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Cyclic Sieving and Plethysm Coefficients

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Abstract. A combinatorial expression for the coefficient of the Schur function s_λ in the expansion of the plethysm $p_{n/d}^d \circ s_\mu$ is given for all d dividing n for the cases in which $n = 2$ or λ is rectangular. In these cases, the coefficient $\langle p_{n/d}^d \circ s_\mu, s_\lambda \rangle$ is shown to count, up to sign, the number of fixed points of an $\langle s_\mu^n, s_\lambda \rangle$-element set under the d^{th} power of an order n cyclic action. If $n = 2$, the action is the Schützenberger involution on semistandard Young tableaux (also known as evacuation), and, if λ is rectangular, the action is a certain power of Schützenberger and Shimozono's *jeu-de-taquin* promotion.

This work extends results of Stembridge and Rhoades linking fixed points of the Schützenberger actions to ribbon tableaux enumeration. The conclusion for the case $n = 2$ is equivalent to the domino tableaux rule of Carré and Leclerc for discriminating between the symmetric and antisymmetric parts of the square of a Schur function.

Résumé. Une expression combinatoire pour le coefficient de la fonction de Schur s_λ dans l'expansion du pléthysme $p_{n/d}^d \circ s_\mu$ est donné pour tous d que disent n, dans les cas où $n = 2$, ou λ est rectangulaire. Dans ces cas, le coefficient $\langle p_{n/d}^d \circ s_\mu, s_\lambda \rangle$ se montre à compter, où l'on ignore le signe, le nombre des point fixés d'un ensemble de $\langle s_\mu^n, s_\lambda \rangle$ éléments sous la puissance d^{e} d'une action cyclique de l'ordre n. Si $n = 2$, l'action est l'involution de Schützenberger sur les tableaux semi-standard de Young (aussi connu sous le nom des évacuations), et si λ est rectangulaire, l'action est une certaine puissance de l'avancement jeu-de-taquin de Schützenberger et Shimozono.

Ce travail étend les résultats de Stembridge et Rhoades, liant les point fixés des actions de Schützenberger aux tableaux de ruban. Pour le cas $n = 2$, la conclusion est équivalent à la règle des tableaux de dominos de Carré et Leclerc, qui distingue entre les parties symétriques et asymétriques du carré d'une fonction de Schur.

Keywords: plethysms, Schützenberger involution, *jeu-de-taquin* promotion, canonical bases, Kashiwara crystals, cyclic sieving phenomenon

1 Introduction

A principal concern of algebraic combinatorics is the identification of collections of combinatorial objects that occur in algebraically significant multiplicities. Perhaps the most celebrated success in this endeavor is the Littlewood–Richardson rule, which gives a combinatorial description for the coefficient of each Schur function arising in the expansion of a product of Schur functions on the Schur basis. For the case of a Schur function s_μ raised to the n^{th} power, there is a natural order n cyclic action on the objects specified by the Littlewood–Richardson rule for the coefficient of s_λ, provided that $n = 2$ or λ is rectangular. In

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this extended abstract of [14], we present an algebraic expression for the number of fixed points under each power of this cyclic action, à la the cyclic sieving phenomenon of [12]. In particular, we show that the cardinality of each fixed point set is given up to sign by the coefficient of s_λ in the expansion of a plethysm involving s_μ. Since the plethysm corresponding to the trivial action is s_μ^n, what we put forth may be viewed as an *accoutrement* to the Littlewood–Richardson rule that endows a series of associated collections of objects with algebraic meaning, and, in so doing, underscores the power of the cyclic sieving paradigm.

Let Λ be the ring of symmetric functions over $\mathbb{Z}$. For all $f, g \in \Lambda$, if V and W are polynomial representations of $GL_m(\mathbb{C})$ with characters $\chi_V = f(x_1, x_2, \ldots, x_m)$ and $\chi_W = g(x_1, x_2, \ldots, x_m)$, respectively, then $\chi_{V\oplus W} = (f+g)(x_1, x_2, \ldots, x_m)$ and $\chi_{V\otimes W} = (fg)(x_1, x_2, \ldots, x_m)$. Plethysm is a binary operation on Λ (so named by Littlewood in 1950) that is compatible with representation composition in the same sense that addition and multiplication correspond to representation direct sum and tensor product, respectively. To wit, if $\rho\colon GL_m(\mathbb{C}) \to GL_M(\mathbb{C})$ is a polynomial representation of $GL_m(\mathbb{C})$ with character $g(x_1, x_2, \ldots, x_m)$, and $\sigma\colon GL_M(\mathbb{C}) \to GL_N(\mathbb{C})$ is a polynomial representation of $GL_M(\mathbb{C})$ with character $f(x_1, x_2, \ldots, x_M)$, then the composition $\sigma\rho\colon GL_m(\mathbb{C}) \to GL_N(\mathbb{C})$ is a polynomial representation of $GL_m(\mathbb{C})$ with character $(f \circ g)(x_1, x_2, \ldots, x_m)$, where $f \circ g \in \Lambda$ denotes the plethysm of f and g. A formal definition is given in section 2.

We are herein concerned with plethysms of the form $p_{n/d}^d \circ s_\mu$, where μ is a partition, s_μ denotes the Schur function associated to μ, d divides n, and $p_{n/d}$ denotes the $(n/d)^{\text{th}}$ power-sum symmetric function, $x_1^{n/d} + x_2^{n/d} + \cdots$. Writing $\langle\ ,\ \rangle$ for the Hall inner product on Λ, we arrive at a combinatorial description of the coefficients $\langle p_{n/d}^d \circ s_\mu, s_\lambda\rangle$ for the cases in which $n = 2$ or λ is rectangular.

Let $\mu = (\mu_1, \mu_2, \ldots, \mu_m)$. If $n = 2$, the Littlewood–Richardson multiplicity $\langle s_\mu^n, s_\lambda\rangle$ is the number of semistandard Young tableaux of shape λ and content $\overline{\mu}\mu := (\mu_m, \ldots, \mu_1, \mu_1, \ldots, \mu_m)$ for which the reading word is anti-Yamanouchi in $\{1, 2, \ldots, m\}$ and Yamanouchi in $\{m+1, m+2, \ldots, 2m\}$. The Schützenberger involution (also known as evacuation) on a semistandard tableau preserves the shape and reverses the content, so it gives an action on the tableaux of shape λ and content $\overline{\mu}\mu$, which restricts to those tableaux with words satisfying the aforementioned Yamanouchi conditions (cf. [14], Remark 4.21).

In general, the Littlewood–Richardson multiplicity $\langle s_\mu^n, s_\lambda\rangle$ is the number of semistandard tableaux of shape λ and content $\mu^n := (\mu_1, \ldots, \mu_m, \mu_1, \ldots, \mu_m, \ldots, \mu_1, \ldots, \mu_m)$ for which the reading word is Yamanouchi in the alphabets $\{km+1, km+2, \ldots, (k+1)m\}$ for all $0 \le k \le n-1$. On a semistandard tableau, *jeu-de-taquin* promotion (introduced in [15]) preserves the shape and permutes the content by the long cycle in $\mathfrak{S}_{mn}$, so m iterations of promotion gives an action on the tableaux of shape λ and content μ^n. If λ is rectangular, this action has order n and restricts to those tableaux with words satisfying the requisite Yamanouchi conditions (cf. [14], Remark 4.31).

We are at last poised to state our main results.

Theorem 1.1. *Let* $\mathrm{EYTab}(\lambda, \overline{\mu}\mu)$ *be the set of all semistandard tableaux of shape* λ *and content* $\overline{\mu}\mu$ *with reading word anti-Yamanouchi in* $\{1, 2, \ldots, m\}$ *and Yamanouchi in* $\{m+1, m+2, \ldots, 2m\}$*, and let* ξ *act on* $\mathrm{EYTab}(\lambda, \overline{\mu}\mu)$ *by the Schützenberger involution. Then*

$$|\{T \in \mathrm{EYTab}(\lambda, \overline{\mu}\mu) : \xi(T) = T\}| = \pm\langle p_2 \circ s_\mu, s_\lambda\rangle.$$

Theorem 1.2. *Let* λ *be a rectangular partition, and let* $\mathrm{PYTab}(\lambda, \mu^n)$ *be the set of all semistandard tableaux of shape* λ *and content* μ^n *with reading word Yamanouchi in the alphabets* $\{km+1, km+$

$2, \ldots, (k+1)m\}$ *for all* $0 \le k \le n-1$. *Let* j *act on* PYTab(λ, μ^n) *by* m *iterations of* jeu-de-taquin *promotion. Then, for all positive integers* d *dividing* n*,*

$$|\{T \in \text{PYTab}(\lambda, \mu^n) : j^d(T) = T\}| = \pm\langle p^d_{n/d} \circ s_\mu, s_\lambda\rangle.$$

From Theorems 3.1 and 3.2 in [8], we see that the Hall–Littlewood symmetric function $Q'_{1^n}(q)$ specializes (up to sign) at $q = e^{\frac{2\pi i \ell}{n}}$ to $p^{\gcd(n,\ell)}_{n/\gcd(n,\ell)}$. Therefore, we may interpret Theorem 1.2 as analogous to exhibiting an instance of the cyclic sieving phenomenon.

Corollary 1.3. *Let* λ *be a rectangular partition. Let* j *act on* PYTab(λ, μ^n) *by* m *iterations of* jeu-de-taquin *promotion. Then, for all integers* ℓ*,*

$$|\{T \in \text{PYTab}(\lambda, \mu^n) : j^\ell(T) = T\}| = \pm\langle Q'_{1^n}(e^{\frac{2\pi i \ell}{n}}) \circ s_\mu, s_\lambda\rangle.$$

Remark 1.4. The signs appearing in Theorems 1.1 and 1.2 are predictable, and depend upon λ, d, and n only. Consult section 4 (or [14] itself), for more details.

[14] is by no means the first attempt at computing the coefficients in the expansions of power-sum plethysms. In [3], a rule for splitting the square of a Schur function into its symmetric and antisymmetric parts was devised, the crux of which was a demonstration that the coefficient $\langle p_2 \circ s_\mu, s_\lambda\rangle$ counted, up to sign, the number of domino tableaux of shape λ and content μ with Yamanouchi reading words. Two years later, [8] introduced a new family of symmetric functions, today referred to as LLT functions, and proposed that the plethysm $p^d_{n/d} \circ s_\mu$ could be expressed as the specialization of an LLT function at an appropriate root of unity (as indeed $p^d_{n/d}$ is the specialization of a Hall–Littlewood function). However, the Lascoux–Leclerc–Thibon conjecture remains unproven, and the Carré–Leclerc rule has not been generalized to cases beyond $n = 2$, for the concept of Yamanouchi reading words has not been extended to n-ribbon tableaux for $n \ge 3$.

Thus, Theorem 1.1 does not give the first combinatorial expression for the coefficient $\langle p_2 \circ s_\mu, s_\lambda\rangle$, but it distinguishes itself from the existing Carré–Leclerc formula by its natural compatibility with the Littlewood–Richardson rule. It is sufficiently robust that the techniques involved in its derivation are applicable to a whole class of plethysm coefficients with $n > 2$, addressed in Theorem 1.2, which is new in content and in form. Furthermore, the author has shown in unpublished work that a bijection of [1] between domino tableaux and tableaux stable under evacuation restricts to a bijection between those tableaux specified in the Carré–Leclerc rule and in Theorem 1.1, respectively. It follows that Theorem 1.1 actually recovers the Carré–Leclerc result.

To prove Theorems 1.1 and 1.2, we turn to the theory of Lusztig canonical bases, which provides an algebraic setting for the Schützenberger actions evacuation and promotion. In particular, we consider an irreducible representation of $GL_{mn}(\mathbb{C})$ for which there exists a basis indexed by the semistandard tableaux of shape λ with entries in $\{1, 2, \ldots, mn\}$ such that, if $n = 2$, the long element $w_0 \in \mathfrak{S}_{\mathfrak{mn}} \hookrightarrow GL_{mn}$ permutes the basis elements (up to sign) by evacuation, and, if λ is rectangular, the long cycle $c_{mn} \in \mathfrak{S}_{\mathfrak{mn}} \hookrightarrow GL_{mn}$ permutes the basis elements (up to sign) by promotion.

With a suitable basis in hand, we proceed to compute the character of the representation at a particular element of GL_{mn}. If $n = 2$, we compute

$$\chi(w_0 \cdot \text{diag}(x_1, x_2, \ldots, x_m, x_m, \ldots, x_2, x_1)),$$

and, if λ is rectangular, we compute

$$\chi(c_{mn}^{md} \cdot \mathrm{diag}(y_1, y_2, \ldots, y_d, y_1, y_2, \ldots, y_d, \ldots, y_1, y_2, \ldots, y_d)),$$

where the block $\mathrm{diag}(y_1, y_2, \ldots, y_d)$ occurs n/d times along the main diagonal, and y_i in turn represents the block $\mathrm{diag}(y_{i,1}, y_{i,2}, \ldots, y_{i,m})$ for all $1 \le i \le d$.

These character evaluations pick out the fixed points of the relevant order n cyclic actions. Furthermore, they may be calculated by diagonalization of the indicated elements, for characters are class functions, and the values of the irreducible characters of GL_{mn} at diagonal matrices are well known. A careful inspection of the resulting formulae yields the desired identities.

The relationship between w_0 and evacuation was first discovered in [2], in the context of a basis dual to Lusztig's canonical basis. Herein we opt for an essentially equivalent basis constructed in [17], which was used by Rhoades to detect the analogous relationship between c_{mn} and promotion. From the observations that w_0 and c_{mn} lift the actions of evacuation and promotion, respectively, with respect to the dual canonical basis (or something like it), [18] and [13] deduced correspondences between fixed points of Schützenberger actions and ribbon tableaux. These theorems inspired our results.

Theorem 1.5 ([18], Corollary 4.2)**.** *Let* $\mathrm{Tab}(\lambda, \overline{\mu}\mu)$ *be the set of all semistandard tableaux of shape* λ *and content* $\overline{\mu}\mu$*, and let* ξ *act on* $\mathrm{Tab}(\lambda, \overline{\mu}\mu)$ *by the Schützenberger involution. Then*

$$|\{T \in \mathrm{Tab}(\lambda, \overline{\mu}\mu) : \xi(T) = T\}|$$

is the number of domino tableaux of shape λ *and content* μ*.*

Theorem 1.6 ([13], proof of Theorem 1.5)**.** *Let* λ *be a rectangular partition, and let* $\mathrm{Tab}(\lambda, \mu^n)$ *be the set of all semistandard tableaux of shape* λ *and content* μ^n*. Let* j *act on* $\mathrm{Tab}(\lambda, \mu^n)$ *by* m *iterations of* jeu-de-taquin *promotion. Then, for all positive integers* d *dividing* n*,*

$$|\{T \in \mathrm{Tab}(\lambda, \mu^n) : j^d(T) = T\}|$$

is the number of (n/d)*-ribbon tableaux of shape* λ *and content* μ^d*.*

Unfortunately, the proofs of Theorems 1.5 and 1.6 cannot be directly adapted to obtain Theorems 1.1 and 1.2. In order for the Yamanouchi restrictions on our tableaux sets to be made to appear in our character evaluations, an additional point of subtlety is needed. We find relief in the insights offered us by the theory of Kashiwara crystals, which provides a framework not only for the study of the Schützenberger actions, but also for the reformulation of the Yamanouchi restrictions in terms of natural operators on semistandard tableaux.

Let $\mathfrak{g}$ be a complex semisimple Lie algebra with weight lattice W, and choose a set of simple roots $\Delta = \{\alpha_1, \alpha_2, \ldots, \alpha_t\}$. A $\mathfrak{g}$-crystal is a finite set B equipped with a weight map $\mathrm{wt}\colon B \to W$ and a pair of raising and lowering operators $e_i, f_i\colon B \to B \sqcup \{0\}$ for each i that obey certain conditions. Most notably, for all $b \in B$, if $e_i \cdot b$ is nonzero, then $\mathrm{wt}(e_i \cdot b) = \mathrm{wt}(b) + \alpha_i$, and if $f_i \cdot b$ is nonzero, then $\mathrm{wt}(f_i \cdot b) = \mathrm{wt}(b) - \alpha_i$.

If $\mathfrak{g} = \mathfrak{sl}_{\mathfrak{mn}}$, then W is a quotient of $\mathbb{Z}^{mn}$, and we may choose for our simple roots the images of the vectors $\epsilon_i - \epsilon_{i+1}$ for $1 \le i \le mn-1$, where ϵ_i denotes the i^{th} standard basis vector for all $1 \le i \le mn$. In this case, we may take B to be the set of semistandard tableaux of shape λ with entries in $\{1, 2, \ldots, mn\}$, with the weight of each tableau encoded in its content. As we see in section 3, there exists a suitable choice of operators e_i and f_i so that B assumes the structure of a $\mathfrak{g}$-crystal, and that the word of a tableau $b \in B$ is Yamanouchi with respect to the letters i and $i+1$ if and only if e_i vanishes at b, and anti-Yamanouchi with respect to i and $i+1$ if and only if f_i vanishes at b. Furthermore, evacuation and promotion act on the set of crystal operators by conjugation (essentially), which explains why they act on the tableaux sets indicated in our main theorems.

We close the introduction with an outline of the rest of the article. In section 2, we introduce plethysms, and we recall the observation of [8] that the classical relationship between tableaux and Schur functions evinces a more general relationship between ribbon tableaux and power-sum plethysms of Schur functions. In section 3, we define Kashiwara crystals for a complex semisimple Lie algebra, before specializing to the $\mathfrak{sl}_{\mathrm{mn}}$ setting, where we show how to assign a crystal structure to the pertinent tableaux sets. We also examine how the Schützenberger actions interact with the raising and lowering crystal operators. Finally, in section 4, we sketch proofs of Theorems 1.1 and 1.2.

2 Background on Plethysms

In this extended abstract, we assume familiarity with the basic facts about Young tableaux and symmetric functions. (Accounts of the fundamentals can be found in [4], Chapters 1-6; for a treatment specific to our needs, see [14].) We start, then, with the rudiments of plethysms, following [11].

Definition 2.1. Let $f, g \in \Lambda$, and let g be written as a sum of monomials, so that $g = \sum_\eta u_\eta x^\eta$, where η ranges over an infinite set of compositions. Let $\{y_i\}_{i=1}^\infty$ be a collection of proxy variables defined by $\prod_{i=1}^\infty (1 + y_i t) = \prod_\eta (1 + x^\eta t)^{u_\eta}$. The *plethysm* of f and g, which we denote by $f \circ g$, is the symmetric function $f(y_1, y_2, \ldots)$.

Remark 2.2. Although the relation $\prod_{i=1}^\infty (1 + y_i t) = \prod_\eta (1 + x^\eta t)^{u_\eta}$ only determines the elementary symmetric functions in the variables $y_1, y_2, \ldots$ (viz., $e_1(y_1, y_2, \ldots) = y_1 + y_2 + \cdots$, $e_2(y_1, y_2, \ldots) = y_1y_2 + y_1y_3 + y_2y_3 + \cdots$, etc.), it is well known that the ring of symmetric functions is generated as a $\mathbb{Z}$-algebra by the elementary symmetric functions, so the plethysm $f \circ g = f(y_1, y_2, \ldots)$ is indeed well-defined.

The following observation follows immediately from Definition 2.1.

Proposition 2.3. *For all $f \in \Lambda$, the map $\Lambda \to \Lambda$ given by $g \mapsto g \circ f$ is a ring homomorphism.*

There exists a family of symmetric functions for which the other choice of map given by plethysm, i.e. $g \mapsto f \circ g$, is also a ring homomorphism, for all f belonging to this family.

Definition 2.4. For all positive integers k, the k^{th} *power-sum symmetric function* in the variables $x_1, x_2, \ldots$ is $p_k := x_1^k + x_2^k + \cdots$.

Proposition 2.5. *Let $g \in \Lambda$, and let k be a positive integer. Then $p_k \circ g = g \circ p_k = g(x_1^k, x_2^k, \ldots)$.*

We may conclude that the map given by $g \mapsto p_k \circ g$ is a ring homomorphism for all positive integers k. We are therefore permitted to introduce an adjoint operator, which we denote by φ_k, given by $f \mapsto \sum_\kappa \langle f, p_k \circ s_\kappa \rangle s_\kappa$, where the sum ranges over all partitions κ. Clearly, the equality $\langle \varphi_k(f), g \rangle = \langle f, p_k \circ g \rangle$ holds for all $f, g \in \Lambda$, which explains the nomenclature.

Let κ be a partition. Just as the ordinary tableaux of shape κ index the monomials of the Schur function s_κ, the k-ribbon tableaux of shape κ index the monomials of the symmetric function $\varphi_k(s_\kappa)$.

Theorem 2.6. *Let κ be a partition, and suppose that the k-core of κ is empty. For all compositions η of $\frac{|\kappa|}{k}$, we denote the monomial $x_1^{\eta_1} x_2^{\eta_2} \cdots$ by x^η, and, for all k-ribbon tableaux T of shape κ and content η, we write x^T for x^η. Then $\varphi_k(s_\kappa) = \epsilon_k(\kappa) \sum_T x^T$, where the sum ranges over all k-ribbon tableaux of shape κ, and $\epsilon_k(\kappa)$ denotes the k-sign of κ.*

Proof. Let $\left(\kappa^{(1)}, \kappa^{(2)}, \ldots, \kappa^{(k)}\right)$ be the k-quotient of κ. Since the k-core of κ is empty, it follows from a result of [10] that $\varphi_k(s_\kappa) = \epsilon_k(\kappa) s_{\kappa^{(1)}} s_{\kappa^{(2)}} \cdots s_{\kappa^{(k)}}$. However, from Equation 24 in [8], we see that $s_{\kappa^{(1)}} s_{\kappa^{(2)}} \cdots s_{\kappa^{(k)}} = \sum_T x^T$, where the sum ranges over all k-ribbon tableaux of shape κ, as desired. $\square$

In view of Theorem 2.6, it is natural to ask if there is an analogue of the Littlewood–Richardson rule that describes the expansion coefficients of the *power-sum plethysms* $p_n \circ s_\mu$, or, more generally, $p_{n/d}^d \circ s_\mu$, for d dividing n. In the following sections, we outline how [14] provides a partial affirmative answer.

3 Crystal Structure on Tableaux

For a complex semisimple Lie algebra $\mathfrak{g}$, Kashiwara's $\mathfrak{g}$-crystals constitute a class of combinatorial models patterned on representations of $\mathfrak{g}$. If $\mathfrak{g}$ is simply laced, there exists a set of axioms, enumerated in [19], that characterize the crystals arising directly from $\mathfrak{g}$-representations, which he calls regular. Given a partition κ with at most s positive parts, we may consider κ as a partition with s parts. The combinatorics of the weight space decomposition of the irreducible $\mathfrak{sl}_\mathfrak{s}$-representation with highest weight encoded in κ is captured in the regular $\mathfrak{sl}_\mathfrak{s}$-crystal structure assigned to the semistandard tableaux of shape κ with entries in $\{1, 2, \ldots, s\}$.

In this section, we review the crystal structure on tableaux, and we observe that it offers a natural setting for the consideration of evacuation and promotion, due to the relationship between these actions and the raising and lowering crystal operators. We also see that the crystal perspective facilitates a recasting of the Yamanouchi conditions on tableaux reading words in terms of the vanishing or nonvanishing of the raising and lowering operators at the corresponding tableaux, viewed as crystal elements. We begin with the definition of a crystal, following [5].

Definition 3.1. Let $\mathfrak{g}$ be a complex semisimple Lie algebra with weight lattice W. Let $\{\alpha_1, \alpha_2, \ldots, \alpha_t\}$ be a choice of simple roots, and let $\{\alpha_1^\vee, \alpha_2^\vee, \ldots, \alpha_t^\vee\}$ be the corresponding simple coroots. A $\mathfrak{g}$-*crystal* is a finite set B equipped with a map $\mathrm{wt}\colon B \to W$ and a pair of operators $e_i, f_i\colon B \to B \sqcup \{0\}$ for each $1 \le i \le t$ that satisfy the following conditions:

(i) $\max\{\ell : f_i^\ell \cdot b \neq 0\} - \max\{\ell : e_i^\ell \cdot b \neq 0\} = \langle \mathrm{wt}(b), \alpha_i^\vee \rangle$ for all $b \in B$;

(ii) $e_i \cdot b \neq 0$ implies $\mathrm{wt}(e_i \cdot b) = \mathrm{wt}(b) + \alpha_i$ and $f_i \cdot b \neq 0$ implies $\mathrm{wt}(f_i \cdot b) = \mathrm{wt}(b) - \alpha_i$ for all $b \in B$;

(iii) $b' = e_i \cdot b$ if and only if $b = f_i \cdot b'$ for all $b, b' \in B$.

We refer to e_i as the *raising operator* associated to α_i, and we refer to f_i as the *lowering operator* associated to α_i. We write $\epsilon_i(b) := \max\{\ell : e_i^\ell \cdot b \neq 0\}$ for the maximum number of times the raising operator e_i may be applied to b without vanishing, and we write $\phi_i(b) := \max\{\ell : f_i^\ell \cdot b \neq 0\}$ for the maximum number of times the lowering operator f_i may be applied to b without vanishing. If a $\mathfrak{g}$-crystal B satisfies the additional conditions (P4), (P5), (P6), (P5'), and (P6') of [19], we say that B is regular.

Definition 3.2. Let B and B' be $\mathfrak{g}$-crystals. A map of sets $\pi\colon B \to B'$ is a *morphism of crystals* if $\pi e_i = e_i \pi$ and $\pi f_i = f_i \pi$ for all $1 \le i \le t$. (Here we tacitly stipulate $\pi(0) := 0$.) If π is bijective, we say π is an *isomorphism*.

Definition 3.3. A $\mathfrak{g}$-crystal B is *connected* if the underlying graph, in which elements b and b' are joined by an edge if there exists i such that $e_i \cdot b = b'$ or $e_i \cdot b' = b$, is connected. We refer to the connected components of the underlying graph as the *connected components* of B.

Remark 3.4. Regular connected $\mathfrak{g}$-crystals should be thought of in analogy with irreducible representations of $\mathfrak{g}$.

Definition 3.5. Let B be a $\mathfrak{g}$-crystal. An element $b \in B$ is a *highest weight element* if e_i vanishes at b for all i. If b is the unique highest weight element of B, then B is a *highest weight crystal* of highest weight $\mathrm{wt}(b)$.

Proposition 3.6 ([19])**.** *Let $\mathfrak{g}$ be simply laced, and let B be a regular connected $\mathfrak{g}$-crystal. Then B is a highest weight crystal.*

Proposition 3.7 ([19])**.** *Let B and B' be regular connected $\mathfrak{g}$-crystals with highest weight elements b and b', respectively. If $\phi_i(b) = \phi_i(b')$ for all $1 \le i \le t$, then B and B' are isomorphic.*

Specializing to the case $\mathfrak{g} = \mathfrak{sl}_s$, we take as our Cartan subalgebra $\mathfrak{h}$ the subspace of traceless diagonal matrices, and we identify $\mathfrak{h}^*$ with the quotient space $\mathbb{C}^s/(\epsilon_1 + \epsilon_2 + \cdots + \epsilon_s)\mathbb{C}$, where ϵ_i denotes the i^{th} standard basis vector for all $1 \le i \le s$. Writing E_i for the image of ϵ_i in $\mathfrak{h}^*$ for all i, we note that the weight lattice W is generated over $\mathbb{Z}$ by $\{E_1, E_2, \ldots, E_s\}$, and we choose the set of simple roots $\{\alpha_1, \alpha_2, \ldots, \alpha_{s-1}\}$ in accordance with the rule $\alpha_i := E_i - E_{i+1}$ for all $1 \le i \le s-1$.

Proposition 3.8 ([6])**.** *Let κ and ι be partitions, each with s parts, such that $\iota_i \le \kappa_i$ for all positive parts ι_i of ι. Let $B_{\kappa/\iota}$ be the set of semistandard skew tableaux of shape κ/ι with entries in $\{1, 2, \ldots, s\}$.*

Let the maps

$$\begin{aligned} \mathrm{wt}\colon B_{\kappa/\iota} &\to \mathbb{Z}^s/(\epsilon_1 + \epsilon_2 + \cdots + \epsilon_s)\mathbb{Z} \\ h_{i,j}, k_{i,j}\colon B_{\kappa/\iota} &\to \mathbb{Z} \\ e_i, f_i\colon B_{\kappa/\iota} &\to B_{\kappa/\iota} \sqcup \{0\} \end{aligned}$$

be given for all $1 \le i \le s-1$ and $j \in \mathbb{N}$ by stipulating, for all $T \in B_{\kappa/\iota}$:

- $\mathrm{wt}(T)$ *to be the image in $\mathbb{Z}^s/(\epsilon_1 + \epsilon_2 + \cdots + \epsilon_n)\mathbb{Z}$ of the content of T;*
- $h_{i,j}(T)$ *to be the number of occurrences of $i+1$ in the j^{th} column of T or to the right minus the number of occurrences of i in the j^{th} column of T or to the right;*
- $k_{i,j}(T)$ *to be the number of occurrences of i in the j^{th} column of T or to the left minus the number of occurrences of $i+1$ in the j^{th} column or to the left;*
- $e_i(T)$ *to be the skew tableau with an i in place of an $i+1$ in the rightmost column for which $h_{i,j}(T)$ is maximal and positive if such a column exists, and 0 otherwise;*
- $f_i(T)$ *to be the skew tableau with an $i+1$ in place of an i in the leftmost column for which $k_{i,j}(T)$ is maximal and positive if such a column exists, and 0 otherwise.*

Then the set $B_{\kappa/\iota}$ equipped with the map wt *and the operators* e_i, f_i *for all* $1 \le i \le s-1$ *is an* $\mathfrak{sl}_\mathfrak{s}$*-crystal.*

Proposition 3.9 ([19])**.** *Let κ be a partition with s parts. The $\mathfrak{sl}_\mathfrak{s}$-crystal $B_\kappa := B_{\kappa/\varnothing}$ defined in Proposition 3.8 is a regular connected crystal of highest weight $\kappa + (\epsilon_1 + \epsilon_2 + \cdots + \epsilon_s)\mathbb{Z}$. The highest weight element is the unique tableau of shape κ and content κ.*

We now observe that Schützenberger's *jeu de taquin* respects the crystal structure on tableaux in the sense that *jeu-de-taquin* slides commute with the raising and lowering operators. Since evacuation and promotion may be defined via *jeu de taquin* (cf. [15] and [16]), it should be no surprise that they inherit this compatibility.

Proposition 3.10 ([9])**.** *Let κ and ι be nonempty partitions such that $\iota_i \le \kappa_i$ for all positive parts ι_i of ι, and let C be an inside corner of κ/ι. For all semistandard skew tableaux T of shape κ/ι, let* $\mathrm{jdt}(T)$ *be the result of a* jeu-de-taquin *slide on T starting from C, and set* $\mathrm{jdt}(0) := 0$*. Then* $e_i \cdot \mathrm{jdt}(T) = \mathrm{jdt}(e_i \cdot T)$ *and* $f_i \cdot \mathrm{jdt}(T) = \mathrm{jdt}(f_i \cdot T)$ *for all* $T \in B_{\kappa/\iota}$ *and* $1 \le i \le s-1$*.*

Proposition 3.11 ([7])**.** *Let κ be a partition with s parts, and let $\xi \colon B_\kappa \to B_\kappa$ be the Schützenberger involution. Set $\xi(0) := 0$. Then, for all $T \in B_\kappa$:*

(i) $\mathrm{wt}(\xi(T)) = w_0 \cdot \mathrm{wt}(T)$*;*

(ii) $\xi(e_i \cdot T) = f_{s-i} \cdot \xi(T)$ *and* $\xi(f_i \cdot T) = e_{s-i} \cdot \xi(T)$ *for all* $1 \le i \le s-1$*.*

Proposition 3.12 ([16])**.** *Let κ be a partition with s parts, and let* $\mathrm{pr}\colon B_\kappa \to B_\kappa$ *be* jeu-de-taquin *promotion. Set* $\mathrm{pr}(0) := 0$*. Then, for all $T \in B_\kappa$:*

(i) $\mathrm{wt}(\mathrm{pr}(T)) = c_s \cdot \mathrm{wt}(T)$*;*

(ii) $\mathrm{pr}(e_i \cdot T) = e_{i+1} \cdot \mathrm{pr}(T)$ *and* $\mathrm{pr}(f_i \cdot T) = f_{i+1} \cdot \mathrm{pr}(T)$ *for all* $1 \le i \le s-1$*.*

The following theorem reveals the reason why we restrict our attention to rectangular partitions in the statement of Theorem 1.2.

Theorem 3.13 ([16])**.** *Let κ be a partition with s parts, and let* $\mathrm{pr}\colon B_\kappa \to B_\kappa$ *be* jeu-de-taquin *promotion. Then* pr^s *acts as the identity if and only if κ is rectangular.*

To close the section, as promised, we reinterpret the Yamanouchi conditions on reading words as vanishing conditions on crystal operators.

Proposition 3.14. *Let κ be a partition with s parts, and let T be a tableau of shape κ. For all $1 \le i < i' \le s-1$, the word of T is Yamanouchi (anti-Yamanouchi) in the subset $\{i, i+1, \ldots, i'\}$ if and only if the raising operators $e_i, e_{i+1}, \ldots, e_{i'-1}$ (lowering operators $f_i, f_{i+1}, \ldots, f_{i'-1}$) all vanish at T.*

4 Proofs of Theorems 1.1 and 1.2

In this section, we sketch proofs of our main theorems. We start by delineating the properties of the basis of *Kazhdan–Lusztig immanants* constructed in [17]. For κ a partition with at most s positive parts, we consider κ as a partition with s parts. We note that the action of the long element $w_0 \in \mathfrak{S}_s \subset GL_s(C)$ on the immanants generating a $GL_s(\mathbb{C})$-representation associated to κ lifts (up to sign) the Schützenberger involution on the tableaux in the $\mathfrak{sl}_\mathfrak{s}$-crystal B_κ, and, analogously, that the action of the long cycle $c_s \in \mathfrak{S}_s \subset GL_s(C)$ on immanants lifts (up to sign) *jeu-de-taquin* promotion if κ is rectangular.

Theorem 4.1 ([13], [14]). *Let κ be a partition of t with s parts, and let $V_{\kappa,s}$ be the dual of the irreducible polynomial $GL_s(\mathbb{C})$-representation with highest weight κ. For all compositions η of t with s parts and semistandard tableaux U of shape κ and content η, let $I_\eta(U) \in V_s$ be the Kazhdan–Lusztig immanant associated to η and U. Set*

$$I_\eta := \{I_\eta(U) : U \text{ is a semistandard tableau of shape } \kappa \text{ and content } \eta\}.$$

Then the following claims hold.

(i) The set $\bigcup_\eta I_\eta$, where η ranges over all compositions of t with s parts, constitutes a basis for $V_{\kappa,s}$.

(ii) For all compositions η of t with s parts, the set I_η constitutes a basis for the weight space of $V_{T,s}$ corresponding to the weight $-\eta$, which we denote by $V_{\kappa,s,\eta}$.

(iii) Let w_0 be the long element in $\mathfrak{S}_s$, and let ξ be the Schützenberger involution. Then

$$w_0 \cdot I_\eta(U) = (-1)^{v(\kappa)} \cdot I_{w_0\cdot\eta}(\xi(U)).$$

(iv) Let c_s be the long cycle in $\mathfrak{S}_s$, and let pr *be* jeu-de-taquin *promotion. If κ is rectangular with exactly a positive parts, then*

$$c_s \cdot I_\eta(U) = (-1)^{\eta_s(a-1)} I_{c_s\cdot\eta}(\mathrm{pr}(U)).$$

4.1 Proof of Theorem 1.1

Let λ be a partition with $2m$ parts. Suppose that 2 divides $|\lambda|$, and let $\mu = (\mu_1, \mu_2, \ldots, \mu_m)$ be a partition of $|\lambda|/2$. Let B_λ be the set of semistandard tableaux of shape λ, endowed with an $\mathfrak{sl}_{2m}$-crystal structure by the sets of raising and lowering operators $\{e_i\}_{i=1}^{2m-1}$ and $\{f_i\}_{i=1}^{2m-1}$, respectively. The key to our proof is the assignment of an $(\mathfrak{sl}_m \oplus \mathfrak{sl}_m)$-crystal structure to B_λ that allows us to inspect the action of ξ on its connected components. This process provides a combinatorial model for the decomposition into irreducible components of the restriction to $GL_m(\mathbb{C}) \times GL_m(\mathbb{C})$ of the irreducible $GL_{2m}(\mathbb{C})$-representation with highest weight λ, which underlies our character evaluation.

Recall that we chose $\{E_1 - E_2, E_2 - E_3, \ldots, E_{2m-1} - E_{2m}\}$ as the set of simple roots for $\mathfrak{sl}_{2m}$. Here we choose $\{E_2 - E_1, E_3 - E_2, \ldots, E_m - E_{m-1}, E_{m+1} - E_{m+2}, E_{m+2} - E_{m+3}, \ldots, E_{2m-1} - E_{2m}\}$ as the set of simple roots for $\mathfrak{sl}_m \oplus \mathfrak{sl}_m$.

Proposition 4.2. *The set B_λ equipped with the map* wt, *the set of raising operators $\{f_i\}_{i=1}^{m-1} \cup \{e_{i+m}\}_{i=1}^{m-1}$, and the set of lowering operators $\{e_i\}_{i=1}^{m-1} \cup \{f_{i+m}\}_{i=1}^{m-1}$ is a regular $(\mathfrak{sl}_m \oplus \mathfrak{sl}_m)$-crystal.*

Proposition 4.3. *Let β and γ be partitions, each with m parts. Equip the set $B_{(\overline{\beta},\gamma)} := B_\beta \times B_\gamma$ with the map* wt × wt. *For all $1 \le i \le m-1$, let e_i and f_i act as the $\mathfrak{sl}_m$-crystal operators e_i and f_i, respectively, on B_β and as the identity on B_γ. For all $m+1 \le i \le 2m-1$, let e_i and f_i act as the identity on B_β and as the $\mathfrak{sl}_m$-crystal operators e_{i-m} and f_{i-m}, respectively, on B_γ. Then $B_{(\overline{\beta},\gamma)}$, together with the set of raising operators $\{f_1, f_2, \ldots, f_{m-1}, e_{m+1}, e_{m+2}, \ldots, e_{2m-1}\}$ and the set of lowering operators $\{e_1, e_2, \ldots, e_{m-1}, f_{m+1}, f_{m+2}, \ldots, f_{2m-1}\}$, is a regular connected $(\mathfrak{sl}_m \oplus \mathfrak{sl}_m)$-crystal.*

Theorem 4.4. *Let $\mathcal{C}$ be a connected component of the $(\mathfrak{sl}_m \oplus \mathfrak{sl}_m)$-crystal B_λ. Let b be the unique highest weight element of $\mathcal{C}$. Then there exist partitions β and γ, each with m parts, such that b is of content $\overline{\beta}\gamma$, and $\mathcal{C}$ is isomorphic to $B_{(\overline{\beta},\gamma)}$.*

Corollary 4.5. *Let $\mathcal{C}$ be a connected component of the $(\mathfrak{sl}_\mathfrak{m} \oplus \mathfrak{sl}_\mathfrak{m})$-crystal B_λ, and let b be the unique highest weight element of $\mathcal{C}$. If $\xi(b) \neq b$, then $\{T \in \mathcal{C} : \xi(T) = T\}$ is empty. Otherwise, there exists a partition $\beta = (\beta_1, \beta_2, \ldots, \beta_m)$ such that b is of content $\overline{\beta}\beta$, and the isomorphism of crystals $\mathcal{C} \xrightarrow{\sim} B_{(\overline{\beta},\beta)}$ restricts to a bijection of sets*

$$\{T \in \mathcal{C} : \xi(T) = T\} \xrightarrow{\sim} \{(U, U') \in B_{(\overline{\beta},\beta)} : \xi(U) = U'\}.$$

We proceed to the proof of Theorem 1.1 itself. Taking characters in the $GL_{2m}(\mathbb{C})$-representation $V_{\lambda,2m}$, we note that

$$\begin{aligned}
&\chi(w_0 \cdot \mathrm{diag}(x_1, x_2, \ldots, x_m, x_m, \ldots, x_2, x_1)) \\
&= (-1)^{v(\lambda)} \cdot \sum_{T \in B_\lambda : \xi(T) = T} x_1^{-2T_1} x_2^{-2T_2} \cdots x_m^{-2T_m} \\
&= (-1)^{v(\lambda)} \cdot \sum_{\theta \vdash |\lambda|/2} |\mathrm{EYTab}^\xi(\lambda, \overline{\theta}\theta)| \cdot s_\theta(x_1^{-2}, x_2^{-2}, \ldots, x_m^{-2}),
\end{aligned}$$

where the first equality follows from Theorems 4.1, and the second equality follows from Corollary 4.5.

Since $\chi\colon GL_{2m}(\mathbb{C}) \to \mathbb{C}$ is a class function, we see also that

$$\begin{aligned}
&\chi(w_0 \cdot \mathrm{diag}(x_1, x_2, \ldots, x_m, x_m, \ldots, x_2, x_1)) \\
&= \chi(\mathrm{diag}(x_1, x_2, \ldots, x_m, -x_m, \ldots, -x_2, -x_1)) \\
&= s_\lambda(x_1^{-1}, x_2^{-1}, \ldots, x_m^{-1}, -x_m^{-1}, \ldots, -x_2^{-1}, -x_1^{-1}) \\
&= (-1)^{v(\lambda)} \sum_D x_1^{-2D_1} x_2^{-2D_2} \cdots x_m^{-2D_m},
\end{aligned}$$

where the sum ranges over all semistandard domino tableaux of shape λ with entries in $\{1, 2, \ldots, m\}$. (Here the second equality follows from Theorem 4.1, and the third from Remark 3.2 of [18].)

By Theorem 2.6,

$$\begin{aligned}
\sum_D x_1^{-2D_1} x_2^{-2D_2} \cdots x_m^{-2D_m} &= \epsilon_2(\lambda) \cdot \phi_2(s_\lambda)(x_1^{-2}, x_2^{-2}, \ldots, x_m^{-2}) \\
&= \epsilon_2(\lambda) \cdot \sum_{\theta \vdash |\lambda|/2} \langle s_\lambda, p_2 \circ s_\theta \rangle s_\theta(x_1^{-2}, x_2^{-2}, \ldots, x_m^{-2}).
\end{aligned}$$

Identifying the coefficients of $s_\mu(x_1^{-2}, x_2^{-2}, \ldots, x_m^{-2})$ in our two expressions, we may conclude that

$$|\mathrm{EYTab}^\xi(\lambda, \overline{\mu}\mu)| = \epsilon_2(\lambda) \cdot \langle s_\lambda, p_2 \circ s_\mu \rangle.$$

□

4.2 Proof of Theorem 1.2

Let λ be a rectangular partition with mn parts. Suppose that n divides $|\lambda|$, and let $\mu = (\mu_1, \mu_2, \ldots, \mu_m)$ be a partition of $|\lambda|/n$. Let B_λ be the set of semistandard tableaux of shape λ, endowed with a $\mathfrak{sl}_{\mathfrak{mn}}$-crystal structure by the sets of raising and lowering operators $\{e_i\}_{i=1}^{mn-1}$ and $\{f_i\}_{i=1}^{mn-1}$, respectively. The key to our proof is the assignment of an $(\mathfrak{sl}_{\mathfrak{m}}{}^{\oplus n})$-crystal structure to B_λ that allows us to inspect the action of j on its connected components. This process provides a combinatorial model for the decomposition into irreducible components of the restriction to $GL_m(\mathbb{C})^{\times n}$ of the irreducible $GL_{mn}(\mathbb{C})$-representation with highest weight λ, which underlies our character evaluation.

Recall that we chose $\{E_1 - E_2, E_2 - E_3, \ldots, E_{mn-1} - E_{mn}\}$ as the set of simple roots for $\mathfrak{sl}_{\mathfrak{mn}}$. Here we choose $\bigcup_{k=0}^{n-1}\{E_{km+1} - E_{km+2}, E_{km+2} - E_{km+3}, \ldots, E_{(k+1)m-1} - E_{(k+1)m}\}$ as the set of simple roots for $\mathfrak{sl}_{\mathfrak{m}}{}^{\oplus n}$.

Proposition 4.6. *The set B_λ equipped with the map* wt*, the set of raising operators $\bigcup_{k=0}^{n-1}\{e_{km+i}\}_{i=1}^{m-1}$, and the set of lowering operators $\bigcup_{k=0}^{n-1}\{f_{km+i}\}_{i=1}^{m-1}$ is a regular $(\mathfrak{sl}_{\mathfrak{m}}{}^{\oplus n})$-crystal.*

Proposition 4.7. *Let $\beta_0, \beta_1, \ldots, \beta_{n-1}$ be partitions, each with m parts. Equip the set $B_{(\beta_0,\beta_1,\ldots,\beta_{n-1})} := B_{\beta_0} \times B_{\beta_1} \times \cdots \times B_{\beta_{n-1}}$ with the map* $\mathrm{wt} \times \mathrm{wt} \times \cdots \times \mathrm{wt}$. *For all $1 \le i \le m-1$ and $0 \le k \le n-1$, let e_{km+i} and f_{km+i} act as the $\mathfrak{sl}_{\mathfrak{m}}$-crystal operators e_i and f_i, respectively, on B_{β_k} and as the identity on B_{β_j} for all $j \ne k$. Then $B_{(\beta_0,\beta_1,\ldots,\beta_{n-1})}$, together with the set of raising operators $\bigcup_{k=0}^{n-1}\{e_{km+i}\}_{i=1}^{m-1}$ and the set of lowering operators $\bigcup_{k=0}^{n-1}\{f_{km+i}\}_{i=1}^{m-1}$, is a regular connected $(\mathfrak{sl}_{\mathfrak{m}}{}^{\oplus n})$-crystal.*

Theorem 4.8. *Let $\mathcal{C}$ be a connected component of the $(\mathfrak{sl}_{\mathfrak{m}}{}^{\oplus n})$-crystal B_λ. Let b be the unique highest weight element of $\mathcal{C}$. Then there exist partitions $\beta_0, \beta_1, \ldots, \beta_{n-1}$, each with m parts, such that b is of content $\beta_0\beta_1\cdots\beta_{n-1}$, and $\mathcal{C}$ is isomorphic to $B_{(\beta_0,\beta_1,\ldots,\beta_{n-1})}$.*

Corollary 4.9. *Let $\mathcal{C}$ be a connected component of the $(\mathfrak{sl}_{\mathfrak{m}}{}^{\oplus n})$-crystal B_λ, and let b be the unique highest weight element of $\mathcal{C}$. If $j^d(b) \ne b$, then $\{T \in \mathcal{C} : j^d(T) = T\}$ is empty. Otherwise, there exist d partitions $\beta_0, \beta_1, \ldots, \beta_d$, each with m parts, such that b is of content $(b_{\beta_0}, b_{\beta_1}, \ldots, b_{\beta_{d-1}})^{(n/d)}$, and the isomorphism of crystals $\mathcal{C} \xrightarrow{\sim} B_{\left((\beta_0,\beta_1,\ldots,\beta_d)^{(n/d)}\right)}$ restricts to a bijection of sets*

$$\{T \in \mathcal{C} : j^d(T) = T\} \xrightarrow{\sim} \{(U_0, U_1, \ldots, U_{n-1}) \in B_{\left((\beta_0,\beta_1,\ldots,\beta_d)^{(n/d)}\right)} : U_j = U_{j'} \;\; \forall j \cong j' \pmod{d}\}.$$

The remainder of the proof of Theorem 1.2 follows from a character evaluation in the $GL_{mn}(\mathbb{C})$-representation $V_{\lambda,mn}$. In the interest of brevity, we suppress the details, but they can be found in [14]. ☐

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