne, \mathcal{F}(\mathcal{M}_2, *, \mathbf{c}))$  and we have the desired counterexample, or
2.  $G(ne, \mathcal{F}(\mathcal{M}_1, *, \mathbf{c})) = G(ne, \mathcal{F}(\mathcal{M}_2, *, \mathbf{c}))$ . But then the example from the previous paragraph implies that the distribution of  $ne$  over 01-fillings in  $\mathcal{F}(\mathcal{M}_1, *, \mathbf{c})$  and  $\mathcal{F}(\mathcal{M}_2, *, \mathbf{c})$  with empty rows are different. Note that the set of fillings of a polyomino  $\mathcal{M}$  with empty rows can be obtained as the union of set  $\mathcal{F}_i(\mathcal{M}, \mathbf{r}, \mathbf{c})$ , which consists of fillings in which the  $i$ -th row is empty. An application of the inclusion-exclusion principle implies that there is a sub-polyomino  $\mathcal{N}_1$  of  $\mathcal{M}_1$  consisting of 4 rows such that  $G(ne, \mathcal{F}(\mathcal{N}_1, *, \mathbf{c})) \neq G(ne, \mathcal{F}(\mathcal{N}_2, *, \mathbf{c}))$  where  $\mathcal{N}_2$  is the sub-polyomino of  $\mathcal{M}_2$  consisting of the same four rows as in  $\mathcal{N}_1$ .

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