Statistics on Lattice Walks and q-Lassalle Numbers

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Abstract. This paper contains two results. First, I propose a q-generalization of a certain sequence of positive integers, related to Catalan numbers, introduced by Zeilberger, see Lassalle (2010). These q-integers are palindromic polynomials in q with positive integer coefficients. The positivity depends on the positivity of a certain difference of products of q-binomial coefficients.
To this end, I introduce a new inversion/major statistics on lattice walks. The difference in q-binomial coefficients is then seen as a generating function of weighted walks that remain in the upper half-plane.

Résumé. Cet document contient deux résultats. Tout d'abord, je vous propose un q -generalization d'une certaine séquence de nombres entiers positifs, liés à nombres de Catalan, introduites par Zeilberger (Lassalle, 2010). Ces q -integers sont des polynômes palindromiques à q à coefficients entiers positifs. La positivité dépend de la positivité d'une certaine différence de produits de q-coefficients binomial.
Pour ce faire, je vous présente une nouvelle inversion/major index sur les chemins du réseau. La différence de q -binomial coefficients est alors considérée comme une fonction de génération de trajets pondérés qui restent dans le demi-plan supérieur.

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1 Introduction to Lassalle's Sequences and their q-analogs.

Michel Lassalle [Las12] has discussed two related sequences of numbers A_k and α_k. $\{A_k\}$ is generated by the following recurrence:

$$A_n = (-1)^{n-1} C_n + \sum_{j=1}^{n-1} (-1)^{n-j-1} \binom{2n-1}{2j-1} A_j C_{n-j}, \quad A_1 = 1$$

The second sequence is

$$\alpha_n = \frac{2A_n}{C_n}$$

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He proved that both A_n and α_n are positive integers and each sequence is increasing (and more). (It turns out that the second sequence is simply related to power sums of zeros of Bessel function $J_0(z)$). It is intriguing to inquire whether there is a natural q-analog of these numbers. It may be generated by a q-analog of the above recurrence (with $C_n(q) = \frac{1}{[n+1]_q} \begin{bmatrix} 2n \\ n \end{bmatrix}_q$, a q-Catalan):

$$A'_n(q) = (-1)^{n-1} q^{n-1} C_n(q) + \sum_{j=1}^{n-1} (-1)^{n-j-1} q^{n-j} \begin{bmatrix} 2n-1 \\ 2j-1 \end{bmatrix}_q A'_j(q) C_{n-j}(q), \tag{1}$$

but a slightly renormalized version looks neater:

$$A_n(q) = (-1)^{n-1} q^{3-2n} C_n(q) + (-1)^n q^{3-2n} \left[2n-1\right]_q C_{n-1}(q) + \sum_{j=2}^{n-1} (-1)^{n-j-1} q^{2j-2n} \begin{bmatrix} 2n-1 \\ 2j-1 \end{bmatrix}_q A_j(q) C_{n-j}(q) \tag{2}$$

with $A_1(q) = 1$. It turns out that $A_n(q)$ are monic unimodal palindromic polynomials in q with positive integer coefficients. Here are some examples:

Example 1

$$\begin{aligned}
&A_2(q) = 1 \\
&A_3(q) = 1 + q + q^2 + q^3 + q^4 \\
&A_4(q) = 1 + 2q + 3q^2 + 5q^3 + 6q^4 + 7q^5 + 8q^6 + 7q^7 + 6q^8 + 5q^9 + 3q^{10} + 2q^{11} + q^{12} \\
&A_5(q) = 1 + 3q + 6q^2 + 12q^3 + 19q^4 + 29q^5 + 41q^6 + 54q^7 + 67q^8 + 80q^9 + 89q^{10} + 96q^{11} + 98q^{12} \\
&+ 96q^{13} + 89q^{14} + 80q^{15} + 67q^{16} + 54q^{17} + 41q^{18} + 29q^{19} + 19q^{20} + 12q^{21} + 6q^{22} + 3q^{23} + q^{24}
\end{aligned}$$

The second Lassalle's sequence α_k has the following q-analog:

$$\alpha_n = \frac{(1+q^n) A_n(q)}{C_n(q)} \tag{3}$$

And each of $\alpha_n(q)$ is also a monic unimodal palindromic polynomial in q with positive integer coefficients. Here are examples of $\alpha_n(q)$

Example 2

$$\begin{aligned}
&\alpha_1(q) = 1 + q \\
&\alpha_2(q) = 1 \\
&\alpha_3(q) = 1 + q \\
&\alpha_4(q) = 1 + 2q + 2q^2 + 2q^3 + q^4 \\
&\alpha_5(q) = 1 + 3q + 5q^2 + 8q^3 + 9q^4 + 9q^5 + 8q^6 + 5q^7 + 3q^8 + q^9
\end{aligned}$$

The proof of positivity of α_k relies (in addition to certain divisibility properties) on the positivity of

$$\binom{n}{k-1}\binom{n}{k} - \binom{n}{k-2}\binom{n}{k+1}$$

and Lassalle used the combinatorial interpretation of this difference of binomial coefficients as a generating function of the number of NSEW walks on a square lattice that start at the origin and finish at $(2k-n-1, 1)$ [GKS92].
Similarly, the positivity of $\alpha_k(q)$ requires the positivity of

$$\begin{bmatrix} n \\ k-1 \end{bmatrix}_q \begin{bmatrix} n \\ k \end{bmatrix}_q - q^2 \begin{bmatrix} n \\ k-2 \end{bmatrix}_q \begin{bmatrix} n \\ k+1 \end{bmatrix}_q \tag{4}$$

However in this case a combinatorial interpretation has to be developed.

2 Introduction to q-enumeration of Lattice walks

To understand the positivity of (4) combinatorially, i.e. as a generating function of certain weighted lattice walks, I first interpret the q-version of the generating function of all NSEW walks as a generating function of a certain (new) inversion statistics on lattice walks.
The total number of lattice walks from $(0, 0)$ to (c, d) of length n is given by [DR84]

$$\binom{n}{\frac{1}{2}(n-c+d)}\binom{n}{\frac{1}{2}(n-c-d)}$$

Think of a given walk as a word w composed of letters N, S, E, W. Then, the *walk inversion* statistics is defined

Definition 1

$$\begin{aligned} winvN(w) &= \sum_{N\in w} \#S \textit{ to the left of } N \\ winvW(w) &= \sum_{W\in w} \#S + \#N + \#E \textit{ to the left of } W \\ winvE(w) &= \sum_{E\in w} \#S + \#N + 2\#W \textit{ to the left of } E \\ wpinv(w) &= winvN(w) + winvW(w) + winvE(w) \end{aligned}$$

Example 3 *Here are the inversions of the walks (from left to right) on Fig. 1:*

$$\begin{aligned} WNEN &= 2 \cdot 1 + 1 = 3 \\ ENWN &= 1 + 1 = 2 \\ NWNE &= 1 + (2 + 2 \cdot 1) = 5 \\ NENW &= 1 + (2 + 1) = 4 \\ WNNE &= 2 + 2 \cdot 1 = 4 \\ ENNW &= 1 + 2 = 3 \end{aligned}$$

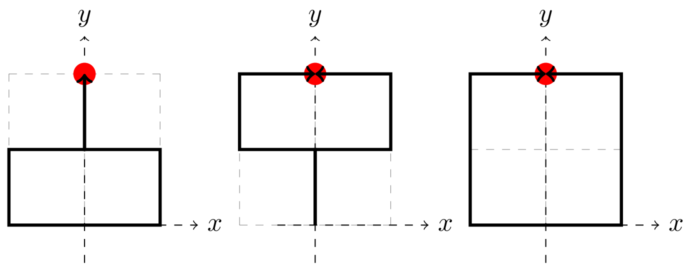

Fig. 1: Some walks from $(0, 0)$ to $(0, 2)$ with $n = 4$ steps.

Denote the set of all lattice walks from (a, b) to (c, d) in n letters (steps) by $P_n((a, b) \mid (c, d))$. The q-analog of the walk enumeration formula is the following generating function.

Proposition 1

$$\begin{bmatrix} n \\ \frac{1}{2}(n-c+d) \end{bmatrix}_q \begin{bmatrix} n \\ \frac{1}{2}(n-c-d) \end{bmatrix}_q = \sum_{w \in P_n((0,0)|(c,d))} q^{winv(w)} \tag{5}$$

The following is true for the walks restricted to the upper half plane. Denote the set of walks starting at $(0, 0)$ and ending at (c, d) in n steps and never going below the x-axis by $P_n^+((0, 0) \mid (c, d))$, then

Proposition 2

$$\begin{bmatrix} n \\ \frac{1}{2}(n+c+d) \end{bmatrix}_q \begin{bmatrix} n \\ \frac{1}{2}(n+c-d) \end{bmatrix}_q - q^{d+1} \begin{bmatrix} n \\ \frac{1}{2}(n+c+d)+1 \end{bmatrix}_q \begin{bmatrix} n \\ \frac{1}{2}(n+c-d)-1 \end{bmatrix}_q = \sum_{w \in P_n^+((0,0)|(c,d))} q^{winv(w)} \tag{6}$$

In Section 6 I will also introduce an analog of a major index, $wmaj(w)$, which is, conjecturally, equally distributed with $winv(w)$ over lattice walks.

3 q-Lassalle Numbers

The purpose of this section is to derive a bilinear recursion relations for $A_k(q)$ and $\alpha_k(q)$, from which the positivity and integrality follow.

The strategy is to rewrite the recursion relation between $A'(q)$ (as in (1)) as a difference equation. Then, using the q-difference equation for the generating function of q-Catalan, to obtain the q-difference equation for the generating function of the q-Lassalle numbers.

$$A'_n(q) = (-1)^{n-1} q^{n-1} C_n(q) + \sum_{j=1}^{n-1} (-1)^{n-j-1} q^{n-j} \begin{bmatrix} 2n-1 \\ 2j-1 \end{bmatrix}_q A'_j(q) C_{n-j}(q)$$

$$- \frac{(-1)^n q^n \left[2n\right]_q C_n(q)}{\left[2n\right]_q!} = \frac{qA'_n(q)}{\left[2n-1\right]_q!} + \sum_{j=1}^{n-1} \frac{qA'_j(q)}{\left[2j-1\right]_q!} \frac{(-1)^{n-j} q^{n-j} C_{n-j}(q)}{\left[2n-2j\right]_q!} \tag{7}$$

Introduce generating functions and the finite q-difference operator:

$$
\begin{aligned}
H(t;q) &= \sum_{k\geq 0} \frac{(-1)^k q^k C_k(q) t^{2k}}{\left[2k\right]_q!} \equiv \sum_{k\geq 0} \frac{(-1)^k q^k t^{2k}}{\left[k\right]_q! \left[k+1\right]_q!} \\
P(t;q) &= \sum_{k\geq 1} \frac{q A_k'(q) t^{2k-1}}{\left[2k-1\right]_q!} \\
D_q f(t) &= \frac{f(t) - f(qt)}{(1-q)t}
\end{aligned}
$$

then (7) is equivalent to

$$
-D_q H(t,q) = P(t;q) H(t;q) \tag{8}
$$

Recall Jackson's basic q-Bessel function ([Ism82]):

$$
J_\nu^{(1)}(x;q) = \frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty} \sum_{n\geq 0} \frac{(-1)^n}{(q;q)_n (q^{\nu+1};q)_n} \left(\frac{x}{2}\right)^{2n+\nu}, \quad \text{where} \quad (a;q)_n = \prod_{j=0}^{n-1} (1 - aq^i) \tag{9}
$$

For $\nu = 1$

$$
J_1^{(1)}(x;q) = \frac{1}{1-q} \sum_{n\geq 0} \frac{(-1)^n}{(q;q)_n (q^2;q)_n} \left(\frac{x}{2}\right)^{2n+1} = \sum_{n\geq 0} \frac{(-1)^n}{\left[n\right]_q! \left[n+1\right]_q!} \left(\frac{x}{2(1-q)}\right)^{2n+1}
$$

So that

$$
H(t;q) = \frac{1}{\sqrt{q}\, t} J_1^{(1)}(2(1-q)\sqrt{q}\, t; q), i.e.
$$

q-Bessel function $J_1^{(1)}(x;q)$ is a generating function for q-Catalan numbers.
$J_1^{(1)}(x;q)$ satisfies the following q-Bessel difference equation:

$$
J_1^{(1)}(qx;q) - \left(q^{\frac{1}{2}} + q^{-\frac{1}{2}}\right) J_1^{(1)}(\sqrt{q}\, x;q) + \left(1 + \frac{x^2}{4}\right) J_1^{(1)}(x;q) = 0 \tag{10}
$$

For $H(t;q)$, (10) translates to:

$$
H(qt;q) - \left(1 + \frac{1}{q}\right) H(\sqrt{q}t;q) + \left(\frac{1}{q} + (1-q)^2 t^2\right) H(t;q) = 0 \tag{11}
$$

Through the q-difference equation (8) this implies the following ungainly looking q-difference equation for $P(t;q)$:

$$
\begin{aligned}
&(1+q+q(1-q)^2t^2)q(1-q)\sqrt{q}tP(\sqrt{q}t;q) + q^2(1-q)^2 tP(t;q)\sqrt{q}tP(\sqrt{q}t;q) + \\
&+ (1+q+q(1-q)^2t^2)q^2(1-q)^2t^2 + q^2(1-q)^2t^2 q(1-q)tP(t;q) \\
&= (1-q^2)tP(t;q) - q(1-q)(1-q^2)t^2
\end{aligned}
$$

But collecting coefficients of t^{2n} on both sides makes things look better:

$$A'_1(q) = 1$$
$$A'_2(q) = q^2$$
$$\frac{[2]_q[s+1]_q}{[2s-1]_q!}A'_s(q) = \frac{q^3+q^{s+1}}{[2s-3]_q!}A'_{s-1}(q) + \sum_{k=2}^{s-2}\frac{q^{k+3}A'_k(q)A'_{s-k}(q)}{[2k-1]_q![2s-2k-1]_q!}$$

It's time to rescale, so let

$$A'_k(q) = q^{3k-4}A_k(q)$$

then the recursion relation for $A_k(q)$ is:

$$A_s(q) = \frac{[2s-1]_q}{[2]_q[s+1]_q}\sum_{k=1}^{s-1} q^{k-1}\frac{[2s-2]_q!}{[2k-1]_q![2s-2k-1]_q!}A_k(q)A_{s-k}(q) \tag{12}$$

Translating this recursion into that for $\alpha_n(q)$ (as in (3)) produces:

$$\alpha_n(q) = \frac{1}{[2]_q\,[n]_q}\sum_{k=1}^{n-1} q^{k-1}\begin{bmatrix} n \\ k+1 \end{bmatrix}_q \begin{bmatrix} n \\ k-1 \end{bmatrix}_q \alpha_k(q)\alpha_{n-k}(q) \tag{13}$$

The ratio of the q-binomial coefficients can be rewritten as

$$\frac{1}{[n]_q}\begin{bmatrix} n \\ k+1 \end{bmatrix}_q \begin{bmatrix} n \\ k-1 \end{bmatrix}_q = \frac{1}{[2]_q}\left(\begin{bmatrix} n-1 \\ r-1 \end{bmatrix}_q \begin{bmatrix} n-1 \\ r \end{bmatrix}_q - q^2\begin{bmatrix} n-1 \\ r-2 \end{bmatrix}_q \begin{bmatrix} n-1 \\ r+1 \end{bmatrix}_q\right)$$

Therefore, the proof depends on

- positivity and integrality of

$$c_{n,k}(q) = \begin{bmatrix} n-1 \\ r-1 \end{bmatrix}_q \begin{bmatrix} n-1 \\ k \end{bmatrix}_q - q^2\begin{bmatrix} n-1 \\ k-2 \end{bmatrix}_q \begin{bmatrix} n-1 \\ k+1 \end{bmatrix}_q$$

- divisibility of $c_{n,k}(q)$ and $\alpha_k(q)$ by powers of $[2]_q$.

As may be seen from (6), the positivity and integrality of $c_{n,k}(q)$ follows from its combinatorial interpretation as the generating function of $winv$ statistics of upper half-plane lattice walks, namely the walks that start at $(0,0)$ and end up at $(2k-n-1,1)$ in $(n-1)$ steps.
Therefore I continue with the lattice walk part of the story.

4 Walk Inversion Generating Function

In order to prove the walk inversion generating function formula, as in (5)

$$\begin{bmatrix} n \\ \frac{1}{2}(n-c+d) \end{bmatrix}_q \begin{bmatrix} n \\ \frac{1}{2}(n-c-d) \end{bmatrix}_q = \sum_{w\in P_n((0,0)|(c,d))} q^{winv(w)}$$

I will need the following lemma. Denote the set of all walks of length n from $(0,0)$ to (c,d) with exactly k W steps by $P_n((c,d);k)$. Of course this fixes the number of S steps, r as well. The total number of steps in n, $(c+k)$ of which are E, k – W, $(d+r)$ – N, and r – S.

$$r = \frac{1}{2}(n-c-d) - k$$

As words, these walks are permutations of each other. Their total number is $\frac{n!}{(c+k)!k!(d+r)!r!}$.

Lemma 1 *The generating function of the walk inversion statistics of n-step walks with k W steps is given by*

$$q^{k^2+ck}\frac{[n]_q!}{[c+k]_q![\frac{1}{2}(n-c+d)-k]_q![k]_q![\frac{1}{2}(n-c-d)-k]_q!} = \sum_{w\in P_n((c,d);k)} q^{winv(w)} \tag{14}$$

Now, in general, to get to (c,d) from $(0,0)$ in n steps one might take just one W step, or two, etc. The maximal number of W steps (i.e. no S steps) is

$$\frac{1}{2}(n-c-d)$$

So that

Lemma 2 *The walk inversion generating function is*

$$\sum_{w\in P_n((0,0)|(c,d))} q^{winv(w)} = \sum_{k=0}^{\frac{1}{2}(n-c-d)} q^{k^2+ck}\frac{[n]_q!}{[c+k]_q![\frac{1}{2}(n-c+d)-k]_q![k]_q![\frac{1}{2}(n-c-d)-k]_q!} \tag{15}$$

Now, consider how a walk can end up at (c,d) in n steps, i.e. where could it be at the previous step?

- either at $(c-1,d)$, with the last step E; adding an E step changes $winv$ by $\#S+\#N+2\#W = r+(d+r)+2k = d+2r+2k = n-c$
  The contribution of walks coming from the West is

$$q^{n-c}\sum_{k=0}^{\frac{1}{2}(n-c-d)} q^{k^2+(c-1)k}\frac{[n-1]_q!}{[c-1+k]_q!\left[\frac{1}{2}(n-c-d)-k\right]_q![k]_q!\left[\frac{1}{2}(n-c+d)-k\right]_q!}$$

- or at $(c,d+1)$ with the last step S; adding an S step does not change $winv$.
  The contribution of the walks coming from the North:

$$\sum_{k=0}^{\frac{1}{2}(n-c-d)-1} q^{k^2+ck}\frac{[n-1]_q!}{[c+k]_q!\left[\frac{1}{2}(n-c+d)-k-1\right]_q![k]_q!\left[\frac{1}{2}(n-c-d)-k-1\right]_q!}$$

- or at $(c+1,d)$ with the last step W; adding a W step changes $winv$ by $\#S+\#N+\#E=r+(d+r)+(k+c+1)=d+2r+c+k+1=n-1-k$
  The contribution of walks coming from the East is

$$q^{n-1}\sum_{k=0}^{\frac{1}{2}(n-c-d)-1} q^{k^2+ck}\frac{[n-1]_q!}{[c+1+k]_q!\left[\frac{1}{2}(n-c+d)-k-1\right]_q![k]_q!\left[\frac{1}{2}(n-c-d)-k-1\right]_q!}$$

- or at $(c,d-1)$ with the last step N; an addition of an N step with r prior S steps changes $winv$ by $\frac{1}{2}(n-c-d)-k.$
  The contribution of the walks coming from the South:

$$q^{\frac{1}{2}(n-c-d)}\sum_{k=0}^{\frac{1}{2}(n-c-d)} q^{k^2+(c-1)k}\frac{[n-1]_q!}{[c+k]_q!\left[\frac{1}{2}(n-c-d)-k\right]_q![k]_q!\left[\frac{1}{2}(n-c+d)-k\right]_q!}$$

The sum of these contributions gives a recursion that establishes (5).

5 Upper Half-Plane walks

Following the logic of walks reflection [GKS92], to every negative path from $(0,0)$ to (c,d) will associate a walk from $(-2,0)$ to (c,d) so that the change in $winv$ is the same for every walk.

Here is the algorithm.

1. Separate each negative walk in two segments $w=w_1\cdot w_2$, where $\cdot$ means concatenation of words:
   (a) w_1: the part of the walk that starts at $(0,0)$ and ends at $(*,0)$ (before it dips below the x-axis the first time)
   (b) w_2: the rest of the walk that runs from $(*,0)$ to (c,d); Notice that w_2 starts with S and necessarily has at least $d+1$ N steps. More precisely, if the walk has k S steps, it has $d+k$ N steps.
2. $\tilde{w_1}$: move w_1 down two steps, so that it starts at $(0,-2)$ and ends at $(*,-2)$;
3. attach $\tilde{w_2}$ to the combined walk. The walk modified this way runs from $(0,-2)$ to (c,d)

Consider the simplest (negative) walk $w=w_1\cdot w_2$: after dipping down below the x-axis it goes straight up to d, i.e. the part after reaching the x-axis (w_2) looks like

$$w_2 = S\underbrace{NN\ldots NN}_{d+1 \text{ times}}$$

Transform this walk into

$$\tilde{w_2} = \underbrace{NN\ldots NN}_{d+1 \text{ times}}S$$

Suppose that w_1 contained k S steps. Then ($\tilde{w}=\tilde{w}_1\cdot\tilde{w}_2$)

$$winv(w)-winv(\tilde{w})=(d+1)(k+1)-(d+1)k=d+1$$

Now consider a generic w with blocks of S and N letters:

$$w_2 = \underbrace{S \dots S}_{s_1 \text{ times}} \underbrace{N \dots N}_{n_1 \text{ times}} \underbrace{S \dots S}_{s_2 \text{ times}} \underbrace{N \dots N}_{n_2 \text{ times}} \dots \underbrace{S \dots S}_{s_r \text{ times}} \underbrace{N \dots N}_{n_r \text{ times}}$$

where

$$\sum_i^r n_i - s_i = d$$

By swapping the last S in the each string of s_i with the last N letter in n_i until $d + 1$ letters N have been moved, transforms w_2

$$\tilde{w}_2 = \underbrace{S \dots SS}_{s_1 - 1 \text{ times}} \underbrace{NN \dots NN}_{n_1 \text{ times}} S \underbrace{S \dots S}_{s_2 - 1 \text{ times}} \underbrace{N \dots N}_{n_2 \text{ times}} S \dots \underbrace{S \dots S}_{s_i - 1 \text{ times}} \underbrace{N \dots N}_{r \text{ times}} S \underbrace{N \dots N}_{n_i - r \text{ times}} \dots \underbrace{S \dots S}_{s_r \text{ times}} \underbrace{N \dots N}_{n_r \text{ times}}$$

with $n_1 + n_2 + \dots + r = d + 1$

$$\begin{aligned} &winv(w) - winv(\tilde{w}) = \{n_1(s_1 + k) - n_1(s_1 - 1 + k)\} + \{n_2(s_1 + s_2 + k) - n_2(s_1 + s_2 - 1 + k)\} \\ &+ \dots + \{r(s_1 + \dots + s_i + k) - r(s_1 + \dots + s_i - 1 + k)\} \\ &+ \{(n_i - r)(s_1 + \dots + s_i + k) - (n_i - r)(s_1 + \dots + s_i + k)\} + \\ &+ \dots + \{n_r(s_1 + \dots + s_r) - n_r(s_1 + \dots + s_r)\} = \sum_i n_i + r = d + 1 \end{aligned}$$

$$winv(\tilde{w}) = q^{-(d+1)} winv(w)$$

So the total contribution of negative walks is

$$q^{d+1} P_n((0, -2) \mid (c, d)) = q^{d+1} \begin{bmatrix} n \\ \frac{1}{2}(n + 2 + c + d) \end{bmatrix}_q \begin{bmatrix} n \\ \frac{1}{2}(n - 2 + c - d) \end{bmatrix}_q$$

Hence (6).

6 Major Walk Index

Setting up the following order $S > N > E > W$, and with the usual definition of a descent set

$$desN(w) = \{i : \text{ S occurs as } i^{\text{th}} \text{ letter and N occurs as } i + 1^{\text{st}} \text{ letter}\}$$

$$majN(w) = \sum_{i \in desN(w)} i$$

$$desE(w) = \{i : \text{ S or N occur as } i^{\text{th}} \text{ letter and E occurs as } i + 1^{\text{st}} \text{ letter}\}$$

$$majE(w) = \sum_{i \in desE(w)} i$$

$$desW(w) = \{i : \text{ S, N, or E occur as } i^{\text{th}} \text{ letter and W occurs as } i + 1^{\text{st}} \text{ letter}\}$$

$$majW(w) = \sum_{i \in desW(w)} i$$

one can give the following

Definition 2

$$wmaj(w) = majN(w) + majE(w) + majW(w) + \#E \times \#W \tag{16}$$

Notice the unusual last term... Nevertheless, experiments show that $winv$ and $wmaj$ are equally distributed over all lattice walks as well as over upper half-plane walks.

Conjecture 1

- $$\sum_{w \in P_n((0,0)|(c,d))} q^{winv(w)} = \sum_{w \in P_n((0,0)|(c,d))} q^{wmaj(w)} \tag{17}$$

- $$\sum_{w \in P_n^+((0,0)|(c,d))} q^{winv(w)} = \sum_{w \in P_n^+((0,0)|(c,d))} q^{wmaj(w)} \tag{18}$$

7 q-Integers Associated with q-super Ballot Numbers

Computer experiments indicate that there is a family of q-numbers related to several generalizations of q-Catalan. For instance, following [Ges92] define q-super Ballot numbers

$$B_{n,k,r}(q) = \frac{[k+2r]_q![2n+k-1]_q!}{(k-1)_q!r_q!n_q![n+k+r]_q!}$$

Then, I venture to make the following conjecture

Conjecture 2 *Define a new sequence $A_{n,k,r}(q)$ with $A_{1,k,r} = B_{0,k,r}(q)$ through the recurrence*

$$(-1)^{n-1}A_{n,k,r}(q) = q^{n-1}B_{n,k,r}(q) + \sum_{j=1}^{n-1}(-1)^j q^{n-j-1} \begin{bmatrix} 2n-1 \\ 2j-1 \end{bmatrix}_q A_{j,k,r}(q)B_{n-j,k,r}(q) \tag{19}$$

then the $A_{n,k,r}(q)$ are polynomials in q with positive integers coefficients for all values of $n, k, r > 0$.

Example 4

$$\begin{aligned}
A_{2,1,1}(q) &= 1 + 2q + 4q^2 + 4q^3 + 3q^4 + q^5 \\
A_{2,2,1}(q) &= 1 + 3q + 7q^2 + 11q^3 + 13q^4 + 12q^5 + 8q^6 + 4q^7 + q^8 \\
A_{2,2,2}(q) &= 1 + 3q + 9q^2 + 18q^3 + 33q^4 + 51q^5 + 72q^6 + 89q^7 + 100q^8 + 101q^9 + 93q^{10} + 77q^{11} \\
&\quad + 57q^{12} + 38q^{13} + 22q^{14} + 11q^{15} + 4q^{16} + q^{17} \\
A_{3,1,1}(q) &= 1 + 3q + 9q^2 + 17q^3 + 28q^4 + 38q^5 + 44q^6 + 43q^7 + 35q^8 + 24q^9 + 13q^{10} + 5q^{11} + q^{12} \\
A_{3,2,1}(q) &= 1 + 5q + 19q^2 + 51q^3 + 110q^4 + 199q^5 + 307q^6 + 412q^7 + 484q^8 + 499q^9 + 452q^{10} \\
&\quad + 358q^{11} + 245q^{12} + 143q^{13} + 69q^{14} + 26q^{15} + 7q^{16} + q^{17} \\
A_{3,2,2}(q) &= 1 + 5q + 22q^2 + 68q^3 + 181q^4 + 414q^5 + 848q^6 + 1567q^7 + 2652q^8 + 4134q^9 + 5980q^{10} \\
&\quad + 8058q^{11} + 10155q^{12} + 11997q^{13} + 13313q^{14} + 13892q^{15} + 13639q^{16} + 12597q^{17} + \\
&\quad + 10937q^{18} + 8913q^{19} + 6802q^{20} + 4845q^{21} + 3206q^{22} + 1958q^{23} + 1094q^{24} \\
&\quad + +552q^{25} + 247q^{26} + 95q^{27} + 30q^{28} + 7q^{29} + q^{30}
\end{aligned}$$

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