Another bijection between 2-triangulations and pairs of non-crossing Dyck paths

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Abstract. A k-triangulation of the n-gon is a maximal set of diagonals of the n-gon containing no subset of $k+1$ mutually crossing diagonals. The number of k-triangulations of the n-gon, determined by Jakob Jonsson, is equal to a $k \times k$ Hankel determinant of Catalan numbers. This determinant is also equal to the number of k non-crossing Dyck paths of semi-length $n-2k$. This brings up the problem of finding a combinatorial bijection between these two sets. In FPSAC 2007, Elizalde presented such a bijection for the case $k=2$. We construct another bijection for this case that is stronger and simpler that Elizalde's. The bijection preserves two sets of parameters, degrees and generalized returns. As a corollary, we generalize Jonsson's formula for $k=2$ by counting the number of 2-triangulations of the n-gon with a given degree at a fixed vertex.

Résumé. Une k-triangulation du n-gon est un ensemble maximal de diagonales du n-gon ne contenant pas de sous-ensemble de $k+1$ diagonales mutuellement croisant. Le nombre de k-triangulations du n-gon, déterminé par Jakob Jonsson, est égal à un déterminant de Hankel $k \times k$ de nombres de Catalan. Ce déterminant est aussi égal au nombre de k chemins de Dyck de largo $n-2k$ que ne pas se croiser. Cela porte le problème de trouver une bijection de type combinatoire entre ces deux ensembles. À la FPSAC 2007, Elizalde a présenté une telle bijection pour le cas $k=2$. Nous construisons une autre bijection pour ce cas qui est plus forte et plus simple que de l'Elizalde. La bijection conserve deux ensembles de paramètres, les degré et les retours généralisée. De ce, nous généralisons la formule de Jonsson pour $k=2$ en comptant le nombre de 2-triangulations du n-gon avec un degré à un vertex fixe.

Keywords: k-triangulations, non-crossing Dyck paths, combinatorial bijection.

1 Introduction

The set of triangulations of n points in convex position on the plane has been studied for a long time because of its interesting combinatorial properties. In recent years, more general structures known as k-triangulations have been shown to satisfy many of the interesting properties of the classical triangulations.

A k-triangulation of the n-gon is a maximal set of diagonals of the n-gon containing no subset of $k+1$ mutually crossing diagonals. Note that the case $k=1$ corresponds to the standard triangulations of the n-gon.

This concept was introduced in 1992 by Capoyleas and Pach [3], who gave a tight bound for the number of diagonals in a k-triangulation. Later, Nakamigawa [13] and independently Dress, Kooolen and Moulton [5], showed that every k-triangulation attains that bound. Nakamigawa also showed that k-triangulations satisfy a flip property similar to the one of ordinary triangulations. This result has been recently strengthened and clarified with the discovery of the analogue of triangles for k-triangulations by Pilaud and Santos [16] and independently in [14].

In 2005, Jonsson [8] proved that the number of k-triangulations of the n-gon is equal to the following $k \times k$ Hankel determinant:

$$\det \begin{bmatrix} C_{n-2} & C_{n-3} & \iddots & \iddots & C_{n-k-1} \\ C_{n-3} & \iddots & \iddots & C_{n-k-1} & \iddots \\ \iddots & \iddots & \iddots & \iddots & \iddots \\ \iddots & C_{n-k-1} & \iddots & \iddots & C_{n-2k+1} \\ C_{n-k-1} & \iddots & \iddots & C_{n-2k+1} & C_{n-2k} \end{bmatrix}, \tag{1}$$

where C_n is the n-th Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$.

The set of all k non-crossing Dyck paths of semilength $n-2k$ is counted by the same determinant. This follows from an almost direct application of the Lindström-Gessel-Viennot theorem ([12], [7]) for counting non-intersecting lattice paths (see [10] for details on the history of this idea). A combinatorial bijection between the set of k-triangulations of the n-gon and the corresponding set of k non-crossing Dyck paths would constitute a simpler proof of the formula for the number of k-triangulations. Elizalde [6] constructed such a bijection for the case $k = 2$. This problem was posed by Jonsson [8] and it has been restated by Krattenthaler [11] and Elizalde [6]. A combinatorial bijection for the general case (and for more general objects) has been constructed by Rubey [17]. Also Krattenthaler [11] found a combinatorial proof using growth diagrams.

Our goal is to construct another bijection for the case $k = 2$. Our bijection is stronger than that in [6] because it transforms two simple parameters for 2-triangulations, the degrees (number of neighbors) at two consecutive vertices, into two simple parameters for pairs of non-crossing Dyck paths, the number of (generalized) returns.

We begin by showing a way to recursively generate the set of k-triangulations and the set of k non-crossing Dyck paths. Then we introduce a k-tuple of parameters for the set of non-crossing Dyck paths that generalizes the usual returns of a single Dyck path. We show, by means of a family of involutions, that the distribution of non-crossing Dyck paths with respect to these generalized returns is independent of their order. In particular, we obtain a generalization for non-crossing Dyck paths of the fact that (single) Dyck paths have the same distribution with respect to the height of the first peak and with respect to the number of returns. This result is a particular case of a theorem originally found by Brak and Essam [2]. Krattenthaler [10] found a combinatorial proof of this theorem using semistandard tableaux. A determinantal formula for the number of non-crossing paths with a given number of returns in the lowermost path follows by applying the Lindström-Gessel-Viennot theorem. Closed product formulas for this determinant are shown in [15]. There are also product formulas for the determinant (1), see [10].

The involutions that we introduce for non-crossing Dyck paths to prove the equidistribution of the generalized returns are also essential in the construction of our bijection between 2-triangulations and pairs of non-crossing Dyck paths. The bijection sends the 2-triangulations having degrees c_0 and c_1 at two fixed consecutive vertices onto the set of all pairs of non-crossing Dyck paths having (generalized) returns c_0 and c_1. Finally, we obtain a formula for the number of 2-triangulations having degree c at a fixed vertex, which refines Jonsson's formula (1) for $k = 2$. We conjecture that a similar formula holds for every k.

2 The set of k-triangulations

Let S be a set of n points in convex position. A straight line segment joining two points in S is a diagonal of S. Two diagonals of S cross if they intersect in a point not in S. A k-triangulation of S is a maximal set of diagonals of S containing no subset of $k+1$ mutually crossing diagonals.

Starting with any point of S and proceeding in the counterclockwise direction, we label the points of S with the numbers $0, 1, \dots, n-1$. In what follows, all operations on the labels of the vertices are performed modulo n, even if this is not explicitly stated. Also, we assume that $n \geq 2k+1$.

We identify the diagonals of S with the set of pairs of elements in S. In this way, the set of diagonals of S, denoted Σ_n is simply $\Sigma_n = \big\{\{i,j\} : \ 0 \leq i < j \leq n-1\big\}$.

It is clear that every diagonal in a set of $k+1$ mutually crossing diagonals must have at least k vertices of S on each side. Therefore, every k-triangulation contains the kn trivial diagonals of the form $\{i, i+h\}$ with $1 \leq h \leq k$. Of these diagonals, we want to keep only those of the form $\{i, i+k\}$.

Let $\Sigma_n^k = \Sigma_n - \big\{\{a, a+i\} : \ a \in \{0, \dots, n-1\}, \quad i \in \{1, \dots, k-1\}\big\}$, and $\Gamma_n^k = \big\{\{a, a+k\} : a \in \{0, \dots, n-1\}\big\}$.

The decomposition theorems in [16] and [14] make clear that Γ_n^k is the natural boundary of a k-triangulation. In what follows, we assume that k-triangulations contain only these trivial diagonals. In other words, we redefine the k-triangulations of S as maximal subsets of Σ_n^k containing no subset of $k+1$ mutually crossing diagonals. We denote the set of all k-triangulations of S by $\mathcal{T}_n^k$.

Given a set of edges $U \subseteq \Sigma_n^k$, the set of neighbors in U of a vertex i is defined by $\mathsf{N}_i(U) = \{j : \ \{i,j\} \in U\}$. Also, for $0 \leq i \leq k-1$, we define the neighbors to the left and right of a given vertex a by $\mathsf{L}_i(a, U) = \{j \in \mathsf{N}_i(U) : \ i < j \leq a\}$ and $\mathsf{R}_i(a, U) = \{j \in \mathsf{N}_i(U) : \ a \leq j < i\}$.

Now we associate to every k-triangulation of S a partial order, by considering the sets of k mutually crossing diagonals incident to the vertices $0, \dots, k-1$.

For $\Gamma_n^k \subseteq U \subseteq \Sigma_n^k$, define

$$\mathcal{C}(U) = \Big\{(a_0, \dots, a_{k-1}) : \ \{0, a_0\}, \cdots, \{k-1, a_{k-1}\} \text{ are mutually crossing diagonals of } U \Big\}.$$

Note that $k \leq a_0 < \cdots < a_{k-1} \leq n-1$ for all $(a_0, \dots, a_{k-1})$ in $\mathcal{C}(U)$.

The set $\mathcal{C}(U)$ is partially ordered by the (direct) product order $(a_0, \dots, a_{k-1}) \leq (b_0, \dots, b_{k-1})$ if and only if $a_i \leq b_i$ for all $i \in \{0, \dots, k-1\}$. In fact, $\mathcal{C}(U)$ is a lattice with this order.

Now suppose we add a point, with label n, to the set S in such a way that the set $S' = S \cup \{n\}$ is in convex position and the point n is located between the points $n-1$ and 0 of S. The k-triangulations of S' can be obtained from those of S by applying a procedure that splits mutually crossing diagonals (incident to the vertices $0, \dots, k-1$) in a k-triangulations of S. Let Σ_{n+1}^k and $\mathcal{T}_{n+1}^k$ be be the set of diagonals and the set of k-triangulations of S', respectively.

Let $\Gamma_n^k \subseteq W \subseteq \Sigma_n^k$. For $(a_0, \dots, a_{k-1}) \in \mathcal{C}(W)$ define $\Psi_{\vec{a}}(W) \subseteq \Sigma_{n+1}^k$ by

$$\begin{aligned}\Psi_{\vec{a}}(W) = \quad & \big\{\{i,j\} \in W : \ k \leq i < j \leq n-1\big\} \quad \cup \\ & \big\{\{n,j\} : \ j \in \mathsf{R}_0(a_0, W)\big\} \quad \cup \\ & \big\{\{i,j\} : \ 0 \leq i \leq k-2, \quad j \in \mathsf{L}_i(a_i, W) \cup \mathsf{R}_{i+1}(a_{i+1}, W)\big\} \quad \cup \\ & \big\{\{k-1,j\} : \ j \in \mathsf{L}_{k-1}(a_{k-1}, W)\big\} \quad \cup \quad \big\{\{n, k-1\}\big\}.\end{aligned}$$

Let $\mathbf{T}_n : \bigcup_{T \in \mathcal{T}_n^k} \{T\} \times \mathcal{C}(T) \longrightarrow \Sigma_{n+1}^k$ be the function defined by $\mathbf{T}_n(T, \vec{a}) = \Psi_{\vec{a}}(T)$.

The importance of the function $\mathbf{T}_n$ is that it generates all the k-triangulations of S' in an injective way.

Theorem 1 *The function $\mathbf{T}_n$ establishes a bijection between $\bigcup_{T\in\mathcal{T}_n^k}\{T\}\times\mathcal{C}(T)$ and $\mathcal{T}_{n+1}^k$.*

Though with a different presentation, this fact was proved by Nakamigawa [13].

3 The set of non-crossing Dyck paths

For our purposes, it is convenient to define Dyck paths as integer functions as follows:

Given $m \geq 1$, a function $f : \{1,\ldots,m\} \longrightarrow \mathbb{N}$ is called a *Dyck path* of length m if it satisfies the following properties: (1) f is non-decreasing, (2) $i \leq f(i)$, for all $i \in \{1,\ldots,m\}$, and (3) $f(m) = m$.

For $m_1 \leq m_2$, let $\mathcal{F}_{m_1,m_2} = \big\{f : \{m_1,\ldots,m_2\} \longrightarrow \mathbb{N} : \quad f \text{ is non-decreasing}\big\}$. We write $f \leq g$ if $f(i) \leq g(i)$ for all $i \in \{m_1,\ldots,m_2\}$.

For any two non-decreasing functions f and g, a *(generalized) return* of g to f is a value i in the domain of f and g for which $f(i) = g(i)$. The set of all returns of g to f is denoted by $\mathrm{ret}(g,f)$.

Define $\mathcal{D}_m^k = \big\{(f_0,\ldots,f_{k-1}) : \quad f_i \text{ is a Dyck path of length } m \text{ for all } i \text{ and} f_{i-1} \geq f_i \text{ for all } i\big\}$.

Note that the upper-most path is listed first in the k-tuple of paths. This is necessary to make the proof of Theorem 7 simpler.

For $i \in \{0,\ldots,k-1\}$ and $D = (f_0,\ldots,f_{k-1})$ define $\mathrm{ret}_i(D) = \mathrm{ret}(f_i, f_{i+1}) \cup \{0\}$, where f_k is the identity function, $f_k = \mathrm{id}$.

The crossings among the returns of $D \in \mathcal{D}_m^k$ are defined by

$$\begin{aligned}\mathcal{C}(D) \quad = \quad \big\{(a_0,\ldots,a_{k-1}) : \quad & a_i \in \mathrm{ret}_i(D) \text{ for } i \in \{0,\ldots,k-1\},\\ & a_{i-1} \leq a_i \text{ for } i \in \{1,\ldots,k-1\}\big\}.\end{aligned}$$

Clearly, $\mathcal{C}(D) \subseteq \mathrm{ret}_0(D) \times \cdots \times \mathrm{ret}_{k-1}(D)$, and $\mathcal{C}(D)$ is a lattice when regarded as a poset with the inherited product order. Note that for a k-triangulation T, $\mathcal{C}(T)$ consists of strictly increasing functions, while $\mathcal{C}(D)$ consists of non-decreasing functions.

Continuing the analogy with our treatment of k-triangulations, we introduce now the set of returns to the left and right of a given value: $\bar{\mathsf{L}}_i(a,D) = \big\{j \in \mathrm{ret}_i(D) : \quad j \leq a\big\}$, and $\bar{\mathsf{R}}_i(a,D) = \big\{j \in \mathrm{ret}_i(D) : a \leq j\big\}$.

For $D = (D_1,\ldots,D_{k-1}) \in \mathcal{D}_m^k$ and $\vec{a} = (a_0,\ldots,a_{k-1}) \in \mathcal{C}(D)$, define $\bar{\Psi}_{\vec{a}}(D) = (\hat{D}_1,\ldots,\hat{D}_{k-1})$ where $\hat{D}_i : \{0,\ldots,m+1\} \to \mathbb{N}$ is given by

$$\hat{D}_i(j) = \begin{cases} D_i(j) & \text{for} \quad j \in \{0,\ldots,a_i\},\\ D_i(j)+1 & \text{for} \quad j \in \{a_i+1,\ldots,m\},\\ m+1 & \text{if} \quad j = m+1.\end{cases}$$

Note that $\bar{\Psi}_{\vec{a}}(D) \in \mathcal{D}_{m+1}^k$, thanks to the condition $a_0 \leq \cdots \leq a_{k-1}$.

Finally, completing the analogy with k-triangulations, the functions $\bar{\Psi}_{\vec{a}}$ can be used to generate recursively the sets $\mathcal{D}_m^k$.

Let $\mathbf{D}_m : \bigcup_{D\in\mathcal{D}_m^k}\big\{D\big\}\times\mathcal{C}(D) \longrightarrow \mathcal{D}_{m+1}^k$ be defined by $\mathbf{D}_m(D,\vec{a}) = \bar{\Psi}_{\vec{a}}(D)$.

Theorem 2 *For every $m \geq 1$, $\mathbf{D}_m$ is a bijection.*

Now we study a family of involutions on the set $\mathcal{F}_{m_1,m_2}$ of non-decreasing integer functions on $\{m_1,\ldots,m_2\}$. For $f,h \in \mathcal{F}_{m_1,m_2}$, with $f \leq h$, the interval between f and g is given by $\mathcal{I}(f,h) = \big\{g \in \mathcal{F}_{m_1,m_2} : \quad f \leq g \leq h\big\}$.

Theorem 3 *For all $m_1 \le m_2$, and for all $f, h \in \mathcal{F}_{m_1,m_2}$ with $f \le h$, there exists an involution $\mu_{f,h}$ on $\mathcal{I}(f,h)$ such that*

$$\big|\text{ret}(g,\ f)\big| = \big|\text{ret}\big(h,\ \mu_{f,h}(g)\big)\big| \quad \textit{and} \quad \big|\text{ret}(\mu_{f,h}(g),\ f)\big| = \big|\text{ret}\big(h,g\big)\big|.$$

This theorem can be proved by applying induction on two parameters, the length of $m_2 - m_1 + 1$ of the domain of the functions, and the minimum distance $\min\big\{h(i) - f(i),\ m_1 \le i \le m_2\big\}$ between the functions, see [14].

The previous theorem implies that the number of chains $f_0 \ge \cdots \ge f_k$ of length k in $\mathcal{F}_{m_1,m_2}$ having a prescribed number of returns $\big|\text{ret}(f_i, f_{i-1})\big| = c_i$, for $i \in \{1,\ldots,k\}$, does not depend on the order of the numbers c_i. We state this fact for chains with fixed endpoints. Define, for $m_1 \le m_2$, $k \ge 1$, and $f_k \le f_{-1} \in \mathcal{F}_{m_1,m_2}$,

$$\begin{array}{rclrll} \mathcal{H}^k_{f_{-1},f_k}(c_0,\ldots,c_k) & = & \big\{\ (f_0,\ldots,f_{k-1}): & f_i \in \mathcal{I}(f_k, f_{-1}) & \text{for } i \in \{0,\ldots,k-1\}, \\ & & & f_{i-1} \ge f_i & \text{for } i \in \{0,\ldots,k\}, \\ & & & \big|\text{ret}(f_{i-1}, f_i)\big| = c_i & \text{for } i \in \{0,\ldots,k\}\big\}. \end{array}$$

Corollary 4 *Let $m_1 \ge m_2$, $k \ge 1$ and $f_{-1}, f_k \in \mathcal{F}_{m_1,m_2}$. Then for every permutation σ on $\{0,\ldots,k\}$,*

$$\big|\mathcal{H}^k_{f_{-1},f_k}(c_0,\ldots,c_k)\big| = \big|\mathcal{H}^k_{f_{-1},f_k}(c_{\sigma(0)},\ldots,c_{\sigma(k)})\big|.$$

Returning to the study of Dyck paths, let $\text{id}(i) = i$ and $\text{cons}_m(i) = m$ for $m \ge 1$. Note that the set of all (single) Dyck paths of length m is equal to the interval $\mathcal{I}(\text{id}, \text{cons}_m)$ of $\mathcal{F}_{0,m}$.

It is well-known that Dyck paths have the same distribution with respect to the number of returns and with respect to the height of the first (or last) peak. This can be shown by means of an involution, see [4]. The following generalization of this property to sets of non-crossing Dyck paths is a simple consequence of Corollary 4. It tell us that the number of k non-crossing Dyck paths of length m for which the height of the first (or last) peak in the the upper-most path is c is equal to the number of paths having c returns in the lower-most path.

Theorem 5 *For all $m \ge 1$, $k \ge 1$ and $c \in \{0,\ldots,m\}$,*

$$\begin{aligned} &\Big|\Big\{(f_0,\ldots,f_{k-1}) \in \mathcal{D}^k_m : f_0(1) = c\ \Big\}\Big| \\ &= \Big|\Big\{(f_0,\ldots,f_{k-1}) \in \mathcal{D}^k_m : \big|\{f_0(i) = m\}\big| = c\ \Big\}\Big| \\ &= \Big|\Big\{(f_0,\ldots,f_{k-1}) \in \mathcal{D}^k_m : \big|\{f_{k-1}(i) = i\}\big| = c\ \Big\}\Big|. \end{aligned}$$

Proof. The first equality is obvious. The second follows from Corollary 4:

$$\begin{aligned} &\Big|\Big\{(f_0,\ldots,f_{k-1}) \in \mathcal{D}^k_m :\ \big|\{f_0(i) = m\}\big| = c\ \Big\}\Big| \\ &\qquad = \sum_{r_0,\ldots,r_{k-1}} \big|\mathcal{H}^k_{\text{id},\text{cons}_m}(c, r_0,\ldots,r_{k-1})\big| = \sum_{r_0,\ldots,r_{k-1}} \big|\mathcal{H}^k_{\text{id},\text{cons}_m}(r_0,\ldots,r_{k-1},c)\big| \\ &\qquad = \Big|\Big\{(f_0,\ldots,f_{k-1}) \in \mathcal{D}^k_m :\ \big|\{f_{k-1}(i) = i\}\big| = c\ \Big\}\Big|. \end{aligned}$$

□

The previous theorem is a particular case of a result by Brak and Essam [2]. A combinatorial proof using semistandard tableaux is given by Krattenthaler [10].

It is not difficult to obtain determinantal formulas for the cardinality of the sets in our previous theorem. The ballot numbers (sometimes called generalized Catalan numbers) $B(n,m) = \frac{m-n+1}{m+1}\binom{m+n}{m}$ count the number of paths on the integer lattice having north and east steps, starting at $(0,0)$, ending at (n,m) and not going under the line $x = y$; see [9, 18]. Hence, ballot numbers count Dyck paths having a given height of the last peak. Combining this idea with the Lindström-Gessel-Viennot Theorem, it is easy to obtain a formula for the number of k non-crossing Dyck paths such that the height of the last peak for the top path is a given value c. By the previous theorem, this formula also counts the paths such that the lowest path has c returns. We state this formula in the following theorem.

Theorem 6 *The number of k non-crossing Dyck paths of length m such that the lowest path has exactly c returns is given by*

$$\det \begin{bmatrix} C_m & C_{m+1} & \iddots & \iddots & B^k_{m+k-1}(c) \\ C_{m+1} & \iddots & \iddots & C_{m+k-1} & \vdots \\ \iddots & \iddots & \iddots & \iddots & \vdots \\ \iddots & C_{m+k-1} & \iddots & \iddots & B^k_{m+2k-3}(c) \\ C_{m+k-1} & \iddots & \iddots & C_{m+2k-3} & B^k_{m+2k-2}(c) \end{bmatrix},$$

where $B^k_m(h) = \frac{2k+h-2}{m}\binom{2m-2k-h+1}{m-1}$.

Product formulas for the determinant in the previous theorem are found in [15] and [10].

4 A strong bijection between $\mathcal{T}^2_n$ and $\mathcal{D}^2_n$

In spite of the similarities that we have found between k-triangulations and non-crossing Dyck paths, we are able to construct a bijection between these sets only[(i)] for the case $k = 2$. As pointed out in the introduction, a combinatorial bijection between $\mathcal{T}^2_n$ and $\mathcal{D}^2_n$ has already been found by Elizalde [6]. Our bijection presents the advantage of sending two simple parameters in $\mathcal{T}^2_n$, the degree (number of neighbors) at two consecutive vertices, into two simple parameter in $\mathcal{D}^2_n$, the number of returns of each path. From this property, we derive as a corollary a formula for the number of 2-triangulations with a given degree at a fixed vertex.

Theorem 7 *For all $n \geq 5$, there exists a bijection ω_n from the set $\mathcal{T}^2_n$ of all 2-triangulations of the n-gon onto the set $\mathcal{D}^2_{n-4}$ of all pairs of non-crossing Dyck paths of length $n-4$, such that for all $T \in \mathcal{T}^2_n$*

(1) $$\begin{aligned} \big|\mathsf{N}_0(T)\big| &= \big|\mathrm{ret}_0\big(\omega_n(T)\big)\big| \quad \textit{and} \\ \big|\mathsf{N}_1(T)\big| &= \big|\mathrm{ret}_1\big(\omega_n(T)\big)\big|. \end{aligned}$$

(2) Moreover, if $\beta^i_T :\ \mathsf{N}_i(T) \longrightarrow \mathsf{ret}_i\big(\omega_n(T)\big)$, for $i = 0, 1$, is the order-preserving bijection guaranteed to exist by (1), then $\beta_T :\ \mathcal{C}(T) \longrightarrow \mathcal{C}\big(\omega_n(T)\big)$ given by $\beta_T\big((i,j)\big) = \big(\beta^0_T(i), \beta^1_T(j)\big)$, is an isomorphism between the crossings of T and the crossings among the returns of $\omega_n(T)$.

(i) The case $k = 1$ is elementary.

Proof. By induction on n:

If $n = 5$, then $\mathcal{T}_5^2$ and $\mathcal{D}_1^2$ contain one element each. Let T be the 2-triangulation in $\mathcal{T}_5^2$ and let D be the element of $\mathcal{D}_1^2$. Then, $\mathsf{N}_0(T) = \{2,3\}, \mathsf{N}_1(T) = \{3,4\}, \mathcal{C}(T) = \{(2,3),\ (2,4),\ (3,4)\}, \mathrm{ret}_0(D) = \{0,1\}, \mathrm{ret}_1(D) = \{0,1\}$ and $\mathcal{C}(D) = \{(0,0),(0,1),(1,1)\}$, so the map $\{T\} \to \{D\}$ satisfies properties (1) and (2).

Let $n \geq 5$, and suppose $\omega_n \colon \mathcal{T}_n^2 \longrightarrow \mathcal{D}_{n-4}^2$ is a bijection such that for all $T' \in \mathcal{T}_n^2$

I
$$\begin{aligned} \big|\mathsf{N}_0(T')\big| &= \big|\mathrm{ret}_0\big(\omega_n(T')\big)\big| \\ \big|\mathsf{N}_1(T')\big| &= \big|\mathrm{ret}_1\big(\omega_n(T')\big)\big| \end{aligned}$$

II $\beta_{T'} \colon \mathcal{C}(T') \longrightarrow \mathcal{C}\big(\omega_n(T')\big)$, given by $\beta_{T'}\big(\,(i,j)\,\big) = (\beta^0_{T'}(i), \beta^1_{T'}(j))$, is an isomorphism between the sets of crossings of T' and $\omega_n(T')$, where $\beta^i_{T'} \colon \mathsf{N}_i(T') \longrightarrow \mathrm{ret}_i\big(\omega_n(T')\big)$, for $i = 0,1$, is the order-preserving bijection between these sets.

Define $\tilde{\omega}_{n+1} \colon \mathcal{T}_{n+1}^2 \longrightarrow \mathcal{D}_{n-3}^2$ by

$$\tilde{\omega}_{n+1}(T) = \bar{\Psi}_{\beta_{T'}(\vec{a})}\big(\omega_n(T')\big),$$

where $(T', \vec{a})$ is the unique element of $\bigcup_{T' \in \mathcal{T}_n^2} \{T'\} \times \mathcal{C}(T')$ such that $\Psi_{\vec{a}}(T') = T$.

By definition of $\tilde{\omega}_{n+1}$, the following diagram commutes:

$$\begin{array}{ccc} \big(T',\, \vec{a}\big) & \xrightarrow{\ (\omega_n,\, \beta_{T'})\ } & \big(\omega_n(T'),\, \beta_{T'}(\vec{a})\big) \\ \Big\downarrow \Psi_{\vec{a}} & & \Big\downarrow \bar{\Psi}_{\beta_{T'}(\vec{a})} \\ T & \xrightarrow{\ \tilde{\omega}_{n+1}\ } & \tilde{\omega}_{n+1}(T) \end{array}$$

Therefore, since the maps $\Psi_{\vec{a}}$, $(\omega_n,\, \beta_{T'})$ and $\bar{\Psi}_{\beta_{T'}(\vec{a})}$ are bijective, it follows that $\tilde{\omega}_{n+1}$ is bijective.

From the definitions of $\Psi_{\vec{a}}$ and $\bar{\Psi}_{\beta_{T'}(\vec{a})}$it is follows that

$$\begin{aligned} \mathsf{N}_0(T) &= \mathsf{L}_0(a_0,\, T') \cup \mathsf{R}_1(a_1,\, T'), \\ \mathsf{N}_1(T) &= \mathsf{L}_1(a_1,\, T') \cup \{n\}. \end{aligned} \tag{2}$$

On the other hand, taking $D' = \omega_n(T')$, $(\bar{a}_0, \bar{a}_1) = \beta_{T'}\big((a_0, a_1)\big)$,

$$\begin{aligned} \mathrm{ret}_0\big(\tilde{\omega}_{n+1}(T)\big) &= \bar{\mathsf{L}}_0(\bar{a}_0,\, D') \cup \bar{\mathsf{R}}_0(\bar{a}_1 + 1,\, D') \cup \{n-3\}, \\ \mathrm{ret}_1\big(\tilde{\omega}_{n+1}(T)\big) &= \bar{\mathsf{L}}_1(\bar{a}_1,\, D') \cup \{n-3\}. \end{aligned} \tag{3}$$

Comparison of (2) and (3) shows that $\tilde{\omega}_{n+1}$ does not have the desired properties (1) and (2). We need to apply the involutions μ_{m_1,m_2}.

For $D \in \mathcal{D}_{n-3}^2$, with $D = (D_0, D_1)$, let $m(D) = \max(\mathrm{ret}_1(D) - \{n-3\})$. Define $\Upsilon(D) \in \mathcal{D}_{n-3}^2$ by $\Upsilon(D) = (D_0, \hat{D}_1)$, where, for $i \in \{0, \dots, n-3\}$,

$$\hat{D}_1(i) = \begin{cases} D_1(i) & \text{if} \quad i \in \{0, \dots, m(D)\}, \\ \mu_{\mathrm{id}+1, D_0'}\big(D_1'\big)(i) & \text{if} \quad i \in \{m(D)+1, \dots, n-4\}, \\ n-3 & \text{if} \quad i = n-3, \end{cases}$$

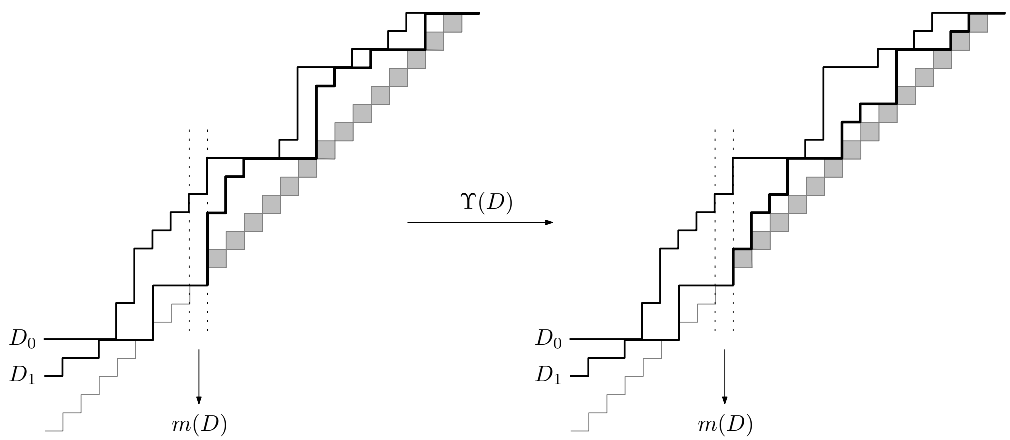

Fig. 1: An element of $\mathcal{D}^2_{n-3}$ and its image under Υ. The portion of the lower path between its last two returns is replaced with a path having a reversed number of intersections with the upper path and with the identity shifted up one unit.

and D'_0, D'_1 are the restrictions of D_0 and D_1 to the interval $\{m(D)+1,\ldots,n-4\}$.

Note that $\Upsilon\big((D_0,D_1)\big)$ is equal to (D_0,D_1) except for the portion of D_1 between its last two returns. This is replaced by a path with reversed number of intersections with D_0 and id $+\,1$; see Figure 1.

Clearly, $\mathrm{ret}_1\big(\Upsilon(D)\big) = \mathrm{ret}_1(D)$, so $m\big(\Upsilon(D)\big) = m(D)$. Therefore, $\Upsilon\big(\Upsilon(D)\big) = D$, because the functions $\mu_{f,h}$ are involutions. Hence, Υ is a bijection from $\mathcal{D}^2_{n-3}$ onto $\mathcal{D}^2_{n-3}$.

By definition of Υ, the returns of $\Upsilon(D)$, where $D=(D_0,D_1)$, are

$$\begin{aligned} \mathrm{ret}_0\big(\Upsilon(D)\big) &= \bar{\mathsf{L}}_0\big(m(D),\,D\big)\cup S\cup\{n-3\},\\ \mathrm{ret}_1\big(\Upsilon(D)\big) &= \mathrm{ret}_1(D), \end{aligned} \tag{4}$$

where S satisfies

$$\begin{aligned} S &\subseteq \{m(D)+1,\ldots,n-4\},\\ \big|S\big| &= \big|\mathrm{ret}(\mathrm{id}+1,\,D_1)\cap\{m(D)+1,\ldots,n-4\}\big|. \end{aligned} \tag{5}$$

Taking $D=\tilde{\omega}_{n+1}(T)=\bar{\Psi}_{\beta_{T'}(\vec{a})}(D')$, it is clear that $m(D)=\bar{a}_1$. Also, $\bar{\mathsf{L}}_0(\bar{a}_0,\,D)=\bar{\mathsf{L}}_0(\bar{a}_1,\,D)$, which combined with (3), (4) and (5) gives us the returns of $\Upsilon\big(\tilde{\omega}_{n+1}(t)\big)$:

$$\begin{aligned} \mathrm{ret}_0\big(\Upsilon(D)\big) &= \bar{\mathsf{L}}_0\big(\bar{a}_0,\,D'\big)\cup S\cup\{n-3\},\\ \mathrm{ret}_1\big(\Upsilon(D)\big) &= \bar{\mathsf{L}}_1\big(\bar{a}_1,\,D'\big)\cup\{n-3\}, \end{aligned}$$

where S satisfies

$$\begin{aligned} S &\subseteq \{\bar{a}_1+1,\ldots,n-4\},\\ \big|S\big| &= \big|\mathrm{ret}(\mathrm{id}+1,\,D_1)\cap\{\bar{a}_1+1,\ldots,n-4\}\big|\\ &= \bar{\mathsf{R}}_1\big(\bar{a}_1+1,\,D'\big). \end{aligned}$$

To summarize, if T, T', D and D' satisfy the following diagram

$$\begin{array}{ccccc} \big(T',\,(a_0,a_1)\big) & \xrightarrow{(\omega_n,\,\beta_{T'})} & \big(D',\,(\bar a_0,\bar a_1)\big) & & \\ \Big\downarrow \Psi_{\vec a} & & \Big\downarrow \bar\Psi_{\beta_{T'}(\vec a)} & & \\ T & \xrightarrow{\tilde\omega_{n+1}} & D & \xrightarrow{\Upsilon} & \Upsilon(D) \end{array}$$

then the sets of neighbors of T satisfy

$$\begin{array}{rcl} \mathsf{N}_0(T) & = & \mathsf{L}_0(a_0,\,T') \cup \mathsf{R}_1(a_1,\,T'), \\ \mathsf{N}_1(T) & = & \mathsf{L}_1(a_1,\,T') \cup \{n\}. \end{array} \tag{6}$$

while the returns of $\Upsilon(D)$ are

$$\begin{array}{rcl} \mathrm{ret}_0\big(\Upsilon(D)\big) & = & \bar{\mathsf{L}}_0\big(\bar a_0,\,D'\big) \cup S \cup \{n-3\}, \\ \mathrm{ret}_1\big(\Upsilon(D)\big) & = & \bar{\mathsf{L}}_1\big(\bar a_1,\,D'\big) \cup \{n-3\}, \end{array} \tag{7}$$

where S satisfies

$$\begin{array}{rcl} S & \subseteq & \{\bar a_1+1,\ldots,n-4\}, \\ \big|S\big| & = & \bar{\mathsf{R}}_1\big(\bar a_1+1,\,D'\big). \end{array} \tag{8}$$

Therefore, the function $\omega_{n+1} = \Upsilon \circ \tilde\omega_{n+1}$ is a bijection from $\mathcal{T}^2_{n+1}$ onto $\mathcal{D}^2_{n-3}$, satisfying (1) and (2):

Note that $\big|\mathsf{L}_0(a_0,\,T')\big| = \big|\bar{\mathsf{L}}_0(\bar a_0,\,D')\big|$ because, according to inductive hypothesis I, $\big|\mathsf{N}_0(T')\big| = \big|\mathrm{ret}_0(D')\big|$ and the map $\beta^0_{T'}$, which sends a_0 to $\bar a_0$, is order-preserving. Similarly, $\big|\mathsf{R}_1(a_1,\,T')\big| = \big|\bar{\mathsf{R}}_1(\bar a_1,\,D')\big|$ because $\big|\mathsf{N}_1(T')\big| = \big|\mathrm{ret}_1(D')\big|$ and $\beta^1_{T'}$ is order-preserving.

But $\big|\bar{\mathsf{R}}_1(a_1+1,\,D')\big| = \big|\bar{\mathsf{R}}_1(\bar a_1,\,D')\big| - 1$, because $\bar a_1 \in \mathrm{ret}_1(D')$ since $(\bar a_0,\bar a_1)$ is an element of $\mathcal{C}(D')$. Hence, by (6), (7) and (8), $\big|\mathsf{N}_0(T)\big| = \big|\mathrm{ret}_0\big(\Upsilon(D)\big)\big|$. For similar reasons, $\big|\mathsf{L}_1(a_1,\,T')\big| = \big|\bar{\mathsf{L}}_1(\bar a_1,\,D')\big|$ and therefore we also have $\big|\mathsf{N}_1(T)\big| = \big|\mathrm{ret}_1\big(\Upsilon(D)\big)\big|$. Hence, ω_{n+1} satisfies (1).

Finally, let β^0_T and β^1_T be the order-preserving bijections from $\mathsf{N}_0(T)$ onto $\mathrm{ret}_0\big(\Upsilon(D)\big)$ and from $\mathsf{N}_1(T)$ onto $\mathrm{ret}_1\big(\Upsilon(D)\big)$, respectively. Let $\beta_T = (\beta^0_T, \beta^1_T)$. The fact that $(i,j) < (i',j')$ if and only if $\beta_T(i,j) < \beta_T(i',j')$ follows immediately because β^0_T and β^1_T are order-preserving.

Hence, we only need to show that β_T is a bijection between $\mathcal{C}(T)$ and $\mathcal{C}(\Upsilon(D))$ to obtain that it is an isomorphism between these sets.

But from (6),(7) and (8), we have the following partitions of $\mathcal{C}(T)$ and $\mathcal{C}(\Upsilon(D))$ into two disjoint sets:

$$\mathcal{C}(T) = \underbrace{\big\{(i,j) \le (a_0,a_1) :\ (i,j) \in \mathcal{C}(T')\big\}}_{\text{(i)}} \quad \cup \quad \underbrace{\big\{(i,n) :\ i \in \mathsf{N}_0(T)\big\}}_{\text{(ii)}}$$

and

$$\mathcal{C}\big(\Upsilon(D)\big) = \underbrace{\big\{(i,j) \le (\bar a_0,\bar a_1) :\ (i,j) \in \mathcal{C}(D')\big\}}_{\text{(iii)}} \quad \cup \quad \underbrace{\big\{(i,n-3) :\ i \in \mathrm{ret}_0\big(\Upsilon(D)\big)\big\}}_{\text{(iv)}}.$$

The sets (i) and (iii) are the lower ideals of (a_0,a_1) in $\mathcal{C}(T')$ and of $(\bar a_0,\bar a_1)$ in $\mathcal{C}(D')$, respectively. By inductive hypothesis II, they are isomorphic under $\beta_{T'}$. It is clear from (6) and (7) that β_T and $\beta_{T'}$ are equal when restricted to (i), so in particular β_T is a bijection between (i) and (iii). We have already seen that $\big|\mathsf{N}_0(T)\big| = \big|\mathrm{ret}_0\big(\Upsilon(D)\big)\big|$, so β_T is also a bijection between (ii) and (iv).

Therefore ω_{n+1} satisfies (2), which completes the proof. $\square$

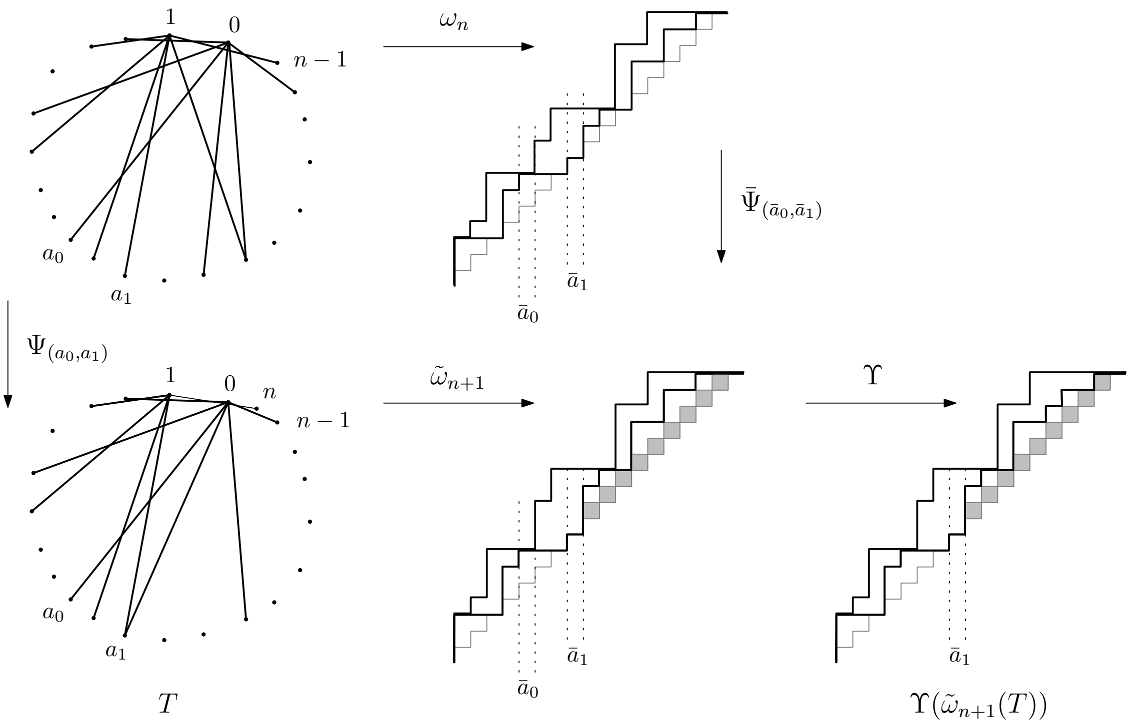

Fig. 2: An example where it can be verified that the poset $\mathcal{C}(T)$ is isomorphic to the poset $\mathcal{C}\big(\Upsilon(\tilde{\omega}_{n+1}(T))\big)$.

Using the property $\big|\mathsf{N}_1(T)\big| = \big|\mathrm{ret}_1\big(\omega_n(T)\big)\big|$ of the bijection ω_n obtained in the previous theorem, and the formula for $|\mathrm{ret}(\mathrm{id}, D_1)|$ from Theorem 6, we obtain a refinement, for the case $k = 2$, of the formula (1) for the number of 2-triangulations. The following formula gives the number of 2-triangulations according to the degree of a fixed vertex. Note that $|\mathrm{ret}(\mathrm{id}, D_1)| = \big|\mathrm{ret}_1\big(\omega_n(T)\big)\big| - 1$ because $0 \in \big|\mathrm{ret}_1\big(\omega_n(T)\big)\big|$. Also, by applying a rotation, the formula can be used to find the number of 2-triangulations with a given degree at any fixed vertex.

Corollary 8 *The number of* 2*-triangulations* T *of the* n*-gon with* $\big|\mathsf{N}_0(T)\big| = c$ *is given by*

$$\det\begin{vmatrix} C_{n-4} & B^2_{n-3}(c-1) \\ C_{n-3} & B^2_{n-2}(c-1) \end{vmatrix},$$

where $B^k_m(h) = \frac{2k+h-2}{m}\binom{2m-2k-h+1}{m-1}$.

The previous theorem cannot be generalized for $k > 2$. No such strong bijection can exist if $k > 2$, because an isomorphism between the crossings $\mathcal{C}(T)$ and $\mathcal{C}\big(\omega_n(T)\big)$ implies in particular that $\big|\mathcal{C}(T)\big| = \big|\mathcal{C}\big(\omega_n(T)\big)\big|$. But the distribution of k-triangulations and non-crossing Dyck paths is different for these parameters when $k > 2$. For example, for $k = 3$ and $n = 10$, $\mathcal{T}^3_{10}$ contains 24 triangulations T of the 10-gon such that $\big|\mathcal{C}(T)\big| = 9$, but there are 32 triples D of non-crossing Dyck paths of length 4 such that $\big|\mathcal{C}(D)\big| = 9$.

Surprisingly, computer experiments suggest that Theorem 7 can be generalized for $k > 2$ if we drop condition (2).

Conjecture 1 *For all $k \geq 1$ and $n \geq 2k+1$, there exists a bijection ω_n^k from the set $\mathcal{T}_n^k$ of all k-triangulations of the n-gon onto the set $\mathcal{D}_{n-2k}^k$ of all k non-crossing Dyck paths of length $n-2k$, such that for all $T \in \mathcal{T}_n^k$*

$$\left|\mathsf{N}_i(T)\right| = \left|\mathrm{ret}_i\left(\omega_n^k(T)\right)\right| \quad \textit{for all } i \in \{0, \ldots, k-1\}.$$

By Theorem 7, this conjecture is true for $k = 2$ (and $k = 1$). If it holds for all k then, by Theorem 6, there is an analogue to Corollary 8 for the general case. We conclude with this conjectured formula for the number of k-triangulations having a given degree at a fixed vertex.

Conjecture 2 *The number of k-triangulations $T \in \mathcal{T}_n^k$ with $\left|\mathsf{N}_0(T)\right| = c$ is given by*

$$\det \begin{bmatrix} C_{n-2k} & C_{n-2k+1} & \iddots & \iddots & B_{n-k-1}^k(c-1) \\ C_{n-2k+1} & \iddots & \iddots & C_{n-k-1} & \vdots \\ \iddots & \iddots & \iddots & \iddots & \vdots \\ \iddots & C_{n-k-1} & \iddots & \iddots & B_{n-3}^k(c-1) \\ C_{n-k-1} & \iddots & \iddots & C_{n-3} & B_{n-2}^k(c-1) \end{bmatrix},$$

where $B_m^k(h) = \frac{2k+h-2}{m}\binom{2m-2k-h+1}{m-1}$.

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