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Permutations Containing and Avoiding 123 *and* 132 *Patterns*

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We prove that the number of permutations which avoid 132-patterns and have exactly one 123-pattern, equals $(n-2)2^{n-3}$, for $n \geq 3$. We then give a bijection onto the set of permutations which avoid 123-patterns and have exactly one 132-pattern. Finally, we show that the number of permutations which contain exactly one 123-pattern and exactly one 132-pattern is $(n-3)(n-4)2^{n-5}$, for $n \geq 5$.

Keywords: Patterns, Words

1 Introduction

In 1990, Herb Wilf asked the following: How many permutations of length n avoid a given pattern, p? By pattern-avoiding we mean the following: Let π be a permutation of length n and let $p = (p_1, p_2, \ldots, p_k)$ be a permutation of length $k \leq n$ (we will call this a pattern of length k). Let J be a set of r integers, and let $j \in J$. Define $place(j, J)$ to be 1 if j is the smallest element in J, 2 if it is the second smallest, ..., and r if it is the largest. The permutation π avoids the pattern p if and only if there does not exist a set of indices $I = (i_1, i_2, \ldots, i_k)$, such that $p = (place(\pi(i_1), I), place(\pi(i_2), I), \ldots, place(\pi(i_k), I))$.

In two beautiful papers ([B1] and [N]), the number of subsequences containing exactly one 132-pattern and exactly one 123-pattern are enumerated. Noonan shows in [N] that the number of permutations containing exactly one 123-pattern is the simple formula $\frac{3}{n}\binom{2n}{n+3}$. Bóna proves that the even simpler formula $\binom{2n-3}{n-3}$ enumerates the number of permutations containing exactly one 132-pattern. Bóna's result proved a conjecture first made by Noonan and Zeilberger in [NZ].

Noonan and Zeilberger considered in [NZ] the number of permutations of length n which contain exactly r p-patterns, for $r \geq 1$. Bóna, in [B2], made further progress concerning the number of permutations with exactly r 132-patterns. In this article we work towards the following generalization: How many permutations of length n avoid patterns p_i, for $i \geq 0$, and contain r_j p_j-patterns, for $j \geq 1$, $r_j \geq 1$? We will first consider the permutations of length n which avoid 132-patterns, but contain exactly one 123-pattern. We then define a natural bijection between these permutations and the permutations of length n which avoid 123-patterns, but contain exactly one 132-pattern. Finally, we will calculate the number of permutations which contain one 123-pattern and one 132-pattern. These results address questions first raised in [NZ].

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2 Known Results

For completeness, two results which are already known are given below.

Lemma 1: *The number of permutations of length n with one 12-pattern is $n-1$.*

Proof: Induct on n. The base case is trivial. A permutation, ϕ, of length n with one 12-pattern must have $n=\phi(1)$ or $n=\phi(2)$. If $n=\phi(1)$, by induction we get $n-2$ permutation. If $n=\phi(2)$, then we must have $n-1=\phi(1)$ (or we would have more than one 12-pattern). The rest of the entries of ϕ must be decreasing. Hence we get 1 more permutation from this second case, for a total of $n-1$.

Lemma 2: *The number of permutations which avoid both the pattern 123 and 132 is 2^{n-1}.*

Proof: Let f_n denote the number of permutations we are interested in. Then $f_n = \sum_{i=1}^{n} f_{n-i}$ with $f_0 = 1$. To see this, let ρ be a permutation of length $n-1$. Insert the element n into the i^{th} position of ρ. Let π be this new permutation of length n. To assure that π avoids the 132-pattern, we must have all entries preceding n in π be larger than the entries following n. To assure that π avoids the 123-pattern, the entries preceding n must be in decreasing order. This argument gives the sum in the recursion. The recursion holds by noting that if $n=1$, there is one permutation which avoids both patterns. To complete the proof note that $f_n = 2^{n-1}$.

3 One 123-pattern, but no 132-pattern

Theorem 1: *The number of permutations of length n which have exactly one 123-pattern, and avoid the 132-pattern is $(n-2)2^{n-3}$.*

Proof: Call a permutation *good* if it has exactly one 123-pattern and avoids the 132-pattern, and let g_n denote the number of good permutations of length n. Let γ be a permutation of length $n-1$. Insert the element n into the i^{th} position of γ. Call this newly constructed permutation of length n, π. To assure that π avoids the 132 pattern, we must have all elements preceding n in π be larger than the elements following n in π. For π to be a good permutation, we must consider two disjoint cases.

Case I: The pattern 123 appears in the elements following n in π. This forces the elements preceding n to be in decreasing order. Summing over i, this case accounts for $\sum_{i=1}^{n} g_{n-i}$ permutations.

Case II: The pattern 123 appears in the elements preceding and including n in π. This forces the 3 in the pattern to be n. Hence the elements preceding n must contain exactly one 12-pattern. (Further there must be at least 2 elements. Hence i must be at least 3). From Lemma 1, this number is $i-2$. We are also forced to avoid both patterns in the elements following n. Lemma 2 implies that there are 2^{n-i-1} such permutations. Summing over i, this case accounts for $\sum_{i=3}^{n-1} (i-2)2^{n-i-1} + n-2$ permutations.

We have established that the recurrence relation

$$g_n = \sum_{i=1}^{n} g_{n-i} + \sum_{i=3}^{n-1} (i-2)2^{n-i-1} + n-2,$$

which holds for $n \geq 3$ ($g_0=0, g_1=0, g_2=0$), enumerates the permutations of length n which avoid the pattern 132 and contain exactly one 123-pattern.

One easy way to proceed would be to find the generating function of g_n. However, in this article we would like to employ a different, and in many circumstances more powerful, tool. We will use the Maple procedure `findrec` in Doron Zeilberger's Maple package EKHAD‡. (The Maple shareware

‡ Available for download at `www.math.temple.edu/~zeilberg/`

package `gfun` could have also been used.) Instructions for its use are available online. To use `findrec` we compute the first few terms of g_n. These are (for $n \geq 4$) $4, 12, 32, 80, 192, 448, 1024$. We type `findrec([4,12,32,80,192,448,1024],0,2,n,N)` and are given the recurrence $h_n = 4(h_{n-1} - h_{n-2})$ for $n \geq 4$. Define $h_0 = 0, h_1 = 0, h_2 = 0$, and $h_3 = 1$, and it is routine to verify that $g_n = h_n$ for $n \geq 0$. Another routine calculation shows us that $h_n = (n-2)2^{n-3}$ for $n \geq 3$, thereby proving the statement of the theorem.

4 One 132-pattern, but no 123-pattern

Theorem 2: *The number of permutations of length n which have exactly one 132-pattern, and avoid the 123-pattern is $(n-2)2^{n-3}$.*
Proof: We prove this by exhibiting a (natural) bijection from the permutations counted in Theorem 1 to the permutations counted in this theorem. Define $S := \{\pi : \pi$ avoids 132-pattern and contains one 123-pattern$\}$ and $T := \{\pi : \pi$ avoids 123-pattern and contains one 132-pattern$\}$. We will show that $| S |=| T |$, by using the following bijection:

Let $\phi : S \longrightarrow T$. Let $s \in S$, and let abc be the 123-pattern in s. Then ϕ acts on the elements of s as follows: $\phi(x) = x$ if $x \notin \{b, c\}$, $\phi(b) = c$, and $\phi(c) = b$. In other words, all elements keep their positions except b and c switch places. An easy examination of several cases shows that this is a bijection, thereby proving the theorem.

5 One 132-pattern and one 123-pattern

Theorem 3: *The number of permutations of length n which have exactly one 132-pattern and one 123-pattern is $(n-3)(n-4)2^{n-5}$.*
Proof: We use the same insertion technique as in the proof of Theorem 1. Call a permutation *good* if it has exactly one 123-pattern and exactly one 132-pattern and let g_n denote the number of good permutations of length n. Let γ be a permutation of length $n-1$. Insert the element n into the i^{th} position of γ. Let π be this newly constructed permutation of length n. We note that the 132-pattern cannot consist of elements only preceding n. If this were the case, we would have two 123-patterns ending with n. For π to be a good permutation, we must consider the following disjoint cases.
Case I: The 132-pattern consists of elements following n. In this case all elements preceding n must be larger than the elements following n.

Subcase A: The 123-pattern consists of elements following n. Summing over i we get $\sum_{i=1}^{n} g_{n-i}$ good permutations in this subcase.

Subcase B: The elements preceding n have exactly one 12-pattern. This gives a 123-pattern where the 3 in the pattern is n. We must also avoid the 123-pattern in the elements following n. Summing over i and using Lemma 1 and Theorem 1, we get $\sum_{i=3}^{n-3} (i-2)(n-i-3)2^{n-i-2}$ good permutations in this subcase.
Case II: The 132-pattern has the first element preceding n, the last element following n, and n as the middle element. The elements preceding n must be $n-1, n-2, \ldots, n-1+2, n-i$, where $n-i$ immediately precedes n in π. See [B1] for a more detailed argument as to why this must be true.

Subcase A: The elements preceding n have exactly one 12-pattern. This gives a 123-pattern where the last element of the pattern is n. We must also avoid both the 123 and the 132 pattern in the elements following

n. Summing over i and using Lemma 1 and Lemma 2 we have $\sum_{i=4}^{n-1}(i-3)2^{n-i-1}$ good permutations in this subcase.

Subcase B: The 123-pattern consists of elements following n. We must have the elements preceding n in π be decreasing to avoid another 123-pattern. Further, the elements following n must not contain a 132-pattern. Using Theorem 1 and summing over i, we get a total of $\sum_{i=2}^{n-3}(n-i-2)2^{n-i-3}$ good permutations in this subcase.

In total, we find that the following recurrence enumerates the permutations of length n which contain exactly one 123-pattern and one 132-pattern.

$$g_n = \sum_{i=1}^{n} g_{n-i} + \sum_{i=1}^{n-4}(2i(n-i-4)+n-3)2^{n-i-4}$$

for $n \geq 5$ and $g_1 = g_2 = g_3 = g_4 = 0$.

Using `findrec` again by typing `findrec([2,12,48,160,480,1344,3584],1,1,n,N)` (where the list is the first few terms of our recurrence for $n \geq 5$) we get the recurrence $f_{n+1} = \frac{2(n+2)}{n} f_n$, with $f_1 = 2$. After reindexing, another routine calculation shows that $f_n = g_n$. Solving f_n for an explicit answer, we find that $g_n = (n-3)(n-4)2^{n-5}$.

We conjecture that the generating function for the number of permutations with exactly zero or exactly one 132-pattern and exactly r 123-patterns is $P(z)/(1-2z)^{r+1}$, where $P(z)$ is a polynomial. For more evidence, and further extensions see [RWZ].

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