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Rigged configurations of type $D_4^{(3)}$ and the filling map

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Abstract. We give a statistic preserving bijection from rigged configurations to a tensor product of Kirillov–Reshetikhin crystals $\bigotimes_{i=1}^{\infty} B^{1,s_i}$ in type $D_4^{(3)}$ by using virtualization into type $D_1^{(1)}$. We consider a special case of this bijection with $B = B^{1,s_1}$, and we obtain the so-called Kirillov–Reshetikhin tableaux model for the Kirillov–Reshetikhin crystal.

Résumé. Nous donnons une bijection préservant les statistiques entre les configurations gréées et les produits tensoriels de cristaux de Kirillov–Reshetikhin $\bigotimes_{i=1}^{\infty} B^{1,s_i}$ de type $D_4^{(3)}$, via une virtualisation en type $D_1^{(1)}$. Nous considérons un cas particulier de cette bijection pour $B = B^{1,s_1}$ et obtenons ainsi les modèles de tableaux appelés Kirillov–Reshetikhin pour le cristal Kirillov–Reshetikhin.

Keywords: rigged configuration, Kirillov–Reshetikhin crystal, bijection

1 Introduction

Rigged configurations were first introduced by Kerov, Kirillov, and Reshetikhin in [14,15] as combinatorial objects that index solutions to the Bethe Ansatz for the Heisenberg spin chains. Rigged configurations were shown to be in bijection with semi-standard tableaux and classical highest weight elements of a tensor power of the vector representation in type $A_n^{(1)}$. This bijection was then extended to Littlewood–Richardson tableaux [16], to non-exceptional types [20], and to type $E_6^{(1)}$ [19]. This bijection $\Phi$ between rigged configurations and the tensor powers has been further expanded to include classically highest weight elements in a tensor product of certain Kirillov–Reshetikhin (KR) crystals [16,21,27,28].

Rigged configurations have been shown to display remarkable representation theoretic properties. A (classical) crystal structure was first given for simply-laced types [26], which was then extended to all finite types [27] and affine types [24]. While $\Phi$ is defined recursively, making it difficult to work with, it preserves certain natural statistics (cocharge and energy), giving a bijective proof of the $X=M$ conjecture of [3,5]. Furthermore, the combinatorial $R$-matrix transforms into the identity map on rigged configurations under $\Phi$. Rigged configurations are also well-behaved under virtualization [21,22,27], a process of realizing a non-simply-laced type crystal inside of a simply-laced type, and embeddings $B(\lambda) \hookrightarrow B(\mu)$ where $\lambda \leq \mu$ component-wise, leading to a model for $B(\infty)$ [24].

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KR crystals in non-exceptional types were given a combinatorial model in \cite{4} using Kashiwara–Nakashima tableaux \cite{13}. The bijection $\Phi$ has also lead to a new tableaux model for KR crystals, coined Kirillov–Reshetikhin (KR) tableaux, using filled rectangular tableaux \cite{18, 25, 27}. The map between Kashiwara–Nakashima tableaux \cite{13} and the KR tableaux is called the filling map.

The goal of this work is to extend $\Phi$ to type $D_4^{(3)}$ and describe the filling map. For this extended abstract, we will be focusing on the KR crystals $B^{1,s}$ and the rigged configurations associated with tensor products of the form $\bigotimes_{i=1}^{N} B^{1,s_i}_i$. In particular, we show $\Phi$ is a classical crystal isomorphism, and we describe the filling map for $B^{1,s}$. We do so by showing the filling map and bijection commute with the virtualization map, proving more special cases of many of the conjectures stated in \cite{27}.

This extended abstract is organized as follows. In Section 2, we give background on crystals, virtualization, and rigged configurations. In Section 3, we describe the bijection $\Phi$. In Section 4, we describe the filling map. In Section 5, we describe the virtualization map and our main results. In Section 6, we give possible extensions to $B^{2,s}$ and some open questions. We conclude in Section 7 with some examples using Sage \cite{29}.

2 Background

2.1 Crystals

For this extended abstract, let $g$ be the Kac–Moody algebra of type $D_4^{(3)}$ with index set $I = \{0, 1, 2\}$, generalized Cartan matrix $A = (A_{ij})_{i,j \in I}$, weight lattice $P$, root lattice $Q$, fundamental weights $\{\Lambda_i \mid i \in I\}$, simple roots $\{\alpha_i \mid i \in I\}$, and simple coroots $\{h_i \mid i \in I\}$. There is a canonical pairing $\langle \ , \rangle : P^\vee \times P \to \mathbb{Z}$ defined by $\langle h_i, \alpha_j \rangle = A_{ij}$, where $P^\vee$ is the dual weight lattice. Let $g_0$ denote the classical subalgebra of type $G_2$ with index set $I_0 = \{1, 2\}$, weight lattice $P$, root lattice $Q$, fundamental weights $\{\Lambda_1, \Lambda_2\}$, and simple roots $\{\tau_1, \tau_2\}$.

An abstract $U_q(g)$-crystal is a nonempty set $B$ together with a weight function $wt : B \to P$, crystal operators $e_a, f_a : B \to B \sqcup \{0\}$, and maps $\varphi_a : B \to \mathbb{Z} \sqcup \{-\infty\}$ for $a \in I$, subject to the conditions

1. $\varphi_a(b) = \varepsilon_a(b) + \langle h_a, wt(b) \rangle$ for all $a \in I$,
2. if $e_a b \in B$, then $\varepsilon_a(e_a b) = \varepsilon_a(b) - 1$, $\varphi_a(e_a b) = \varphi_a(b) + 1$, and $wt(e_a b) = wt(b) + \alpha_a$,
3. if $f_a b \in B$, then $\varepsilon_a(f_a b) = \varepsilon_a(b) + 1$, $\varphi_a(f_a b) = \varphi_a(b) - 1$, and $wt(f_a b) = wt(b) - \alpha_a$,
4. $f_a b = b'$ if and only if $b = e_a b'$ for $b, b' \in B$ and $a \in I$,
5. if $\varphi_a(b) = -\infty$ for $b \in B$, then $e_a b = f_a b = 0$.

We define for all $b \in B$

$$\varepsilon_a(b) = \max \{k \in \mathbb{Z}_{\geq 0} \mid e_a^k b \neq 0\}, \quad \varphi_a(b) = \max \{k \in \mathbb{Z}_{\geq 0} \mid f_a^k b \neq 0\}.$$  \hspace{1cm} (2.1)
An important class of finite dimensional \( U_2 \). Kirillov–Reshetikhin crystals by Kang and Misra [9].

An abstract \( U_q(\mathfrak{g}) \)-crystal with \( \varepsilon_a \) and \( \varphi_a \) defined as above is called a regular crystal.

Let \( B_1 \) and \( B_2 \) be abstract \( U_q(\mathfrak{g}) \)-crystals. The tensor product of crystals \( B_2 \otimes B_1 \) is defined to be the Cartesian product \( B_2 \times B_1 \) with the crystal structure

\[
e_i(b_2 \otimes b_1) = \begin{cases} e_i(b_2) \otimes b_1 & \text{if } \varepsilon_i(b_2) > \varphi_i(b_1), \\ b_2 \otimes e_i b_1 & \text{if } \varepsilon_i(b_2) \leq \varphi_i(b_1), \\ \varepsilon_i(b_2 \otimes b_1) = \max \{ \varepsilon_i(b_2), \varepsilon_i(b_1) - \langle h_i, \text{wt}(b_2) \rangle \} & \text{if } \varepsilon_i(b_2) \leq \varepsilon_i(b_1) \end{cases}
\]

\[
f_i(b_2 \otimes b_1) = \begin{cases} f_i(b_2) \otimes b_1 & \text{if } \varepsilon_i(b_2) \geq \varphi_i(b_1), \\ b_2 \otimes f_i b_1 & \text{if } \varepsilon_i(b_2) < \varphi_i(b_1), \\ \varphi_i(b_2 \otimes b_1) = \max \{ \varphi_i(b_1), \varphi_i(b_2) + \langle h_i, \text{wt}(b_2) \rangle \} & \text{if } \varepsilon_i(b_2) \leq \varepsilon_i(b_1) \end{cases}
\]

\[\text{wt}(b_2 \otimes b_1) = \text{wt}(b_2) + \text{wt}(b_1).\]

\[\text{Remark 2.1} \quad \text{Our tensor product convention is the opposite to that given in [12].}\]

Let \( B_1 \) and \( B_2 \) be two abstract \( U_q(\mathfrak{g}) \)-crystals. A crystal morphism \( \psi : B_1 \rightarrow B_2 \) is a map \( B_1 \cup \{0\} \rightarrow B_2 \cup \{0\} \) with \( \psi(0) = 0 \) such that for \( b \in B_1 \)

1. if \( \psi(b) \in B_2 \), then \( \text{wt}(\psi(b)) = \text{wt}(b), \varepsilon_i(\psi(b)) = \varepsilon_i(b), \) and \( \varphi_i(\psi(b)) = \varphi_i(b); \)
2. we have \( \psi(e_i b) = e_i \psi(b) \) provided \( \varepsilon_i b \neq 0 \) and \( e_i \psi(b) \neq 0; \)
3. we have \( \psi(f_i b) = f_i \psi(b) \) provided \( f_i \psi(b) \neq 0\).

A crystal embedding or isomorphism is a crystal morphism such that the induced map \( B_1 \cup \{0\} \rightarrow B_2 \cup \{0\} \) is an embedding or bijection respectively. A crystal morphism is strict if it commutes with all crystal operators.

If an abstract \( U_q(\mathfrak{g}) \)-crystal \( B \) is isomorphic to the crystal basis of an integrable \( U_q(\mathfrak{g}) \)-module, we simply say \( B \) is a \( U_q(\mathfrak{g}) \)-crystal. In particular, an irreducible highest weight \( U_q(\mathfrak{g}_0) \)-module with highest weight \( \lambda \) admits a crystal basis [11], which we denote by \( B(\lambda) \). Moreover there is a unique element \( u_\lambda \in B(\lambda) \) such that \( \text{wt}(u_\lambda) = \lambda \) and \( e_a u_\lambda = 0 \) for all \( a \in I_0 \). For each dominant integral weight \( \lambda = k_1 \lambda_1 + k_2 \lambda_2 \), we can associate a partition \( (k_1 + k_2, k_2) \). We can realize \( B(\lambda) \) as semistandard tableaux of shape \( \lambda \) filled with entries in \( B(\lambda_1) \) whose crystal structure is given by embedding into \( B(\lambda_1) \otimes [\lambda] \) using the reverse far-eastern reading word. The resulting tableaux were explicitly described by Kang and Misra [9].

### 2.2 Kirillov–Reshetikhin crystals

An important class of finite dimensional \( U'_q(\mathfrak{g}) \)-representations are Kirillov–Reshetikhin (KR) modules \( W^{r,s} \) indexed by \( r \in I_0 \) and \( s \in \mathbb{Z}_{>0} \). KR modules are characterized by their Drinfeld polynomials [23].
and correspond to the minimal affinization of $B(s\Lambda_{c})$ [1]. The KR modules $W^{1,s}$ admit a crystal basis called Kirillov–Reshetikhin (KR) crystals and denoted by $B^{1,s}$. As $U_{q}(g_{0})$-crystals, we have $B^{1,s} \cong \bigoplus_{k=1}^{\infty} B(k\Lambda_{1})$, and $B^{1,s}$ is a perfect crystal [10]. This means we can use a semi-infinite tensor product of $B^{1,s}$ to realize highest weight $U_{q}(g)$-crystals, see [7] for details.

There is a statistic called energy defined on $B = \bigotimes_{i=1}^{N} B^{1,s}$ [5]. First we define the local energy function on $B^{1,s} \otimes B^{1,t}$ as follows. The combinatorial $R$-matrix is the unique $U_{q}^{r}(g)$-crystal isomorphism $R: B^{1,s} \otimes B^{1,t} \rightarrow B^{1,t} \otimes B^{1,s}$ [10]. Let $c' \otimes c = R(b \otimes b')$. We define

$$H(e_{i}(b \otimes b')) = H(b \otimes b') + \begin{cases} -1 & i = 0 \text{ and } e_{0}(b \otimes b') = e_{0}b \otimes b' \text{ and } e_{0}(c' \otimes c) = c' \otimes e_{0}c, \\ 1 & i = 0 \text{ and } e_{0}(b \otimes b') = e_{0}b \otimes b' \text{ and } e_{0}(c' \otimes c) = e_{0}c', \\ 0 & \text{otherwise}. \end{cases} \tag{2.2}$$

The local energy function is defined up to an additive constant [8], and so we normalize $H$ by the condition $H(1^{s} \otimes 1^{t}) = 0$ where $1^{k}$ is row of length $k$ filled with 1. Next we define $D_{B^{1,s}}: B^{1,s} \rightarrow \mathbb{Z}$ by

$$D_{B^{1,s}}(b) = H(b \otimes b') - H(1^{s} \otimes b''), \tag{2.3}$$

where $b''$ is the unique element such that $\varphi(b'') = s\Lambda_{0}$. Then we define

$$D(b_{N} \otimes \ldots \otimes b_{1}) = \sum_{1 \leq i < j \leq N} H_{i}R_{i+1}R_{i+2} \cdots R_{j-1} + \sum_{j=1}^{N} D_{B^{1,s}}(b_{j}R_{1}R_{2} \cdots R_{j-1}), \tag{2.4}$$

where $R_{i}$ and $H_{i}$ are the combinatorial $R$-matrix and local energy function, respectively, acting on the $i$-th and $(i + 1)$-th factors and $D_{B^{1,s}}$ acts on the rightmost factor. We say the energy of an element $b \in B$ is $D(b)$.

### 2.3 Rigged configurations

Let $H_{0} = I_{0} \times \mathbb{Z}_{>0}$. Consider a multiplicity array $L = \{ L_{i}^{(a)} \in \mathbb{Z}_{\geq 0} \mid (a, i) \in H_{0} \}$ and a dominant integral weight $\lambda$ of $g_{0}$. A $(L; \lambda)$-configuration is a sequence of partitions $\nu = \{ \nu^{(a)} \mid a \in I \}$ such that

$$\sum_{(a, i) \in H_{0}} iL_{i}^{(a)}\mu_{a} = \sum_{(a, i) \in H_{0}} iL_{i}^{(a)}\mu_{a} - \lambda, \tag{2.5}$$

where $m_{i}^{(a)}$ is the number of parts of length $i$ in the partition $\nu^{(a)}$. We denote the set of $(L, \lambda)$-configurations by $C(L, \lambda)$. The vacancy numbers of $\nu \in C(L, \lambda)$ are defined as

$$p_{i}^{(a)} = \sum_{j \geq 1} \min(i, j)L_{j}^{(a)} - \sum_{(b, j) \in H_{0}} A_{ab} \min(i, j)m_{j}^{(b)}. \tag{2.6}$$

A rigged configuration of classical weight $\lambda$ is a $(L; \lambda)$-configuration $\nu$, along with a sequence of multisets of integers $J = \{ J_{i}^{(a)} \mid (a, i) \in H_{0} \}$ such that $|J_{i}^{(a)}| = m_{i}^{(a)}$ (the size of $J_{i}^{(a)}$) and $\max J_{i}^{(a)} \leq p_{i}^{(a)}$. (Often each $J_{i}^{(a)}$ will be sorted in weakly decreasing order.) So to each row of length $i$, we have an integer $x \in J_{i}^{(a)}$ and we call the pair $(i, x)$ a string. The integers $x \in J_{i}^{(a)}$ are called label, rigging, or
quantum number. The colabel of a string \((i, x)\) is defined as \(p_i^{(a)} - x\). A rigged configuration is highest weight if \(\min J_i^{(a)} \geq 0\) for all \((a, i) \in \mathcal{H}_0\) and is valid if \(\max J_i^{(a)} \leq p_i^{(a)}\). We say a string \((i, a)\) is singular if \(p_i^{(a)} = x\) and is quasi-singular if \(p_i^{(a)} = x - 1\) and \(J_i^{(a)} \neq p_i^{(a)}\).

**Example 2.2** R rigged configurations will be depicted with vacancy numbers on the left and labels on the right. For example,

\[
\begin{array}{cccc}
5 & 4 & 3 & 2 \\
1 & 1 & 1 & 1 \\
\end{array}
\]

is a rigged configuration of weight \(2\Lambda_1 + \Lambda_2\) with \(L\) is given by \(L_1^{(1)} = L_2^{(1)} = 1\) with all other \(L_i^{(a)} = 0\). See Section 7 on how to construct this example in Sage.

Denote by \(RC^*(L; \lambda)\) the set of valid highest weight rigged configurations. Rigged configurations have an abstract \(U_q(\mathfrak{g}_0)\)-crystal structure [27]. To obtain the weight, we first note that we can compute the classical weight by

\[
\overline{\text{wt}}(\nu, J) = \sum_{(a, i) \in \mathcal{H}_0} i(L_i^{(a)} - m_i^{(a)} + \mathfrak{L}_i^{(a)}).
\]  

We can extend this to \(\text{wt} : RC(L; \lambda) \rightarrow P\) by \(\text{wt}(\nu, J) = k_0 \Lambda_0 + \overline{\text{wt}}(\nu, J)\), where \(k_0\) is such that \(\langle \overline{\text{wt}}(\nu, J), c \rangle = 0\) with \(c\) the canonical central element of \(\mathfrak{g}\) (i.e., we make \(\text{wt}(\nu, J)\) be level \(0\)). Explicitly, if \(\overline{\text{wt}}(\nu, J) = c_1 \Lambda_1 + c_2 \Lambda_2\), then we have \(k_0 = -2c_1 - 3c_2\). Next we recall the crystal operators.

**Definition 2.3** Let \(\mathfrak{g}_0\) be a Lie algebra of finite type and \(L\) a multiplicity array. Let \((\nu, J)\) be a valid rigged configuration. Fix \(a \in I_0\) and let \(x\) be the smallest label of \((\nu, J)^{(a)}\), the strings associated to \(\nu^{(a)}\).

1. If \(x \geq 0\), then set \(e_a(\nu, J) = 0\). Otherwise, let \(\ell\) be the minimal length of all strings in \((\nu, J)^{(a)}\) which have label \(x\). The rigged configuration \(e_a(\nu, J)\) is obtained by replacing the string \((\ell, x)\) with the string \((\ell - 1, x + 1)\) and changing all other labels so that all colabels remain fixed.

2. If \(x > 0\), then add the string \((1, -1)\) to \((\nu, J)^{(a)}\). Otherwise, let \(\ell\) be the maximal length of all strings in \((\nu, J)^{(a)}\) which have label \(x\) and replace the string \((\ell, x)\) by the string \((\ell + 1, x - 1)\). In both cases, change all other labels so that all colabels remain fixed. If the result is a valid rigged configuration, then it is \(f_a(\nu, J)\). Otherwise \(f_a(\nu, J) = 0\).

**Remark 2.4** The condition for highest weight rigged configurations matches with the usual crystal theoretic definition; i.e., \(e_a(\nu, J) = 0\) for all \((\nu, J) \in RC^*(L; \lambda)\).

**Example 2.5** Let \((\nu, J)\) be the rigged configuration from Example 2.2 Then

\[
e_1(\nu, J) = 0, \quad e_2(\nu, J) = 2 \begin{array}{cccc}
3 & 2 & 1 & 0 \\
-1 & -1 & -1 & 1 \\
\end{array}, \quad f_1(\nu, J) = -1 \begin{array}{cccc}
3 & 2 & 1 & 0 \\
-1 & -1 & -1 & 1 \\
\end{array}, \quad f_2(\nu, J) = 0.
\]

Let \(RC(L; \lambda)\) denote the set generated from \(RC^*(L; \lambda)\) by the crystal operators. Let \(RC(L)\) be the closure under the crystal operators of the set \(RC^*(L) = \bigcup_{\lambda \in P^+} RC^*(L; \lambda)\).
Theorem 2.6 ([27]) Let \( \mathfrak{g}_0 \) be a Lie algebra of finite type. For \((\nu, J) \in \text{RC}^\star(L; \lambda)\), let \( X_{(\nu, J)} \) be the closure of \((\nu, J)\) under \( e_a, f_a \) for \( a \in I_0 \). Then \( X_{(\nu, J)} \cong B(\lambda) \) as \( U_q(\mathfrak{g}_0) \)-crystals.

There is a statistic called cocharge on rigged configurations given by

\[
\text{cc}(\nu, J) = \frac{1}{2} \sum_{a,b \in I_0} \sum_{i,j \in \mathbb{Z}_{>0}} (\alpha_a | \alpha_b) \min(i, j) m_i^{(a)} m_j^{(b)} + \sum_{(a,i) \in H_0} \sum_{x \in J_i^{(a)}} x. \tag{2.8}
\]

Moreover cocharge is invariant under \( e_a \) and \( f_a \) for \( a \in I_0 \) [27].

2.4 Virtual crystals

Let \( \hat{\mathfrak{g}} \) be the Kac–Moody algebra with index set \( \hat{I} \) of type \( D_4^{(1)} \) and \( \hat{\mathfrak{g}}_0 \) be of type \( D_4 \). We consider the diagram folding \( \phi: \mathcal{I} \setminus J \) defined by \( \phi(0) = 0, \phi(2) = 1, \) and \( \phi(1) = \phi(3) = \phi(4) = 2 \). The folding \( \phi \) restricts to a diagram folding of type \( \hat{\mathfrak{g}}_0 \setminus \mathfrak{g}_0 \), and by abuse of notation, we also denote this folding by \( \phi \).

Remark 2.7 To simplify our notation, for any object \( X \) or \( \mathcal{X} \) of \( \mathfrak{g}_0 \), we denote the corresponding object of \( \hat{\mathfrak{g}}_0 \) by \( \mathcal{X} \).

Furthermore, the folding \( \phi \) induces an embedding of weight lattices \( \Psi: \mathcal{F} \to \hat{\mathcal{F}} \) given by

\[
\Lambda_a \mapsto \sum_{b \in \phi^{-1}(a)} \hat{\lambda}_b, \quad \bar{\pi}_a \mapsto \sum_{b \in \phi^{-1}(a)} \hat{\alpha}_b. \tag{2.9}
\]

This gives an embedding of crystals as sets \( v: B(\lambda) \to B(\Psi(\lambda)) \), and let \( V(\lambda) \) denote the image of \( v \).

We can define a crystal structure on \( V \) which is induced from the crystal \( B(\Psi(\lambda)) \) by

\[
e^v := \prod_{b \in \phi^{-1}(a)} \hat{c}_b, \quad f^v := \prod_{b \in \phi^{-1}(a)} \hat{f}_b, \quad \varepsilon^v_a := \varepsilon_x, \quad \varphi^v_a := \varphi_x, \quad \text{wt} := \Psi^{-1} \circ \text{wt}, \tag{2.10}
\]

where we fix some \( x \in \phi^{-1}(a) \). We say the pair \( \left(V(\lambda), B(\Psi(\lambda))\right) \) is a virtual crystal and the isomorphism \( v \) is the virtualization map.

Proposition 2.8 ([27]) Let \( \mathfrak{g}_0 \) be of finite type. Then we have \( B(\lambda) \cong V(\lambda) \) as \( U_q(\mathfrak{g}_0) \)-crystals.

In particular, we can define a virtualization map on rigged configurations by

\[
\hat{\nu}^{(b)} = \nu^{(a)}, \quad \hat{J}_i^{(b)} = J_i^{(a)} \tag{2.11a}
\]

for all \( b \in \phi^{-1}(a) \) [27].

3 The bijection \( \Phi \)

Consider a tensor product of KR crystals \( B = \bigotimes_{i=1}^N B^{\gamma_i; \alpha_i} \). We write \( \text{RC}(B) \) for \( \text{RC}(L) \) with \( L_i^{(n)} \) equal to the number of factors \( B^{\alpha, \gamma_i} \) occurring in \( B \). In this section, we describe the map \( \Phi: \text{RC}(B) \to B \).
3.1 The basic algorithm $\delta$

We begin by describing the basic step $\delta$: $\text{RC}(B^{1,1} \otimes B^*) \rightarrow \text{RC}(B^*)$, where $B^*$ is some tensor product of KR crystals. Each step $\delta$ returns some element $b \in B^{1,1}$, which we use to create $B$. We note that this is the special case of the algorithm given in [17] for type $D_4^{(3)}$.

Set $\ell_0 = 1$. Do the following process for $a = 1$. Find the minimal integer $i \geq \ell_{a-1}$ such that $\nu^{(a)}$ has a singular string of length $i$. If no such $i$ exists, then set $b = a$ and $\ell_a = \infty$ and terminate. Otherwise set $\ell_a = i$ and repeat the above process for $a = 2$.

Suppose the process has not terminated. We remove the selected (singular) string of length $\ell_1$ from consideration. If there are no singular or quasi-singular strings in $\nu^{(a)}$ larger than $\ell_2$ or if $\ell_2 = \ell_1$ and there is only one string of length $\ell_1$ in $\nu^{(a)}$, then set $b = 3$ and terminate. Otherwise find the smallest $i \geq \ell_2$ that satisfies one of the following three mutually exclusive conditions:

- **(S)** $J^{(1,i)}$ is singular and $i > 1$;
- **(P)** $J^{(1,i)}$ is singular and $i = 1$;
- **(Q)** $J^{(1,i)}$ is quasi-singular.

If (P) holds, set $b = 0$, and $\ell_3 = i$ and terminate. If (S) holds, set $\ell_3 = i - 1, \ell_3 = i$, say case (S) holds for $a = n$, and continue. If (Q) holds, find the minimal index $i \geq j$ such that (S) holds. If no such $j$ exists, set $b = 0$ and terminate. Else set $\ell_3 = j$ and say case (Q, S) holds and continue.

Suppose the process has not terminated, and let $a = 2$. If $\ell_a = \ell_{a+1}$, then set $\ell_a = \ell_a$, afterwards reset $\ell_a = \ell_a - 1$, and say case (S2) holds for $a$. Otherwise find the minimal index $i \geq \ell_{a+1}$ such that $\nu^{(a)}$ has a singular string of length $i$. If no such $i$ exists, set $b = a + 1$ and terminate. Otherwise set $\ell_a = i$ and repeat this for $a = 1$ (there must exist at least two singular strings if $\ell_3 = \ell_1$ and case (S2) does not hold). If the process has not terminated, set $b = \bar{1}$.

Set all undefined $\ell_a$ and $\ell_a$ for $a = 1, 2, 3$ to $\infty$.

3.2 Change in the rigged configuration

The rigged configurations change under $\delta$ as follows. We first remove a box from $\ell_a$ in $\nu^{(a)}$ for $a = 1, 2$, and if case (S2) holds for $a$, we remove another box from that particular row, otherwise we remove a box from $\ell_a$. If case (S) holds, then remove two boxes from $\ell_3$ and make the resulting string singular. If case (Q) holds, remove a box from $\ell_3$ and make the resulting string singular. If case (Q, S) holds, then we remove both boxes corresponding to $\ell_3$ and $\ell_3$, but we make the smaller one (i.e. the row corresponding to $\ell_3$) singular and the larger one quasi-singular. Also make all the changed strings in $\nu^{(a)}$ singular.

**Remark 3.1** We can determine the inverse algorithm by roughly doing the opposite of the above; in particular, selecting largest (quasi)singular strings at most as long as before.

**Example 3.2** Using the rigged configuration $(\nu, J)$ from Example 2.2 and $B = B^{1,1} \otimes B^{1,2} \otimes B^{2,1}$. Applying the map $\delta$, we get $b = 3$ and

$$\delta(\nu, J) = \begin{array}{c|c|c|c|c|c} & & & & & \\ \hline \text{3} & & & & & \text{2} \\ \hline & & & & & \\ \end{array}$$
3.3 Extending to arbitrary rectangles

We now extend $\Phi$ to $B = \bigotimes_{i=1}^{N} B^{1,s_i}$ by defining the map

$$ls: \text{RC}(B^{1,s} \otimes B^*) \to \text{RC}(B^{1,1} \otimes B^{1,s-1} \otimes B^*),$$

which is known as left-split. On the rigged configurations, the map $ls$ is the identity (but perhaps increases the vacancy numbers) and a strict crystal embedding. Thus iterating $ls$ with $\delta$, we obtain a map $\Phi: \text{RC}(B) \to B$.

4 Filling map

We determine the highest weight rigged configurations for $B^{1,s}$ by using the virtual Kleber algorithm [22].

**Lemma 4.1** Consider the KR crystal $B^{1,s}$. We have $\text{RC}(B^{1,s}) = \bigoplus_{k=0}^{s} \text{RC}(B^{1,s}; k\Lambda_1)$. Moreover the highest weight rigged configurations in $\text{RC}(B^{1,s}; k\Lambda_1)$ are given by $\nu^{(1)} = (s-k, s-k)$ and $\nu^{(2)} = (s-k)$ with all labels 0.

From Lemma 4.1 and the $U_q(g_0)$-crystal decomposition of $B^{1,s}$ is multiplicity free, there exists a natural $U_q(g_0)$-crystal isomorphism $\iota: \text{RC}(B^{1,s}) \to B^{1,s}$. For type $D_4^{(3)}$, we note that $k\Lambda_1$ can be considered as the partition $(k)$.

**Definition 4.2** Let $B^{1,s}$ be a KR crystal of type $D_4^{(3)}$ and consider the classical component $B(k\Lambda_1) \subseteq B^{1,s}$. The filling map fill: $B^{1,s} \to (B^{1,1})^{\otimes s}$ is given by adding \( \lfloor \frac{s-k}{2} \rfloor \) copies of the horizontal domino \( \begin{array}{l} 1 \end{array} \) and an additional \( \emptyset \) if $s-k$ is odd.

Let $T^{1,s}$ denote the image of $B^{1,s}$ under fill written as a $1 \times s$ rectangle. We note that $T^{1,s}$ inherits a classical crystal structure from $(B^{1,1})^{\otimes s}$.

**Example 4.3** Consider the element

$$b = \begin{array}{c} 3 \ 0 \ 2 \ 2 \ T \end{array} \in B(5\Lambda_1) \subseteq B^{1,9},$$

then we have

$$\text{fill}(b) = \begin{array}{c} 3 \ 0 \ 2 \ 2 \ T \ T \ T \ T \ T \ T \ T \end{array}.$$

Now suppose $b \in B^{1,8}$, then we have

$$\text{fill}(b) = \begin{array}{c} 3 \ 0 \ 2 \ 2 \ T \ T \ T \ 1 \ 0 \end{array}.$$

We give a $U'_q(g)$-crystal structure to $T^{1,s}$ by following [10, 30] as the conditions for $e_0$ and $f_0$ are preserved under the filling map.

**Proposition 4.4** The filling map fill: $B^{1,s} \to T^{1,s}$ given in Definition 4.2 is a $U'_q(g)$-crystal isomorphism.

We also can show the following.

**Proposition 4.5** Let $B = B^{1,s}$. Then $\Phi = \text{fill} \circ \iota$ with fill as in Definition 4.2 on highest weight elements.
5 Virtualization Map

Lemma 5.1 The virtualization map \( v : B^{1,s} \rightarrow B^{2,s} \) for types \( D^{(3)}_4 \) is given column-by-column by

\[
\begin{array}{cccccc}
1 & \rightarrow & 1 & 2 & \rightarrow & 1 \\
3 & \rightarrow & 2 & 2 & \rightarrow & 3 \\
0 & \rightarrow & 0 & 0 & \rightarrow & 1 \\
\end{array}
\]

Using Lemma 5.1 and the analogue of \( \Phi \) in type \( D^{(1)}_4 \) [18, 25], we can show the following.

Theorem 5.2 Consider a tensor product of KR crystals \( B = \bigotimes_{i=1}^N B^{1,s}_i \) of type \( D^{(3)}_4 \). The virtualization map \( v \) commutes with the map \( \Phi \).

We need to define the complement rigging map \( \theta : RC(B) \rightarrow RC(B^r) \) by sending \( (\nu, J) \rightarrow (\nu, J') \), where \( J' \) is obtained by \( x' = p_i^{(a)} - x \) for all labels \( x \) and \( B_i \) are the factors of \( B \) in reverse order. That is to say \( \theta \) maps each label \( x \) to its colabel. We can define \( \bar{\delta} := \theta \circ \delta \circ \theta \), and using the virtualization map, Proposition 4.5 and the results of [25], we can show the following.

Lemma 5.3 We have \( \delta \circ \bar{\delta} = \bar{\delta} \circ \delta \).

Using the results on the combinatorial \( R \)-matrix in [30], we can show the following.

Lemma 5.4 Consider \( B = B^{1,s} \otimes B^{1,1} \). We have \( \Phi^{-1} \circ R \circ \Phi \) is the identity map on \( RC(B) \).

Then following [28, Sec. 8], the map \( rs := \theta \circ is \circ \theta \) preserves statistics using [30]. From Lemma 5.4 the \( R \)-matrix preserves statistics. Thus iterating \( rs \) and \( R \)-matrices, we preserve statistics to \( \bigotimes_{i=1}^{N'} B^{1,1} \). Then we use the results of [23, 25] and Theorem 5.2 to obtain our main result.

Theorem 5.5 Let \( B = \bigotimes_{i=1}^N B^{1,s}_i \) of type \( D^{(3)}_4 \). The map \( \Phi : RC(B) \rightarrow B \) is a \( U_q(g_0) \)-crystal isomorphism and \( \Phi \circ \theta \) sends cocharge to energy.

From Proposition 4.4, Lemma 5.1, Theorem 5.5, and the filling map for type \( D^{(1)}_4 \) given in [18], we can show the following.

Theorem 5.6 Let \( B = B^{1,s} \). Then \( \Phi = fill \circ \iota \) with \( fill \) as in Definition 4.2 as \( U_q(g_0) \)-crystal morphisms.

Thus we can define a \( U'_q(g) \)-crystal structure on \( RC(B) \) by extending \( \Phi \) to be a \( U'_q(g) \)-crystal isomorphism. Thus we have a special case in type \( D^{(3)}_4 \) of the conjectures given in [27].

6 Extensions and questions

The \( U_q(g_0) \)-crystal decomposition of \( B^{2,s} \) and the highest weight rigged configurations will appear in the full version of this work. The author hopes to use this to determine the filling map for \( B^{2,s} \).

There is a map \( \iota : RC(B^{2,1} \otimes B^*) \rightarrow RC(B^{1,1} \otimes B^{1,1} \otimes B^*) \) called \( left-top \) which adds a singular string of length 1 to \( \nu^{(1)} \). In the full version, this is used to extend the \( U_q(g_0) \)-crystal isomorphism \( \Phi \) to tensor products also containing \( B^{2,1} \).
Example 6.1 Continuing from Example 3.2 we obtain

\[ \Phi(\nu, J) = 3 \otimes 2 \otimes 3 \otimes 1 \otimes 2 \]

The computations for the Kleber algorithm can be modified to determine the \( U_q(\mathfrak{g}_0) \)-crystal decomposition of \( B^{r,s} \) of type \( G_2^{(1)} \). However there is a difficulty with determining what the map \( \delta \) should be. This would need to be overcome to define the filling map for type \( G_2^{(1)} \).

There is a conjecture [22, Conj. 3.7] that we can realize \( B^{1,s} \) of type \( D_4^{(3)} \) as a virtual crystal in \( B^{1,s} \) of type \( D_4^{(1)} \). Therefore obtaining a direct description of \( e_0 \) and \( f_0 \) on rigged configurations could lead to an answer to this conjecture using the results of [18]. The author hopes to have this description and prove this conjecture in this special case in the full version of this work.

7 Examples using Sage

The bijection \( \Phi \) and the rigged configurations have been implemented by the author in Sage [29]. We begin by setting up the Sage environment to give a more concise printing.

```
sage: RiggedConfigurations.global_options(display="horizontal")
```

We construct our the rigged configuration from Example 2.2 by specifying the partitions and corresponding labels.

```
sage: nu = RC(partition_list=[[4,1], [4]], rigging_list=[[3,1], [-2]]); nu
5[ ][ ][ ]3 -2[ ][ ][ ][ ]-2
1[ ][ ]1
```

We apply the full bijection and print the output using Sage’s ASCII art.

```
sage: ascii_art(nu.to_tensor_product_of_kirillov_reshetikhin_tableaux())
3 # 2 3 # 1
-2
```

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References


[17] Bin Mohammad Mahathir, Soliton cellular automata constructed from a $U_q(g)$-Crystal $B_n^{1,1}$ and Kirillov-Reshetikhin type bijection for $U_q(E_6^{(1)})$-Crystal $B_6^{1,1}$. Thesis (Ph.D.)–Osaka University, 2012.


