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Permutations Containing and Avoiding 123 and 132 Patterns

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We prove that the number of permutations which avoid 132-patterns and have exactly one 123-pattern, equals \((n - 2)2^{n-3}\), for \(n \geq 3\). We then give a bijection onto the set of permutations which avoid 123-patterns and have exactly one 132-pattern. Finally, we show that the number of permutations which contain exactly one 123-pattern and exactly one 132-pattern is \((n - 3)(n - 4)2^{n-5}\), for \(n \geq 5\).

Keywords: Patterns, Words

1 Introduction

In 1990, Herb Wilf asked the following: How many permutations of length \(n\) avoid a given pattern, \(p\)? By pattern-avoiding we mean the following: Let \(\pi\) be a permutation of length \(n\) and let \(p = (p_1, p_2, \ldots, p_k)\) be a permutation of length \(k \leq n\) (we will call this a pattern of length \(k\)). Let \(J\) be a set of \(r\) integers, and let \(j \in J\). Define \(\text{place}(j, J)\) to be 1 if \(j\) is the smallest element in \(J\), 2 if it is the second smallest, ..., and \(r\) if it is the largest. The permutation \(\pi\) avoids the pattern \(p\) if and only if there does not exist a set of indices \(I = (i_1, i_2, \ldots, i_k)\), such that \(p = (\text{place}(\pi(i_1), I), \text{place}(\pi(i_2), I), \ldots, \text{place}(\pi(i_k), I))\).

In two beautiful papers ([B1] and [N]), the number of subsequences containing exactly one 132-pattern and exactly one 123-pattern are enumerated. Noonan shows in [N] that the number of permutations containing exactly one 123-pattern is the simple formula \(\frac{2}{n} \binom{2n}{n+3}\). Bóna proves that the even simpler formula \(\binom{2n-3}{n-3}\) enumerates the number of permutations containing exactly one 132-pattern. Bóna’s result proved a conjecture first made by Noonan and Zeilberger in [NZ].

Noonan and Zeilberger considered in [NZ] the number of permutations of length \(n\) which contain exactly \(r\) \(p\)-patterns, for \(r \geq 1\). Bóna, in [B2], made further progress concerning the number of permutations with exactly \(r\) 132-patterns. In this article we work towards the following generalization: How many permutations of length \(n\) avoid patterns \(p_i\), for \(i \geq 0\), and contain \(r_j\) \(p_j\)-patterns, for \(j \geq 1, r_j \geq 1\)? We will first consider the permutations of length \(n\) which avoid 132-patterns, but contain exactly one 123-pattern. We then define a natural bijection between these permutations and the permutations of length \(n\) which avoid 123-patterns, but contain exactly one 132-pattern. Finally, we will calculate the number of permutations which contain one 123-pattern and one 132-pattern. These results address questions first raised in [NZ].

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2 Known Results

For completeness, two results which are already known are given below.

Lemma 1: The number of permutations of length \( n \) with one 12-pattern is \( n - 1 \).

Proof: Induct on \( n \). The base case is trivial. A permutation, \( \phi \), of length \( n \) with one 12-pattern must have \( n = \phi(1) \) or \( n = \phi(2) \). If \( n = \phi(1) \), by induction we get \( n - 2 \) permutation. If \( n = \phi(2) \), then we must have \( n - 1 = \phi(1) \) (or we would have more than one 12-pattern). The rest of the entries of \( \phi \) must be decreasing. Hence we get 1 more permutation from this second case, for a total of \( n - 1 \).

Lemma 2: The number of permutations which avoid both the pattern 123 and 132 is \( 2^{n-1} \).

Proof: Let \( f_n \) denote the number of permutations we are interested in. Then \( f_n = \sum_{i=1}^{n} f_{n-i} \) with \( f_0 = 1 \). To see this, let \( \rho \) be a permutation of length \( n - 1 \). Insert the element \( n \) into the \( i^{th} \) position of \( \rho \). Let \( \pi \) be this new permutation of length \( n \). To assure that \( \pi \) avoids the 132-pattern, we must have all entries preceding \( n \) in \( \pi \) be larger than the entries following \( n \). To assure that \( \pi \) avoids the 123-pattern, the entries preceding \( n \) must be in decreasing order. This argument gives the sum in the recursion. The recursion holds by noting that if \( n = 1 \), there is one permutation which avoids both patterns. To complete the proof note that \( f_n = 2^{n-1} \).

3 One 123-pattern, but no 132-pattern

Theorem 1: The number of permutations of length \( n \) which have exactly one 123-pattern, and avoid the 132-pattern is \( (n - 2)2^{n-3} \).

Proof: Call a permutation good if it has exactly one 123-pattern and avoids the 132-pattern, and let \( g_n \) denote the number of good permutations of length \( n \). Let \( \gamma \) be a permutation of length \( n - 1 \). Insert the element \( n \) into the \( i^{th} \) position of \( \gamma \). Call this newly constructed permutation of length \( n \), \( \pi \). To assure that \( \pi \) avoids the 132-pattern, we must have all elements preceding \( n \) in \( \pi \) be larger than the elements following \( n \) in \( \pi \). For \( \pi \) to be a good permutation, we must consider two disjoint cases.

Case I: The pattern 123 appears in the elements following \( n \) in \( \pi \). This forces the elements preceding \( n \) to be in decreasing order. Summing over \( i \), this case accounts for \( \sum_{i=1}^{n} g_{n-i} \) permutations.

Case II: The pattern 123 appears in the elements preceding and including \( n \) in \( \pi \). This forces the 3 in the pattern to be \( n \). Hence the elements preceding \( n \) must contain exactly one 12-pattern. (Further there must be at least 2 elements. Hence \( i \) must be at least 3). From Lemma 1, this number is \( i - 2 \). We are also forced to avoid both patterns in the elements following \( n \). Lemma 2 implies that there are \( 2^{n-i-1} \) such permutations. Summing over \( i \), this case accounts for \( \sum_{i=3}^{n-1} (i - 2)2^{n-i-1} + n - 2 \) permutations.

We have established that the recurrence relation

\[
g_n = \sum_{i=1}^{n} g_{n-i} + \sum_{i=3}^{n-1} (i - 2)2^{n-i-1} + n - 2,
\]

which holds for \( n \geq 3 \) (\( g_0 = 0, g_1 = 0, g_2 = 0 \)), enumerates the permutations of length \( n \) which avoid the pattern 132 and contain exactly one 123-pattern.

One easy way to proceed would be to find the generating function of \( g_n \). However, in this article we would like to employ a different, and in many circumstances more powerful, tool. We will use the Maple procedure \texttt{findrec} in Doron Zeilberger’s Maple package \texttt{EKHAD} (The Maple shareware

\[\ddagger\] Available for download at www.math.temple.edu/~zeilberg/
package gfun could have also been used.) Instructions for its use are available online. To use find-rec we compute the first few terms of \( g_n \). These are (for \( n \geq 4 \)) 4, 12, 32, 80, 192, 448, 1024. We type `findrec([4, 12, 32, 80, 192, 448, 1024], 0, 2, n, N)` and are given the recurrence \( h_n = 4(h_{n-1} - h_{n-2}) \) for \( n \geq 4 \). Define \( h_0 = 0, h_1 = 0, h_2 = 0, \) and \( h_3 = 1 \), and it is routine to verify that \( g_n = h_n \) for \( n \geq 0 \). Another routine calculation shows us that \( h_n = (n - 2)2^{n-3} \) for \( n \geq 3 \), thereby proving the statement of the theorem.

4 One 132-pattern, but no 123-pattern

**Theorem 2:** The number of permutations of length \( n \) which have exactly one 132-pattern, and avoid the 123-pattern is \( (n - 2)2^{n-3} \).

**Proof:** We prove this by exhibiting a (natural) bijection from the permutations counted in Theorem 1 to the permutations counted in this theorem. Define \( S := \{ \pi : \pi \text{ avoids 123-pattern and contains one 132-pattern} \} \) and \( T := \{ \pi : \pi \text{ avoids 123-pattern and contains one 123-pattern} \} \). We will show that \( |S| = |T| \), by using the following bijection:

Let \( \phi : S \longrightarrow T \). Let \( s \in S \), and let \( abc \) be the 123-pattern in \( s \). Then \( \phi \) acts on the elements of \( s \) as follows: \( \phi(x) = x \) if \( x \notin \{b, c\}, \phi(b) = c, \) and \( \phi(c) = b \). In other words, all elements keep their positions except \( b \) and \( c \) switch places. An easy examination of several cases shows that this is a bijection, thereby proving the theorem.

5 One 132-pattern and one 123-pattern

**Theorem 3:** The number of permutations of length \( n \) which have exactly one 132-pattern and one 123-pattern is \( (n - 3)(n - 4)2^{n-5} \).

**Proof:** We use the same insertion technique as in the proof of Theorem 1. Call a permutation *good* if it has exactly one 123-pattern and exactly one 132-pattern and let \( S_n \) denote the number of good permutations of length \( n \). Let \( \gamma \) be a permutation of length \( n - 1 \). Insert the element \( b \) into the \( i^{th} \) position of \( \gamma \). Let \( \pi \) be this newly constructed permutation of length \( n \). We note that the 132-pattern cannot consist of elements only preceding \( b \). If this were the case, we would have two 123-patterns ending with \( b \). For \( \pi \) to be a good permutation, we must consider the following disjoint cases.

**Case I:** The 132-pattern consists of elements following \( n \). In this case all elements preceding \( n \) must be larger than the elements following \( n \).

**Subcase A:** The 132-pattern consists of elements following \( n \). Summing over \( i \) we get \( \sum_{i=1}^{n} g_{n-i} \) good permutations in this subcase.

**Subcase B:** The elements preceding \( n \) have exactly one 12-pattern. This gives a 123-pattern where the 3 in the pattern is \( n \). We must also avoid the 123-pattern in the elements following \( n \). Summing over \( i \) and using Lemma 1 and Theorem 1, we get \( \sum_{i=3}^{n-3} (i - 2)(n - i - 3)2^{n-i-2} \) good permutations in this subcase.

**Case II:** The 132-pattern has the first element preceding \( n \), the last element following \( n \), and \( n \) as the middle element. The elements preceding \( n \) must be \( n - 1, n - 2, \ldots, n - 1 + 2, n - i \), where \( n - i \) immediately precedes \( n \) in \( \pi \). See [B1] for a more detailed argument as to why this must be true.

**Subcase A:** The elements preceding \( n \) have exactly one 12-pattern. This gives a 123-pattern where the last element of the pattern is \( n \). We must also avoid both the 123 and the 12 pattern in the elements following
n. Summing over $i$ and using Lemma 1 and Lemma 2 we have $\sum_{i=4}^{n-1} (i-3)2^{n-i-1}$ good permutations in this subcase.

**Subcase B:** The 123-pattern consists of elements following $n$. We must have the elements preceding $n$ in $\pi$ be decreasing to avoid another 123-pattern. Further, the elements following $n$ must not contain a 132-pattern. Using Theorem 1 and summing over $i$, we get a total of $\sum_{i=2}^{n-3} (n-i-2)2^{n-i-3}$ good permutations in this subcase.

In total, we find that the following recurrence enumerates the permutations of length $n$ which contain exactly one 123-pattern and one 132-pattern.

$$g_n = \sum_{i=1}^{n} g_{n-i} + \sum_{i=1}^{n-4} (2i(n-i-4) + n - 3)2^{n-i-4}$$

for $n \geq 5$ and $g_1 = g_2 = g_3 = g_4 = 0$.

Using findrec again by typing findrec([2,12,48,160,480,1344,3584],1,1,n,N) (where the list is the first few terms of our recurrence for $n \geq 5$) we get the recurrence $f_{n+1} = \frac{2(n+2)}{n}f_n$. with $f_1 = 2$. After reindexing, another routine calculation shows that $f_n = g_n$. Solving $f_n$ for an explicit answer, we find that $g_n = (n-3)(n-4)2^{n-5}$.

We conjecture that the generating function for the number of permutations with exactly zero or exactly one 132-pattern and exactly $r$ 123-patterns is $P(z)/(1-2z)^{r+1}$, where $P(z)$ is a polynomial. For more evidence, and further extensions see [RWZ].

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**References**


