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Generalised cluster algebras and q-characters at roots of unity

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Abstract. Shapiro and Chekhov (2011) have introduced the notion of *generalised cluster algebra*; we focus on an example in type C_n. On the other hand, Chari and Pressley (1997), as well as Frenkel and Mukhin (2002), have studied the *restricted integral form* $U_\varepsilon^{\mathrm{res}}(\widehat{\mathfrak{g}})$ of a quantum affine algebra $U_q(\widehat{\mathfrak{g}})$ where $q = \varepsilon$ is a root of unity. Our main result states that the Grothendieck ring of a tensor subcategory $\mathcal{C}_{\varepsilon^\mathbb{Z}}$ of representations of $U_\varepsilon^{\mathrm{res}}(L\mathfrak{sl}_2)$ is a generalised cluster algebra of type C_{l-1}, where l is the order of ε^2. We also state a conjecture for $U_\varepsilon^{\mathrm{res}}(L\mathfrak{sl}_3)$, and sketch a proof for $l = 2$.

Résumé. Shapiro et Chekhov (2011) ont introduit la notion d'*algèbre amassée généralisée*; nous étudions un exemple en type C_n. Par ailleurs, Chari et Pressley (1997), ainsi que Frenkel et Mukhin (2002), ont étudié la *forme entière restreinte* d'une algèbre affine quantique $U_q(\widehat{\mathfrak{g}})$ où $q = \varepsilon$ est une racine de l'unité. Notre résultat principal affirme que l'anneau de Grothendieck d'une sous-catégorie tensorielle $\mathcal{C}_{\varepsilon^\mathbb{Z}}$ de représentations de $U_\varepsilon^{\mathrm{res}}(L\mathfrak{sl}_2)$ est une algèbre amassée généralisée de type C_{l-1}, où l est l'ordre de ε^2. Nous conjecturons une propriété similaire pour $U_\varepsilon^{\mathrm{res}}(L\mathfrak{sl}_3)$ et donnons un aperçu de la preuve pour $l = 2$.

Keywords: representation theory; (generalised) cluster algebras; quantum affine algebras; restricted integral form; q-characters; ε-characters; Kirillov-Reshetikhin modules; quantum loop algebras

1 Introduction

Cluster algebras have been introduced in 2001 by Fomin and Zelevinski [7]. These rings have special generators, called *cluster variables*. For every cluster $\mathbf{x}$, and every cluster variable $x \in \mathbf{x}$, there is a unique cluster $(\mathbf{x} \setminus \{x\}) \cup \{x'\}$, and an exchange relation

$$xx' = m_+ + m_- \tag{1.1}$$

where $m_\pm$ are exchange monomials in $\mathbf{x} \setminus \{x\}$. Fomin and Zelevinsky [8] have proved a classification theorem for cluster algebras with finitely many clusters (also called of finite type), in terms of Cartan matrices. We are interested in *generalised cluster algebras*, introduced by Shapiro and Chekhov in 2011 [4], which differ from standard cluster algebras by the form of their exchange relations. Finite type classification and combinatorial behaviour stay the same [4]. We exhibit some interesting bases of a generalised cluster algebra $\mathcal{A}_n$ of Cartan type C_n, which will be relevant in representation theory.

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On the other hand, for a finite-dimensional complex Lie algebra $\mathfrak{g}$, let $U_\varepsilon(L\mathfrak{g})$ denote the quantum enveloping algebra of the loop algebra of $\mathfrak{g}$, where the quantum parameter ε is a root of unity in $\mathbb{C}^*$. In the spirit of Hernandez and Leclerc's papers [11, 12], we consider a certain tensor category $\mathcal{C}_{\varepsilon^{\mathbb{Z}}}$ of the category of finite-dimensional representations of $U_\varepsilon(L\mathfrak{g})$, and prove that when $\mathfrak{g} = \mathfrak{sl}_2$, the Grothendieck ring of $\mathcal{C}_{\varepsilon^{\mathbb{Z}}}$ is isomorphic to $\mathcal{A}_{l-1}$ (see Theorem 4.2), where l is the order of ε^2. Moreover, this isomorphism maps the basis of classes of simple objects of $\mathcal{C}_{\varepsilon^{\mathbb{Z}}}$ to the basis of (generalised) cluster monomials, multiplied by Tchebychev polynomials in the single generator of the coefficient ring. For $\mathfrak{g} = \mathfrak{sl}_3$ and $l = 2$, we prove a similar result, where $\mathcal{A}_{l-1}$ is replaced by a generalised cluster algebra of type G_2. Extensive computations with Maple allow us to formulate a conjecture for $\mathfrak{g} = \mathfrak{sl}_3$ and $l > 2$.

2 Generalised cluster algebras

2.1 Background

We recall, following [4], the definition and the main structural properties of generalised cluster algebras (see also [16]). For a fixed integer $n \in \mathbb{N}^*$, let $B = (b_{ij}) \in \mathcal{M}_n(\mathbb{Z})$ be a skew-symmetrisable matrix, i.e. such that there exists an integer diagonal matrix $\tilde{D} = \operatorname{diag}(\tilde{d}_1 \dots \tilde{d}_n)$ such that $\tilde{D}B$ is skew-symmetric.

Suppose that for each index $k \in [\![1, n]\!]$, there is an integer $d_k \in \mathbb{N}$ that divides all coefficients b_{jk} in the k-th column. Introduce the notation

$$\beta_{jk} := \frac{b_{jk}}{d_k} \in \mathbb{Z}. \tag{2.1}$$

Let $(\mathbb{P}, \cdot, \oplus)$ be a commutative semifield, called the *coefficient group*. For example, one can take for $\mathbb{P}$ the tropical semifield $\operatorname{Trop}(\lambda_1, \dots, \lambda_n)$ generated by some indeterminates $\lambda_1, \dots, \lambda_n$. This is by definition the set of Laurent monomials in the λ_i's, with ordinary multiplication and tropical addition

$$\left(\prod_i \lambda_i^{a_i}\right) \oplus \left(\prod_i \lambda_i^{b_i}\right) = \left(\prod_i \lambda_i^{\min(a_i, b_i)}\right).$$

Let $\mathcal{F} = \mathbb{ZP}(t_1, \dots, t_n)$ be the ambient field of rational functions in n independent variables, where $\mathbb{ZP}$ is the integer group ring of $\mathbb{P}$. For a collection of variables $\mathbf{p}_i = (p_{i,0}, p_{i,1}, \dots, p_{i,d_i}) \in \mathbb{P}^{d_i+1}$ ($i \in [\![1, n]\!]$), define the corresponding homogeneous *exchange polynomial*

$$\theta_i[\mathbf{p}_i](u, v) := \sum_{r=0}^{d_i} p_{i,r} u^r v^{d_i - r} \in \mathbb{ZP}[u, v]. \tag{2.2}$$

Definition 2.1 *A* generalised seed *is a triple* $(\mathbf{x}, \bar{\mathbf{p}}, B)$ *where*

(i) the tuple $\mathbf{x} = \{x_1, \dots, x_n\}$*, called a* cluster*, is a collection of algebraically independent elements of* $\mathcal{F}$*, called* cluster variables*, which generate* $\mathcal{F}$ *over* $\operatorname{Frac}\mathbb{ZP}$*;*

(ii) the matrix $B = (b_{ij}) \in \mathcal{M}_n(\mathbb{Z})$*, called the* exchange matrix*, is skew-symmetrisable;*

(iii) $\bar{\mathbf{p}} = (\mathbf{p}_1, \dots, \mathbf{p}_n)$ *is a* coefficient tuple*, where for every* $i \in [\![1, n]\!]$*, the tuples* $\mathbf{p}_i = (p_{i,0}, p_{i,1}, \dots, p_{i,d_i})$ *in* $\mathbb{P}^{d_i+1}$ *are the coefficients of the* i*-th exchange polynomial* θ_i*.*

The triple $(\mathbf{x}, \{\theta_1, \dots, \theta_n\}, B)$ is also called a generalised seed.

Definition 2.2 *The* generalised mutation in direction $k \in [\![1, n]\!]$*, is the operation that transforms a generalised seed* $(\mathbf{x}, \bar{\mathbf{p}}, B)$ *into another generalised seed* $\mu_k(\mathbf{x}, \bar{\mathbf{p}}, B) := (\mathbf{x}', \bar{\mathbf{p}'}, B')$ *given by*

(i) matrix mutation*: the matrix* $B' = (b'_{ij})$ *is defined by*

$$b'_{ij} = \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k \\ b_{ij} + \dfrac{1}{2}\left(|b_{ik}|b_{kj} + b_{ik}|b_{kj}|\right) & \text{otherwise} \end{cases} \tag{2.3}$$

(ii) cluster mutation:

$$\begin{cases} x'_i = x_i & \text{if } i \neq k \\ x_k x'_k = \theta_k[\mathbf{p}_k](u_k^+, u_k^-) \end{cases} \tag{2.4}$$

where we define

$$u_k^+ := \prod_{j=1}^{n} x_j^{[\beta_{jk}]_+} \quad \text{and} \quad u_k^- := \prod_{j=1}^{n} x_j^{[-\beta_{jk}]_+}. \tag{2.5}$$

(iii) coefficient mutation:

$$\begin{cases} p'_{k,r} = p_{k,d_k - r} \\ \dfrac{p'_{i,r}}{p'_{i,r-1}} = \begin{cases} (p_{k,d_k})^{\beta_{ki}} \dfrac{p_{i,r}}{p_{i,r-1}} & \text{if } i \neq k \text{ and } b_{ki} \geq 0 \\ (p_{k,0})^{\beta_{ki}} \dfrac{p_{i,r}}{p_{i,r-1}} & \text{if } i \neq k \text{ and } b_{ki} \leq 0 \end{cases} \end{cases} \tag{2.6}$$

It follows easily from the definition of matrix mutation that for each $k, r \in [\![1, n]\!]$, the integer d_k divides all coefficients in the k-th column of $\mu_r(B)$. Moreover, note that μ_r is an involution. We say that two generalised seeds are *mutation-equivalent* if one can be obtained from the other by performing a finite sequence of mutations.

Definition 2.3 *The* generalised cluster algebra $\mathcal{A}(\bar{\mathbf{p}}, B) = \mathcal{A}(\mathbf{x}, \{\theta_1, \ldots, \theta_n\}, B)$ *of* rank n*, corresponding to the generalised seed* $(\mathbf{x}, \{\theta_1, \ldots, \theta_n\}, B)$*, is the* $\mathbb{ZP}$*-subalgebra of* $\mathcal{F}$ *generated by all cluster variables from all the seeds that are mutation-equivalent to the initial seed* $(\mathbf{x}, \{\theta_1, \ldots, \theta_n\}, B)$*.*

A *cluster monomial* in $\mathcal{A}(\bar{\mathbf{p}}, B)$ is a monomial in the cluster variables involving only variables belonging to a single cluster. We say that a generalised cluster algebra is of *finite type* if it has finitely many cluster variables. The Laurent phenomenon from [7] remains true for generalised cluster algebras.

Theorem 2.4 ([4, Theorem 2.5]) *Every generalised cluster variable is a Laurent polynomial in the initial cluster variables.*

Generalised cluster algebras of finite type can also be classified in terms of Cartan matrices.

Theorem 2.5 ([4, Theorem 2.7]) *Generalised cluster algebras of finite type follow the same Cartan-Killing classification as standard cluster algebras.*

2.2 A generalised cluster algebra of type C_n

Recall that the *exchange graph* of a cluster algebra is the graph whose vertices are the clusters, and two clusters are linked by an edge if they can be obtained from each other by one mutation. We know from [8, Section 12.3] that for a cluster algebra of type C_n, the exchange graph is isomorphic to the n-dimensional cyclohedron. It has a nice description in terms of triangulations of a regular $(2n+2)$-gon $\mathbf{P}_{2n+2}$. Theorem 2.5 allows us to use the same labeling system for a generalised cluster algebra of type C_n. In particular, mutations can be seen as flips between triangulations of $\mathbf{P}_{2n+2}$. Each vertex of the exchange graph corresponds to a centrally symmetric triangulation, and two such triangulations are linked by an edge if they can be obtained from each other either by a flip involving two diameters, or by a pair of centrally symmetric flips. Note that each centrally symmetric triangulation contains a unique diameter.

Let $\mathscr{C}$ be the circle in which $\mathbf{P}_{2n+2}$ is inscribed, and let Θ be the central symmetry around the center of $\mathscr{C}$. Let us identify the set Σ of vertices of $\mathbf{P}_{2n+2}$ with the cyclic group

$$\mathbb{Z}/(2n+2)\mathbb{Z} \cong 2\mathbb{Z}/(4n+4)\mathbb{Z},$$

by labelling the vertices clockwise: 0,2,4,...,$2n$, $2n+2$,$2n+4$,..., $4n+2$, with the natural additive law induced by the cyclic group. We now rename half of the vertices: for each $k \in [\![0,n]\!]$, write

$$(2n+2)+2k := \overline{2k}. \tag{2.7}$$

In particular, $2n+2=\bar{0}$ and $\overline{2n}+2=0$.

Consider a pair $\{[a,b],[\bar{a},\bar{b}]\}$ of centrally symmetric diagonals. We may choose $[a,b]$ to represent the Θ-orbit of this pair. The segment $[a,b]$ divides the circle $\mathscr{C}$ into two arcs. The Θ-orbits of the vertices of $\mathbf{P}_{2n+2}$ that lie on the smallest arc form a set denoted by $\mathcal{O}_{ab}$. For example, if $a<b \in [\![0,2n]\!]$, the set $\mathcal{O}_{ab}$ consists of the Θ-orbits of $a+2$, $a+4$,..., $b-2$. In general, we have $\mathcal{O}_{ab}=\mathcal{O}_{ba}=\mathcal{O}_{\bar{a}\bar{b}}=\mathcal{O}_{\bar{b}\bar{a}}$.

We label cluster variables by the corresponding Θ-orbits of diagonals. Namely, if $b \neq \bar{a}$, the variable x_{ab} corresponds to the pair of diagonals $\{[a,b],[\bar{a},\bar{b}]\}$. If $b=\bar{a}$, the variable $x_{a\bar{a}}$ corresponds to the diameter $\{[a,\bar{a}]\}$. Thus we have

$$x_{ab}=x_{ba}=x_{\bar{a}\bar{b}}=x_{\bar{b}\bar{a}}. \tag{2.8}$$

By convention, if a and b are neighbours in $\mathbf{P}_{2n+2}$, we set $x_{ab}=1$. Note that each cluster variable x_{ab} may also be labelled by the set $\mathcal{O}_{ab}$.

Definition 2.6 *Let $\mathbb{P}=\mathrm{Trop}(\lambda)$. For an integer $n \in \mathbb{N}, n \geq 2$, we denote by $\overline{\mathcal{A}}_n=\mathcal{A}(\mathbf{x},\{\theta_1^0,\dots,\theta_n^0\},B)$ the generalised cluster algebra defined by the initial seed: $\mathbf{x}=\{x_1,\dots,x_n\}$,*

$$\theta_i^0(u,v)=u+v \ \ (i \in [\![1,n-1]\!]), \qquad \theta_n^0(u,v)=u^2+\lambda uv+v^2, \tag{2.9}$$

and

$$B := \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \dots & 0 \\ -1 & 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & -1 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & -1 & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & 1 & 0 \\ 0 & 0 & \dots & 0 & -1 & 0 & 2 \\ 0 & 0 & 0 & \dots & 0 & -1 & 0 \end{pmatrix}. \tag{2.10}$$

Thus $\overline{\mathcal{A}}_n$ is the $\mathbb{Z}[\lambda^{\pm 1}]$-subalgebra of $\mathcal{F}$ generated by the cluster variables. We will rather work with a variant of $\overline{\mathcal{A}}_n$, in which the coefficient λ is not assumed to be invertible.

Definition 2.7 *Let $\mathcal{A}_n$ be the $\mathbb{Z}[\lambda]$-subalgebra of $\mathcal{F}$ generated by the cluster variables of $\overline{\mathcal{A}}_n$.*

As above, we label the cluster variables of $\overline{\mathcal{A}}_n$ (or $\mathcal{A}_n$) by Θ-orbits of diagonals of $\mathbf{P}_{2n+2}$. The initial cluster variables corresponding to the initial seed are as follows (see Figure 1):

$$x_k := x_{\overline{2n},2k} \quad (k \in [\![1,n]\!]). \tag{2.11}$$

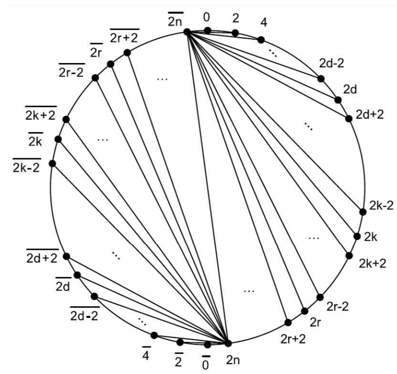

Figure 1: The initial cluster of $\mathcal{A}_n$

Note that the exchange polynomials θ_i^0, $i < n$, are not affected by mutation; they coincide with standard exchange relations. Moreover, flipping a diameter while keeping a non-crossing, centrally-symmetric triangulation of $\mathbf{P}_{2n+2}$, gives another diameter (it is easy to see that the quadrilaterals in which diameters are flipped, are always rectangles). This implies that there is only one variable of the form $x_{a,\bar{a}}$ in each cluster, and it is obtained by mutating another $x_{b,\bar{b}}$. We show the following explicit result in [9].

Proposition 2.8 *In the generalised cluster algebra $\mathcal{A}_n$, the following exchange relations between variables x_{ab} and x_{cd} hold, up to rotation (i.e. index shifting):*

1. *If $a \neq \bar{b}$, $c \neq \bar{d}$, and the quadrilateron $[acbd]$ is contained in one half of the circle $\mathscr{C}$, we have a standard exchange relation of the form*

$$x_{\overline{2n},2k+2}x_{2d-2,2r+2} = x_{\overline{2n},2r+2}x_{2d-2,2k+2} + x_{\overline{2n},2d-2}x_{2k+2,2r+2}, \tag{2.12}$$

which corresponds to the Ptolemy rule in the first diagram of Figure 2.

2. *if $a = \bar{b}$ and $c = \bar{d}$, we have a generalised exchange relation of the form*

$$x_{\overline{2n},2n}x_{\overline{2k},2k} = x_{\overline{2n},2k}^2 + x_{\overline{2k},2n}^2 + \lambda x_{\overline{2n},2k}x_{2n,2k}. \tag{2.13}$$

The monomials with coefficient 1 correspond to the Ptolemy rule in the second diagram of Figure 2.

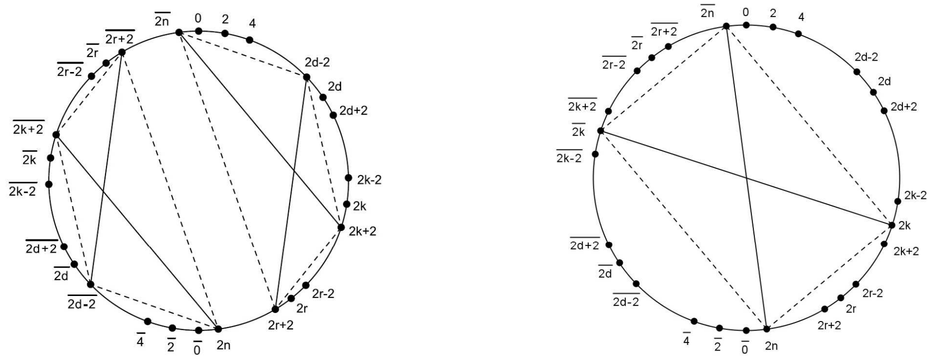

Figure 2: Generalised exchange relations for $\mathcal{A}_n$

We also have the following useful identity for multiplying a diagonal by a diameter: if $a \neq \bar{b}$ and $c = \bar{d}$, relations are of the form

$$x_{\overline{2n},2k}x_{2d,\overline{2d}} = \lambda x_{\overline{2n},2d}x_{2d,2k} + x_{\overline{2n},2d}x_{2k,\overline{2d}} + x_{2d,2k}x_{\overline{2d},\overline{2n}}. \tag{2.14}$$

We call a pair of centrally symmetric diagonals of $\mathbf{P}_{2n+2}$ *small* if the corresponding set $\mathcal{O}_{ab}$ contains only one element. The attached variables x_{ab} are also called *small*. We show that $\mathcal{A}_n$ has some interesting bases. For $k \in \mathbb{N}$, denote by $S_k(u) \in \mathbb{Z}[u]$ the k-th Tchebychev polynomial of the second kind, given by

$$S_k(u)^2 = S_{k-1}(u)S_{k+1}(u) + 1 \tag{2.15}$$

with initial conditions $S_0(u) = 1$ and $S_1(u) = u$.

Proposition 2.9 *The set $\mathcal{S}$ of all monomials in the small variables forms a $\mathbb{Z}$-basis of $\mathcal{A}_n$. Equivalently, $\mathcal{A}_n$ is the polynomial ring with coefficients in $\mathbb{Z}$ in the small variables. Moreover, the set*

$$\mathcal{B} := \{S_k(\lambda) \cdot m,\ k \in \mathbb{N}, m \in \mathcal{M}_0\} \tag{2.16}$$

is a $\mathbb{Z}$-basis of $\mathcal{A}_n$, where $\mathcal{M}_0$ is the set of cluster monomials of $\mathcal{A}_n$.

The basis $\mathcal{B}$ will be meaningful in representation theory.

3 Quantum loop algebras and ε-characters

Let l be an integer, $l \geq 2$. Introduce the root of unity

$$\varepsilon := \begin{cases} \exp\left(\dfrac{i\pi}{l}\right) & \text{if } l \text{ is even} \\[2ex] \exp\left(\dfrac{2i\pi}{l}\right) & \text{if } l \text{ is odd} \end{cases} \tag{3.1}$$

Thus l is the order of ε^2, and we have $\varepsilon^{2l} = 1$. Following [5], let us also write $\varepsilon^* := \varepsilon^{l^2} = 1$.

Let $\mathfrak{g}$ be a finite-dimensional complex simple Lie algebra of simply-laced type, with Dynkin diagram δ, vertex set $I = [\![1, n]\!]$ and Cartan matrix $C = (a_{ij})_{i,j \in I}$. Denote by α_i the simple roots, by ϖ_i the fundamental weights and by P the weight lattice [1].

Let q be an indeterminate; then $\mathbb{C}(q)$ is the field of rational functions of q with complex coefficients, and $\mathbb{C}[q, q^{-1}]$ is the ring of complex Laurent polynomials in q.

Let $U_q(\widehat{\mathfrak{g}})$ be the quantum affine algebra associated with $\mathfrak{g}$ [6]. This is a Hopf algebra over $\mathbb{C}(q)$. Denote by $U_q(L\mathfrak{g})$ the quantum loop algebra, which is isomorphic to a quotient of $U_q(\widehat{\mathfrak{g}})$ where the central charge is mapped to 1. Therefore, $U_q(L\mathfrak{g})$ inherits a Hopf algebra structure.

We will be interested in finite-dimensional representations of $U_q(\widehat{\mathfrak{g}})$, on which the central charge acts trivially. It is then sufficient to consider finite-dimensional representations of $U_q(L\mathfrak{g})$, and we will focus on these onwards.

Let $U_q^{\mathrm{res}}(L\mathfrak{g})$ be the restricted integral form obtained from $U_q(L\mathfrak{g})$ [3]; it is a $\mathbb{C}[q, q^{-1}]$-Hopf algebra.

Let us now specialise $U_q^{\mathrm{res}}(L\mathfrak{g})$ at the root of unity ε, by setting

$$U_\varepsilon^{\mathrm{res}}(L\mathfrak{g}) := U_q^{\mathrm{res}}(L\mathfrak{g}) \otimes_{\mathbb{C}[q,q^{-1}]} \mathbb{C} \tag{3.2}$$

via the algebra homomorphism $f_\varepsilon : \mathbb{C}[q, q^{-1}] \to \mathbb{C}$ such that $f_\varepsilon(q) = \varepsilon$.

Consider the category $\mathcal{C}_q$ of finite-dimensional type 1 representations of the quantum loop algebra $U_q(L\mathfrak{g})$ for q an indeterminate (see [2, Section 11.2]); it is monoidal, abelian and non semisimple. An object V in $\mathcal{C}_q$ has a q-character $\chi_q(V)$ [6], which is an element of $\mathbb{Z}[Y_{i,a}^{\pm 1},\ i \in I,\ a \in \mathbb{C}(q)]$. The simple objects of $\mathcal{C}_q$ are parametrised [6] by the highest monomial of their q-characters, which is a *dominant monomial*, i.e. it contains only positive exponents. If S is a simple object of $\mathcal{C}_q$ such that the highest monomial of $\chi_q(S)$ is m, then S will be denoted by $L(m)$ [14]. For $i \in I,\ k \in \mathbb{N}^*,\ a \in \mathbb{C}(q)$, the simple object

$$W_{k,a}^{(i)} = L(Y_{i,a} Y_{i,aq^2} \dots Y_{i,aq^{2(k-1)}}) \tag{3.3}$$

is called a *Kirillov-Reshetikhin module*. In particular $W_{1,a}^{(i)} = L(Y_{i,a})$ is a *fundamental module*. By convention, $W_{0,a}^{(i)}$ is the trivial representation for any a, i. The following result was conjectured in [13] and proved in [15] (see also [10]):

Theorem 3.1 ([15, 10]) *The classes $[W_{k,a}^{(i)}]$ in $K_0(\mathcal{C}_q)$ satisfy a system of equations called T-system:*

$$[W_{k,a}^{(i)}][W_{k,aq^2}^{(i)}] = [W_{k+1,a}^{(i)}][W_{k-1,aq^2}^{(i)}] + \prod_{j \sim i} [W_{k,aq}^{(j)}] \quad (i \in I, k \in \mathbb{N}^*, a \in \mathbb{C}(q)), \tag{3.4}$$

where $j \sim i$ means that j is a neighbour of i in the Dynkin diagram δ.

Example 3.2 *If $\mathfrak{g} = \mathfrak{sl}_2$, the T-system reads*

$$[W_{k,a}^{(1)}][W_{k,aq^2}^{(1)}] = [W_{k+1,a}^{(1)}][W_{k-1,aq^2}^{(1)}] + 1 \quad (k \in \mathbb{N}^*). \tag{3.5}$$

Let $\mathcal{C}_\varepsilon$ be the category of finite-dimensional type 1 $U_\varepsilon^{\mathrm{res}}(L\mathfrak{g})$-modules. Let $K_0(\mathcal{C}_\varepsilon)$ be its Grothendieck ring. An object V in $\mathcal{C}_\varepsilon$ has an ε-character $\chi_\varepsilon(V)$ [5], which is an element of $\mathbb{Z}[Y_{i,a}^{\pm 1},\ i \in I,\ a \in \mathbb{C}^*]$. The parametrisation of the simple objects by their highest dominant monomials also holds on $\mathcal{C}_\varepsilon$, with q replaced by ε (see [3, 5]). The simple module whose highest weight monomial is m will be written $L(m)$.

In particular, the *fundamental modules* of $U_\varepsilon^{\mathrm{res}}(L\mathfrak{g})$ are the simple objects $L(Y_{i,a})$, where $i \in I, a \in \mathbb{C}^*$, and the *standard modules* are the tensor products of fundamental modules. The simple module $L(m)$ is called *prime* if it cannot be written as a tensor product of non-trivial modules.

The ring $K_0(\mathcal{C}_\varepsilon)$ is the ring of polynomials $\mathbb{Z}[[L(Y_{i,a})]\ i \in I, a \in \mathbb{C}^*]$ (see [5, Section 3.1]).

For a simple object V of $\mathcal{C}_q$, with highest weight vector v, it is known [5, Proposition 2.5] that the $U_q^{\mathrm{res}}(L\mathfrak{g})$-module $V^{\mathrm{res}} := U_q^{\mathrm{res}}(L\mathfrak{g}) \cdot v$ is a free $\mathbb{C}[q, q^{-1}]$-module. Put $V_\varepsilon^{\mathrm{res}} = V^{\mathrm{res}} \otimes_{\mathbb{C}[q,q^{-1}]} \mathbb{C}$, where as above q acts on $\mathbb{C}$ by multiplication by ε. This is a $U_\varepsilon^{\mathrm{res}}(L\mathfrak{g})$-module called the *specialisation of V at $q = \varepsilon$*. Frenkel and Mukhin [5] prove that for a simple module V in $\mathcal{C}_q$, the ε-character $\chi_\varepsilon(V_\varepsilon^{\mathrm{res}})$ is obtained by substituting $q \mapsto \varepsilon$ in $\chi_q(V)$. The ε-characters χ_ε thus satisfy combinatorial properties similar to q-characters ([5, Section 3.2]).

Since the Dynkin diagram δ is a bipartite graph, we may choose a partition of the vertices $I = I_0 \sqcup I_1$, where each edge connects a vertex of I_0 with a vertex of I_1. For $i \in I_d$ $(d = 0, 1)$, set $\xi_i := d$. For $i \in I$ and $a \in \mathbb{C}^*$, introduce the following notation:

$$\mathbf{Y}_{i,a} := \prod_{j=0}^{l-1} Y_{i,a\varepsilon^{2j+\xi_i}}\,. \tag{3.6}$$

Note that since $\varepsilon^{2l} = 1$, we have $\mathbf{Y}_{i,\varepsilon^{2r}} = \mathbf{Y}_{i,1}$ for any $r \in \mathbb{Z}$. A monomial in the variables $Y_{i,a}$ is called *l-acyclic* if it is not divisible by $\mathbf{Y}_{j,b}$ for any $j \in I,\ b \in \mathbb{C}^*$.

Frenkel and Mukhin [5] describe a *quantum Frobenius map* $\mathrm{Fr} : U_\varepsilon^{\mathrm{res}}(L\mathfrak{g}) \to U_{\varepsilon^*}^{\mathrm{res}}(L\mathfrak{g})$ that gives rise to the *Frobenius pullback*

$$\mathrm{Fr}^* : K_0(\mathrm{Rep}\, U_{\varepsilon^*}^{\mathrm{res}}(L\mathfrak{g})) \longrightarrow K_0(\mathcal{C}_\varepsilon), \tag{3.7}$$

and prove that this is the injective ring homomorphism such that $\mathrm{Fr}^*([L(Y_{i,a})]) = [L(\mathbf{Y}_{i,a})]$.

The following theorem was proved by Chari and Pressley for roots of unity of odd order [3] and generalised by Frenkel and Mukhin [5] to roots of unity of arbitrary order.

Theorem 3.3 ([5, Theorem 5.4]) *Let $L(m)$ be a simple object of $\mathcal{C}_\varepsilon$. Then*

$$L(m) \cong L(m^0) \otimes L(m^1), \tag{3.8}$$

using the decomposition $m = m^0 m^1$, where m^1 is a monomial in the variables $\mathbf{Y}_{i,a}$, and m^0 is l-acyclic.

Note that $L(m^1)$ is the Frobenius pullback of an irreducible $U_{\varepsilon^*}^{\mathrm{res}}(L\mathfrak{g})$-module $L(\widetilde{m}^1)$, and the ε-character $\chi_\varepsilon(\mathrm{Fr}^*(L(\widetilde{m}^1)))$ is obtained from $\chi_{\varepsilon^*}(L(\widetilde{m}^1))$ by replacing each $Y_{i,a^l}^{\pm 1}$ by $\mathbf{Y}_{i,a\varepsilon^{\xi_i}}^{\pm 1}$. In fact, since $\varepsilon^* = 1$, the category $\mathrm{Rep}\, U_{\varepsilon^*}^{\mathrm{res}}(L\mathfrak{g})$ is equivalent to $\mathrm{Rep}\, L\mathfrak{g}$, so ε^*-characters are easy to compute.

Therefore, the computation of ε-characters of simple objects of $\mathcal{C}_\varepsilon$ is reduced, by Theorem 3.3, to understanding the ε-characters of all representations $L(m^0)$ where m^0 is l-acyclic.

Let $\mathcal{C}_{\varepsilon^{\mathbb{Z}}}$ be the full subcategory of $\mathcal{C}_\varepsilon$ whose objects M have all their composition factors $L(m)$ such that m contains only variables of the form $Y_{i,\varepsilon^{2k+\xi_i}}$, where $i \in I$ and $k \in \mathbb{Z}$. For example, the modules $z_i := L(\mathbf{Y}_{i,1})$ are objects of $\mathcal{C}_{\varepsilon^{\mathbb{Z}}}$. Let $K_0(\mathcal{C}_{\varepsilon^{\mathbb{Z}}})$ be the Grothendieck ring of $\mathcal{C}_{\varepsilon^{\mathbb{Z}}}$. This is the subring of $K_0(\mathcal{C}_\varepsilon)$ generated by the classes $[L(Y_{i,\varepsilon^{2k+\xi_i}})], i \in I,\ k \in \mathbb{Z}$.

4 Cluster structure on $K_0(\mathcal{C}_{\varepsilon^{\mathbb{Z}}})$ in type A_1

Let us introduce the following notation for *specialised Kirillov-Reshetikhin modules*:

$$W_\varepsilon(k,a,i) := \left(W_{k,a}^{(i)}\right)_\varepsilon^{\mathrm{res}} \quad (i \in I,\ k \in \mathbb{N},\ a \in \mathbb{C}^*). \tag{4.1}$$

In type A_1, that is, for $\mathfrak{g} = \mathfrak{sl}_2$, we have $I = \{1\}$, so we drop the index i in the above notation. We also drop the index $i = 1$ in the variables Y_{1,ε^n} and introduce the notation

$$Y_n := Y_{1,\varepsilon^n} \quad \text{and} \quad \mathbf{Y}_1 = Y_0 Y_2 \ldots Y_{2l-2} \tag{4.2}$$

for any integer n. We then have $Y_{2l+n} = Y_n$ for every n. With this new notation, we have

$$W_\varepsilon(k,\varepsilon^{2r}) = L(Y_{2r}Y_{2r+2}Y_{2r+4}\ldots Y_{2(r+k-1)}) \quad (k,r \in [\![0,l-1]\!]), \quad z_1 = L(\mathbf{Y}_1). \tag{4.3}$$

To each module $W_\varepsilon(k,\varepsilon^{2r})$ with $k < l$ and $r \in [\![1,l]\!]$, attach the diagonal $[2r-2, 2r+2k]$ of the $2l$-gon $\mathbf{P}_{2l}$ defined in Section 2. The following result is a reformulation of a theorem of Chari and Pressley [3].

Theorem 4.1 ([3, Theorem 9.6]) *For $\mathfrak{g} = \mathfrak{sl}_2$, the simple objects of $\mathcal{C}_{\varepsilon^{\mathbb{Z}}}$ are exactly the modules below:*

$$\bigotimes_{t=1}^{r} L(Y_{2d_t}\ldots Y_{2(d_t+k_t-1)})^{\otimes a_t} \otimes L(\mathbf{Y}_1^a) = \bigotimes_{t=1}^{r} W_\varepsilon(k_t,\varepsilon^{2d_t})^{\otimes a_t} \otimes L(\mathbf{Y}_1^a) \tag{4.4}$$

where $r \in \mathbb{N}^*,\ k_1,\ldots,k_r \in [\![0,l-1]\!],\ d_1,\ldots,d_r \in \mathbb{Z},\ a_1,\ldots,a_r,a \in \mathbb{N}$, with the condition that for every $t \neq s \in [\![1,r]\!]$, the diagonals $[2d_t-2, 2d_t+2k_t]$ and $[2d_s-2, 2d_s+2k_s]$ do not cross inside $\mathbf{P}_{2l}$.

It follows from Theorem 4.1 that the *prime* simple modules of $U_\varepsilon^{\mathrm{res}}(L\mathfrak{sl}_2)$ are the modules $W_\varepsilon(k,\varepsilon^{2d})$ $(k < l,\ r \in [\![1,l]\!])$ and the Frobenius pullbacks $L(\mathbf{Y}_1^a)$ $(a \in \mathbb{N}^*)$.

The modules $W_\varepsilon(k,\varepsilon^{2r})$, for $k < l-1$, satisfy the T-system (3.5), and the ε-characters behave like their q-character counterparts. The $(l-1)$-th relation is different from the T-system:

$$\begin{aligned}&\chi_\varepsilon(L(Y_0Y_2Y_4\ldots Y_{2(l-2)}))\chi_\varepsilon(L(Y_2Y_4\ldots Y_{2(l-1)}))\\ &\quad = \chi_\varepsilon(L(Y_2Y_4\ldots Y_{2(l-2)}))\cdot\chi_\varepsilon(z_1) + 1 + \chi_\varepsilon(L(Y_2Y_4\ldots Y_{2(l-2)}))^2.\end{aligned} \tag{4.5}$$

We can now state our main result, proved in [9]. Recall the generalised cluster algebra $\mathcal{A}_n$ defined in Section 2, Definition 2.7. Let $R = K_0(\mathcal{C}_{\varepsilon^{\mathbb{Z}}})$ be the Grothendieck ring of $\mathcal{C}_{\varepsilon^{\mathbb{Z}}}$ for $\mathfrak{g} = \mathfrak{sl}_2$ and ε as in (3.1).

Theorem 4.2 *There exists a ring isomorphism $\varphi : \mathcal{A}_{l-1} \to R$, such that*

$$\varphi(x_{2r,2d}) = [L(Y_{2r+2}\ldots Y_{2d-2})]\ (r,d \in [\![0,l-1]\!], |r-d| < l), \quad \varphi(\lambda) = [z_1]. \tag{4.6}$$

The $\mathbb{Z}$-basis $\mathcal{S}$ of $\mathcal{A}_{l-1}$ is mapped by φ to the basis of classes of standard modules in R. The $\mathbb{Z}$-basis $\mathcal{B}$ of $\mathcal{A}_{l-1}$ consisting in generalised cluster monomials (see (2.16)) is mapped to the basis B of classes of simple modules in R.

5 Type A_2

For any integer $n \in \mathbb{Z}$ and any vertex $i = 1, 2$ of the Dynkin diagram, we set $Y_{i,n} := Y_{i,\varepsilon^n}$, where we consider the second index modulo 4. We also write $\mathbf{Y}_1 = Y_{1,0}Y_{1,2}$ and $\mathbf{Y}_2 = Y_{2,1}Y_{2,3}$.

Theorem 5.1 *Any simple finite-dimensional $U_\varepsilon^{\mathrm{res}}(L\mathfrak{sl}_3)$-module $L(m)$ can be written*

$$L(m) = L(\mathbf{Y}_1^k \mathbf{Y}_2^\ell) \otimes L(m^0),$$

where $L(m^0)$ is one of the following eight tensor products:

$$\begin{array}{llll} (i) & L(Y_{1,0})^{\otimes a} \otimes L(Y_{1,0}Y_{2,1})^{\otimes b} & (ii) & L(Y_{2,1})^{\otimes a} \otimes L(Y_{1,0}Y_{2,1})^{\otimes b} \\ (iii) & L(Y_{1,0})^{\otimes a} \otimes L(Y_{1,0}Y_{2,3})^{\otimes b} & (iv) & L(Y_{2,3})^{\otimes a} \otimes L(Y_{1,0}Y_{2,3})^{\otimes b} \\ (v) & L(Y_{1,2})^{\otimes a} \otimes L(Y_{1,2}Y_{2,1})^{\otimes b} & (vi) & L(Y_{2,1})^{\otimes a} \otimes L(Y_{1,2}Y_{2,1})^{\otimes b} \\ (vii) & L(Y_{1,2})^{\otimes a} \otimes L(Y_{1,2}Y_{2,3})^{\otimes b} & (viii) & L(Y_{2,3})^{\otimes a} \otimes L(Y_{1,2}Y_{2,3})^{\otimes b}. \end{array}$$

This follows from Theorem 3.3 and some direct computations of ε-characters. The following identities can also be checked directly.

Proposition 5.2 *The following identities hold, for $i \in \{0, 2\}$ and $j \in \{1, 3\}$.*

$$\begin{aligned} \chi_\varepsilon(L(Y_{1,i}))\chi_\varepsilon(L(Y_{2,j})) &= \chi_\varepsilon(L(Y_{1,i}Y_{2,j})) + 1 \\ \chi_\varepsilon(L(Y_{1,i}Y_{2,1}))\chi_\varepsilon(L(Y_{1,i}Y_{2,3})) &= \chi_\varepsilon(L(Y_{1,i}))^3 + \chi_\varepsilon(L(Y_{1,i}))^2\chi_\varepsilon(L(\mathbf{Y}_2)) \\ &\quad + \chi_\varepsilon(L(Y_{1,i}))\chi_\varepsilon(L(\mathbf{Y}_1)) + 1 \\ \chi_\varepsilon(L(Y_{1,0}Y_{2,j}))\chi_\varepsilon(L(Y_{1,2}Y_{2,j})) &= \chi_\varepsilon(L(Y_{2,j}))^3 + \chi_\varepsilon(L(Y_{2,j}))^2\chi_\varepsilon(L(\mathbf{Y}_1)) \\ &\quad + \chi_\varepsilon(L(Y_{2,j}))\chi_\varepsilon(L(\mathbf{Y}_2)) + 1. \end{aligned} \tag{5.1}$$

The Grothendieck ring $R := K_0(\mathcal{C}_{\varepsilon^{\mathbb{Z}}})$ for $\mathfrak{g} = \mathfrak{sl}_3,\ l = 2$, is isomorphic to a polynomial ring:

$$R \cong \mathbb{Z}[[L(Y_{1,0})], [L(Y_{1,2})], [L(Y_{2,1})], [L(Y_{2,3})]]. \tag{5.2}$$

Recall that for $\mathfrak{g} = \mathfrak{sl}_3$, the Grothendieck ring of the finite-dimensional representations of $\mathfrak{g}$ is isomorphic to the polynomial ring $\mathbb{Z}[[V(\varpi_1)], [V(\varpi_2)]]$. In this case, the simple modules are of the form $V(a_1\varpi_1 + a_2\varpi_2)$ with $(a_1, a_2 \in \mathbb{N})$. They are mapped by the Frobenius pullback to polynomials $S_{a_1,a_2}([L(\mathbf{Y}_1)], [L(\mathbf{Y}_2)])$ that can be computed inductively using the Littlewood-Richardson rule.

Let $\overline{\mathcal{G}}$ be the generalised cluster algebra of type G_2, with $\mathbb{P} = \mathrm{Trop}(\lambda_1, \lambda_2)$, initial cluster $\{x_1, x_2\}$, exchange matrix

$$B = \begin{pmatrix} 0 & 3 \\ -1 & 0 \end{pmatrix},$$

and initial exchange polynomials $\theta_1^0(u, v) = u + v$, $\theta_2^0(u, v) = u^3 + \lambda_1 u^2 v + \lambda_2 u v^2 + v^3$. There are eight cluster variables $x_1, \ldots, x_8$, organised as in Figure 3.

Let $\mathcal{G}$ be the $\mathbb{Z}[\lambda_1, \lambda_2]$-subalgebra of the ambient field $\mathcal{F}$ generated by the cluster variables of $\overline{\mathcal{G}}$. As for $\mathcal{A}_n$, one can also check that $\mathcal{G}$ is isomorphic to a polynomial ring:

$$\mathcal{G} \cong \mathbb{Z}[x_1, x_3, x_5, x_7]. \tag{5.3}$$

Denote by $\mathcal{M}_0$ the set of generalised cluster monomials of $\mathcal{G}$. Then the set

$$\mathcal{H} := \{S_{a_1,a_2}(\lambda_1, \lambda_2) \cdot m,\ a_1, a_2 \in \mathbb{N},\ m \in \mathcal{M}_0\} \tag{5.4}$$

is a $\mathbb{Z}$-basis of $\mathcal{G}$. Let us now exhibit a cluster structure on R.

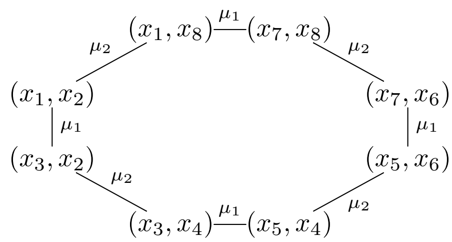

Figure 3: The clusters in type G_2

Theorem 5.3 *For $l = 2$ and $\mathfrak{g} = \mathfrak{sl}_3$, there exists a ring isomorphism $\eta : \mathcal{G} \to R$ such that*

$$\begin{array}{llll} \eta(x_1) = [L(Y_{1,0})] & \eta(x_2) = [L(Y_{1,0}Y_{2,3})] & \eta(x_3) = [L(Y_{2,3})] & \eta(x_4) = [L(Y_{1,2}Y_{2,3})] \\ \eta(x_5) = [L(Y_{1,2})] & \eta(x_6) = [L(Y_{1,2}Y_{2,1})] & \eta(x_7) = [L(Y_{2,1})] & \eta(x_8) = [L(Y_{1,0}Y_{2,1})] \\ & \eta(\lambda_1) = [L(\mathbf{Y}_1)] & \eta(\lambda_2) = [L(\mathbf{Y}_2)] & \end{array} \tag{5.5}$$

The $\mathbb{Z}$-basis $\mathcal{E} \subset \mathcal{G}$ of monomials in x_1, x_3, x_5, x_7 is mapped by η to the basis of classes of standard modules in R. Moreover, the $\mathbb{Z}$-basis $\mathcal{H}$ of generalised cluster monomials of $\mathcal{G}$ is mapped to the basis B of classes of simple modules in R.

Our various computations have led us to the following conjecture.

Conjecture 5.4 *For $l \geq 2$, the Grothendieck ring of $\mathcal{C}_{\varepsilon^{\mathbb{Z}}}$, is isomorphic to a generalised cluster algebra $\mathcal{G}_l$ of rank $2l - 2$. An initial seed of $\mathcal{G}_l$ is given by the exchange matrix*

$$\begin{pmatrix} 0 & -1 & 1 & 0 & \dots & 0 & \dots & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & -1 & 1 & 0 & \ddots & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & -1 & 1 & 0 & \vdots & \vdots & \vdots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 & 0 & 0 \\ 0 & \dots & 0 & -1 & 1 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & \dots & & 0 & -1 & 1 & 0 & -1 & 1 & 0 & 0 \\ & & & 0 & 0 & -1 & 1 & 0 & -1 & 1 & 0 \\ \vdots & \ddots & & 0 & 0 & 0 & -1 & 1 & 0 & -2 & 3 \\ & & & 0 & 0 & 0 & 0 & -1 & 2 & 0 & -3 \\ 0 & & \dots & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \end{pmatrix}, \tag{5.6}$$

the cluster variables correspond to

$$\begin{aligned} x_{2k+1} &= [L(Y_{1,0}Y_{1,2l-2} \dots Y_{1,2l-2k})] \quad (k \in [\![0, l-2]\!]), \\ x_{2k} &= [L(Y_{2,2l-1}Y_{2,2l-3} \dots Y_{2,2l-2k+1})] \quad (k \in [\![1, l-2]\!]), \\ x_{2l-2} &= [L(Y_{1,0}Y_{1,2l-2}Y_{1,2l-4} \dots Y_{1,4}Y_{2,2l-1}Y_{2,2l-3}Y_{2,2l-5} \dots Y_{2,5}Y_{2,3})]. \end{aligned}$$

The coefficients are $\lambda_i = [L(\mathbf{Y}_i)], i = 1, 2,$ where $\mathbf{Y}_1 = Y_{1,0}Y_{1,2} \dots Y_{1,2l-2}$ and $\mathbf{Y}_2 = Y_{2,1}Y_{2,3} \dots Y_{2,2l-1}$.

The initial exchange polynomials are

$$\theta_r^0(u,v) = u + v \; (r \in [\![1, 2l-3]\!]) \quad \textit{and} \quad \theta_{2l-2}^0(u,v) = u^3 + \lambda_1 u^2 v + \lambda_2 u v^2 + v^3.$$

Moreover, the generalised cluster monomials are mapped to classes of simple modules.

Remark 5.5 *For $l > 2$, the above generalised cluster algebras are of infinite type, so we only hope for an inclusion of the cluster monomials in the set of classes of simple modules $L(m)$ where m is l-acyclic.*

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