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On k-simplexes in $(2k-1)$-dimensional vector spaces over finite fields

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Abstract. We show that if the cardinality of a subset of the $(2k-1)$-dimensional vector space over a finite field with q elements is $\gg q^{2k-1-\frac{1}{2k}}$, then it contains a positive proportional of all k-simplexes up to congruence.

Résumé. Nous montrons que si la cardinalité d'un sous-ensemble de l'espace vectoriel à $(2k-1)$ dimensions sur un corps fini à q éléments est $\gg q^{2k-1-\frac{1}{2k}}$, alors il contient une proportion non-nulle de tous les k-simplexes de congruence.

Keywords: distance problem, finite Euclidean graphs, finite non-Euclidean graphs, spectral graphs

1 Introduction

A classical result due to Furstenberg, Katznelson and Weiss (6) states that if $E \subset \mathbb{R}^2$ has positive upper Lebesgue density, then for any $\delta > 0$, the δ-neighborhood of E contains a congruent copy of a sufficiently large dilate of every three-point configuration. For higher dimensional simplexes, Bourgain (4) showed that if $E \subset \mathbb{R}^d$ has positive upper density, and Δ is a k-simplex with $k < d$, then E contains a rotated and translated image of every large dilate of Δ. The cases $k = d$ and $k = d+1$ still remain open. Magyar (10; 11) studied related problems in the integer lattice $\mathbb{Z}^d$. He showed (11) that if $d > 2k+4$, and $E \subset \mathbb{Z}^d$ has positive upper densitiy, then all large (depending on density of E) dilates of a k-simplex in $\mathbb{Z}^d$ can be embedded in E.

Hart and Iosevich (7) made the first investigation in an analog of this question in finite field geometries. They showed that if $E \subset \mathbb{F}_q^d$, $d \geq \binom{k+1}{2}$ of cardinality $|E| \geq Cq^{\frac{kd}{k+1}}q^{\frac{k}{2}}$ for a sufficiently large constant $C > 0$, then E contains an isometric copy of every k-simplex. Using graph theoretic method, the author (14) showed that the same result holds for $d \geq 2k$ and $|E| \gg q^{\frac{d-1}{2}+k}$ (cf. Theorem 1.4 in (14)).

Note that serious difficulties arise when the size of the simplex is sufficiently large with respect to the ambient dimension. Even in the case of triangles, the result in (14) is only non-trivial for $d \geq 4$. In (5), Covert, Hart, Iosevich, and Uriarte-Tuero addressed the case of triangles in two-dimensional vector spaces over finite fields. They showed that if E has density $\geq \rho$, for some $\frac{C}{\sqrt{q}} \leq \rho \leq 1$ with a sufficiently large constant $C > 0$, then the set of triangles determined by E, up to congruence, has density $\geq c\rho$. In (15), the author studied the remaining case; triangles in three-dimensional vector spaces over finite fields. Using a combination of graph theory method and Fourier analysis, the author showed that if $E \subset \mathbb{F}_q^d$,

$d \geq 3$, such that $|E| \gg q^{\frac{d+2}{2}}$, then E determines almost all triangles up to congruence. The arguments in (15), however, do not work for $k \geq 5$.

In this paper, we will study the case of k-simplexes in $(2k-1)$-dimensional vector spaces with $k \geq 3$. Given $E_1, \ldots, E_k \subset \mathbb{F}_q^d$, where $\mathbb{F}_q$ is a finite field of q elements, define

$$T_k(E_1, \ldots, E_k) = \{(x_1, \ldots x_k) \in E_1 \times \ldots \times E_k\} / \sim \tag{1.1}$$

with the equivalence relation $\sim$ such that $(x_1, \ldots, x_k) \sim (x'_1, \ldots, x'_k)$ if there exist $\tau \in \mathbb{F}_q^d$ and $O \in SO_d(\mathbb{F}_q)$, the set of d-by-d orthogonal matrices over $\mathbb{F}_q$ with determinant 1, such that

$$(x'_1, \ldots, x'_k) = (O(x_1) + \tau, \ldots, O(x_k) + \tau). \tag{1.2}$$

The main result of this paper is the following.

Theorem 1.1 *Let $E \subset \mathbb{F}_q^{2k-1}$ with $k \geq 3$, and suppose that*

$$|E| \gg q^{2k-1-\frac{1}{2k}}.$$

There exists $c > 0$ such that

$$|T_{k+1}(E)| \geq cq^{\binom{k+1}{2}}.$$

In other words, we always get a positive proportion of all k-simplexes if $E \gg q^{k-1-\frac{1}{2k}}$ and $k \geq 3$. The rest of this short paper is organized as follows. In Section 2, we establish some results about the occurrences of colored subgraphs in a pseudo-random coloring of a graph. In Section 3, we construct our main tools to study simplexes in vector spaces over finite fields, the finite Euclidean and non-Euclidean graphs. We then prove our main result, Theorem 1.1 in Section 4.

2 Subgraphs in expanders

We call a graph $G = (V, E)$ (n, d, λ)-graph if G is a d-regular graph on n vertices with the absolute values of each of its eigenvalues but the largest one are at most λ. Suppose that a graph G of order n is colored by t colors. Let G_i be the induced subgraph of G on the i^{th} color. We call a t-colored graph G (n, d, λ)-r.c. (regularly colored) graph if G_i is an (n, d, λ)-regular graph for each color $i \in \{1, \ldots, t\}$. In this section, we will study the occurrences of colored subgraphs in an (n, d, λ)-r.c. graph.

2.1 Colored subgraphs

It is well-known that if $\lambda \ll d$ then an (n, d, λ)-graph behaves similarly as a random graph $G_{n,d/n}$. Precisely, we have the following result.

Theorem 2.1 *((1, Theorem 9.2.4)) Let G be an (n, d, λ)-graph. For a vertex $v \in V$ and a subset B of V, denote by $N(v)$ the set of all neighbors of v in G, and let $N_B(v) = N(v) \cap B$ denote the set of all neighbors of v in B. For every subset B of V, we have*

$$\sum_{v \in V} (|N_B(v)| - \frac{d}{n}|B|)^2 \leqslant \frac{\lambda^2}{n}|B|(n - |B|). \tag{2.1}$$

The following result is an easy corollary of Theorem 2.1.

Theorem 2.2 *((1, Corollary 9.2.5)) Let G be an (n,d,λ)-graph. For every set of vertices B and C of G, we have*

$$|e(B,C) - \frac{d}{n}|B||C|| \leqslant \lambda\sqrt{|B||C|}, \tag{2.2}$$

where $e(B,C)$ is the number of edges in the induced bipartite subgraph of G on (B,C) (i.e. the number of ordered pair (u,v) where $u \in B$, $v \in C$ and uv is an edge of G).

Let H be a fixed graph of order s with r edges. Let $\mathrm{Aut}(H)$ be an automorphism group of H. It is well-known that for every constant $p \in (0,1)$, the random graph $G(n,p)$ contains

$$(1+o(1))p^r(1-p)^{\binom{s}{2}-r}\frac{n^s}{|\mathrm{Aut}(H)|} \tag{2.3}$$

induced copies of H. Alon extended this result to (n,d,λ)-graph. He proved that every large subset of the set of vertices of a (n,d,λ)-graph contains the "correct" number of copies of any fixed small subgraph.

Theorem 2.3 *((9, Theorem 4.10)) Let H be a fixed graph with r edges, s vertices and maximum degree Δ, and let $G=(V,E)$ be an (n,d,λ)-graph, where, say, $d \leqslant 0.9n$. Let $m < n$ satisfies $m \gg \lambda\left(\frac{n}{d}\right)^{\Delta}$. Then, for every subset $U \subset V$ of cardinality m, the number of (not necessrily induced) copies of H in U is*

$$(1+o(1))\frac{m^s}{|\mathrm{Aut}(H)|}\left(\frac{d}{n}\right)^r. \tag{2.4}$$

In (14), we observed that Theorem 2.3 can be extended to (n,d,λ)-r.c. graph. Precisely, we showed that every large subset of the set of vertices of an (n,d,λ)-r.c. graph contains the "correct" number of copies of any fixed small colored graph. We present here a multiset version of this statement.

Theorem 2.4 *Let H be a fixed t-colored graph with r edges, s vertices, maximum degree Δ (with the vertex set is ordered), and let G be a t-colored graph of order n. Suppose that G is an (n,d,λ)-r.c graph, where, say, $d \ll n$. Let $E_1,\ldots,E_s \subset V$ satisfy $|E_i| \gg \lambda\left(\frac{n}{d}\right)^{\Delta}$. Then the number of (not necessrily induced) copies of H in $E_1\times\ldots\times E_s$ (one vertex in each set) is*

$$(1+o(1))\prod_{i=1}^{s}|E_i|\left(\frac{d}{n}\right)^r. \tag{2.5}$$

The proof of this theorem is similar to the proofs of (9, Theorem 4.10) and (14, Theorem 2.3). Note that going from one color formulation ((9, Theorem 4.10)) and one set formulation ((14, Theorem 2.3)) to a multicolor-multiset formulation (Theorem 2.4) is just a matter of inserting different letters in a couple of places.

2.2 Colored stars

Given any k colors $r_1,\ldots,r_k$, a k-star of type $(r_1,\ldots,r_k)$ has $k+1$ vertices, one center vertex x_0 and k leaves $x_1,\ldots,x_k$, with the edge (x_0,x_i) is colored by the color r_i. The following result gives us an estimate for the number of colored k-stars in an (n,d,λ)-r.c. graph G (see (15) for an earlier version).

Theorem 2.5 *Let G be an (n, d, λ)-r.c. graph. Given any k colors $r_1, \ldots, r_k$ in the color set. Suppose that $E_0, E_1, \ldots, E_k \subset V(G)$ with*

$$|E_0|^2 \prod_{i \in I} |E_i| \gg \left(\frac{n}{d}\lambda\right)^{2|I|} \tag{2.6}$$

for all $I \subset \{1, \ldots, k\}$, $|I| \geq 2$, and

$$|E_0||E_i| \gg \left(\frac{n}{d}\lambda\right)^2 \tag{2.7}$$

for all $i \in \{1, \ldots, k\}$. Let $e_{\{r_1,\ldots,r_k\}}(E_0; \{E_1, \ldots, E_k\})$ denote the number of k-stars of type $(r_1, \ldots, r_k)$ in $E_0 \times E_1 \times \ldots \times E_k$ (with the center in E_0). We have

$$e_{\{r_1,\ldots,r_k\}}(E_0; \{E_1, \ldots, E_k\}) = (1 + o(1)) \left(\frac{d}{n}\right)^k \prod_{i=0}^{k} |E_i|, \tag{2.8}$$

where k is fixed and $n, d, \lambda \gg 1$.

Proof: The proof proceeds by induction. We first consider the bast case, $k = 1$. Since $|E| \gg \frac{n}{d}\lambda$ and the number of 1-stars of type a in $E_0 \times E_1$ is just the number of a-colored edges in $E_0 \times E_1$, the statement follows immediately from Theorem 2.2 and (2.7).

Suppose that the statement holds for all colored l-stars with $l < k$. For a vertex $v \in V$ and a color r, let $N_E^r(v)$ denote the set of all r-colored neighbors of v in E. From Theorem 2.1, we have

$$\sum_{v \in E_0} (|N_{E_i}^{r_i}(v)| - \frac{d}{n}|E_i|)^2 \leqslant \sum_{v \in V} (|N_{E_i}^{r_i}(v)| - \frac{d}{n}|E_i|)^2 \leqslant \frac{\lambda^2}{n}|E_i|(n - |E_i|) \leqslant \lambda^2|E_i|. \tag{2.9}$$

For $k \geqslant 2$, by the Cauchy-Schwartz inequality, we have

$$\prod_{i=1}^{k} \left(\sum_{j=1}^{n} a_{i.j}^2\right) \geqslant \left(\sum_{j=1}^{n} \prod_{i=1}^{k-1} a_{i,j}^2\right) \left(\sum_{j=1}^{n} a_{k.j}^2\right) \geqslant \left(\sum_{j=1}^{n} \prod_{i=1}^{k} a_{i,j}\right)^2. \tag{2.10}$$

It follows from (2.9) and (2.10) that

$$\left(\sum_{v \in E_0} \prod_{i=1}^{k} (N_{E_i}^{r_i}(v) - \frac{d}{n}|E_i|)\right)^2 \leqslant \prod_{i=1}^{k} \sum_{v \in E_0} (|N_{E_i}^{r_i}(v)| - \frac{d}{n}|E_i|)^2 \leqslant \lambda^{2k} \prod_{i=1}^{k} |E_i|.$$

It can be written as

$$\left| \sum_{I \subset \{1,\ldots,k\}} (-1)^{k-|I|} \left(\frac{d}{n}\right)^{k-|I|} \prod_{j \notin I} |E_j| \sum_{v \in E_0} \prod_{i \in I} N_{E_i}^{r_i}(v) \right| \leqslant \lambda^k \sqrt{\prod_{i=1}^{k} |E_i|}. \tag{2.11}$$

For any $I \subset \{1, \ldots, k\}$ with $0 < |I| < k$, by the induction hypothesis, we have

$$\sum_{v \in E_0} \prod_{i \in I} N_{E_i}^{r_i}(v) = e_I(E_0; \{E_i\}_{i \in I}) = (1 + o(1)) \left(\frac{d}{n}\right)^{|I|} |E_0| \prod_{i \in I} |E_i|. \tag{2.12}$$

Putting (2.11) and (2.12) together, we have

$$\left| \sum_{v\in E_0} \prod_{i=1}^{k} N_{E_i}^{r_i}(v) - (1+o(1)) \left(\frac{d}{n}\right)^k \prod_{i=0}^{k} |E_i| \right| \leqslant \lambda^k \sqrt{\prod_{i=1}^{k} |E_i|}.$$

Since $|E_0|^2 \prod_{i=1}^k |E_i| \gg (\frac{n}{d}\lambda)^{2k}$, the left hand side is dominated by $(1+o(1))\left(\frac{d}{n}\right)^k \prod_{i=0}^k |E_i|$. This implies that

$$e_{\{r_1,\ldots,r_k\}}(E_0;\{E1,\ldots,E_k\}) = \sum_{v\in E_0} \prod_{i=1}^{k} N_{E_i}^{r_i}(v) = (1+o(1)) \left(\frac{d}{n}\right)^k \prod_{i=0}^{k} |E_i|,$$

completing the proof of the theorem. □

3 Finite Euclidean and non-Euclidean graphs

In this section, we construct our main tools to study simplexes in vector spaces over finite fields, the graphs associated to finite Euclidean and non-Euclidean spaces. The construction of finite Euclidean graphs follows one of Medrano et al. in (12) and the construction of finite non-Euclidean graphs follows one of Bannai, Shimabukuro, and Tanaka in (3).

3.1 Finite Euclidean graphs

Let $\mathbb{F}_q$ be a finite field with q elements where $q \gg 1$ is an odd prime power. For any $x = (x_1, \ldots, x_d) \in \mathbb{F}_q^d$, let

$$\|x\| = x_1^2 + \ldots + x_d^2.$$

For a fixed $a \in \mathbb{F}_q$, the finite Euclidean graph $G_q(a)$ in $\mathbb{F}_q^d$ is defined as the graph with the vertex set $\mathbb{F}_q^d$, and the edge set

$$\{(x,y) \in \mathbb{F}_q^d \times \mathbb{F}_q^d \mid x \neq y, ||x-y|| = a\}.$$

Medrano et al. (12) studied the spectrum of these graphs and showed that these graphs are asymptotically Ramanujan graphs. Precisely, they proved the following result.

Theorem 3.1 *((12)) The finite Euclidean graph $G_q(a)$ is a regular graph with $n(q,a) = q^d$ vertices of valency*

$$k(q,a) = \begin{cases} q^{d-1} + \chi((-1)^{(d-1)/2}a)q^{(d-1)/2} & a \neq 0, \ d \textit{ odd,} \\ q^{d-1} - \chi((-1)^{d/2}))q^{(d-2)/2} & a \neq 0, \ d \textit{ even,} \\ q^{d-1} & a = 0, \ d \textit{ odd,} \\ q^{d-1} - \chi((-1)^{d/2}))(q-1)q^{(d-2)/2} & a = 0, \ d \textit{ even.} \end{cases}$$

where χ is the quadratic character

$$\chi(a) = \begin{cases} 1 & a \neq 0, \ a \textit{ is square in } \mathbb{F}_q, \\ -1 & a \neq 0, \ a \textit{ is nonsquare in } \mathbb{F}_q, \\ 0 & a = 0. \end{cases}$$

Let λ be any eigenvalues of the graph $G_q(a)$ with $\lambda \neq$ valency of the graph then

$$|\lambda| \leq 2q^{\frac{d-1}{2}}. \tag{3.1}$$

3.2 Finite non-Euclidean graphs

Let $V = \mathbb{F}_q^{2k-1}$ be the $(2k-1)$-dimensional vector space over the finite field $\mathbb{F}_q$ (q is an odd prime power). For each element x of V, we denote the 1-dimensional subspace containing x by $[x]$. Let Ω be the set of all square type non-isotropic 1-dimensional subspaces of V with respect to the quadratic form $Q(x) = x_1^2 + \ldots + x_{2k-1}^2$. The simple orthogonal group $O_{2k-1}(\mathbb{F}_q)$ acts transtively on Ω, and yields a symmetric association scheme $\Psi(O_{2k-1}(\mathbb{F}_q), \Omega)$ of class $(q+1)/2$. The relations of $\Psi(O_{2k-1}(\mathbb{F}_q), \Omega)$ are given by

$$\begin{aligned}
R_1 &= \{([U],[V]) \in \Omega \times \Omega \mid (U+V)\cdot(U+V) = 0\},\\
R_i &= \{([U],[V]) \in \Omega \times \Omega \mid (U+V)\cdot(U+V) = 2 + 2\nu^{-(i-1)}\}\ (2 \leqslant i \leqslant (q-1)/2)\\
R_{(q+1)/2} &= \{([U],[V]) \in \Omega \times \Omega \cdot (U+V)\cdot(U+V) = 2\},
\end{aligned}$$

where ν is a generator of the field $\mathbb{F}_q$ and we assume $U \cdot U = 1$ for all $[U] \in \Omega$ (see (2) for more details).

The graphs (Ω, R_i) are asymptotic Ramanujan for large q. The following theorem summaries the results from Section 2 in (3) in a rough form.

Theorem 3.2 *(3) The graphs (Ω, R_i) $(1 \leq i \leq (q+1)/2)$ are regular of order $q^{2k-2}(1+o_q(1))/2$ and valency Kq^{2k-3}. Let λ be any eigenvalue of the graph (Ω, R_i) with $\lambda \neq$ valency of the graph then*

$$|\lambda| \leq kq^{(2k-3)/2},$$

for some $k, K > 0$ (In fact, we can show that $k = 2 + o_q(1)$ and $K = 1 + o_q(1)$ or $1/2 + o_q(1)$).

4 Proof of Theorem 1.1

We now give a proof of Theorem 1.1. For any $\{a_{ij}\}_{1 \leq i < j \leq k+1} \in \mathbb{F}_q^{\binom{k+1}{2}}$, define

$$T_{\{a_{ij}\}_{1 \leq i < j \leq k+1}}(E) = \{(x_i)_{i=1}^{k+1} \in E^{k+1} : \|x_i - x_j\| = a_{ij}\}.$$

Hart and Iosevich (7) observed that in vector spaces over finite fields, a (non-degenerate) simplex is defined uniquely (up to translation and rotation) by the norms of its edges.

Lemma 4.1 *((7)) Let P be a (non-degenerate) simplex with vertices $V_0, V_1, \ldots, V_k$ where $V_j \in \mathbb{F}_q^d$. Let P' be another (non-degenerate) simplex with vertices $V_0', \ldots, V_k'$. Suppose that*

$$\|V_i - V_j\| = \|V_i' - V_j'\| \tag{4.1}$$

for all i, j. There exists $\tau \in \mathbb{F}_q^d$ and $O \in SO_d(\mathbb{F}_q)$ such that $\tau + O(P) = P'$.

Therefore, it suffices to show that if $E \subset \mathbb{F}_q^{2k-1}$ $(k \geq 3)$ of cardinality $|E| \gg q^{2k-1-\frac{1}{2k}}$, then

$$\left|\left\{\{a_{ij}\}_{1 \leq i < j \leq k+1} \in \mathbb{F}_q^{\binom{k+1}{2}} : |T_{\{a_{ij}\}_{1 \leq i < j \leq k+1}}(E)| > 0\right\}\right| \geqslant cq^{\binom{k+1}{2}}. \tag{4.2}$$

Consider the set of colors $L=\{c_0,\dots,c_{q-1}\}$ corresponding to elements of $\mathbb{F}_q$. We color the complete graph G_q with the vertex set $\mathbb{F}_q^{2k-1}$, by q colors such that $(x,y)\in\mathbb{F}_q^{2k-1}\times\mathbb{F}_q^{2k-1}$ is colored by c_i whenever $||x-y||=i$.

Suppose that $|E|\gg q^{2k-1-\frac{1}{2k}}$, we have

$$|E|\gg\left(\frac{q^{2k-1}\cdot 2q^{k-1}}{q^{2k-2}(1+o(1))}\right)^{\frac{2i}{i+2}},$$

for all $2\le i\le k$. From Theorem 3.1, G_q is a $(q^{2k-1},q^{2k-2}(1+o(1)),2q^{k-1})$-r.c. graph when $k\ge 3$. Therefore, applying Theorem 2.5 for the number of k-stars of type $(a_{12},\dots,a_{1(k+1)})$ in E^{k+1}, we have

$$e_{a_{12},\dots,a_{1(k+1)}}(E;\{E,\dots,E\})=\left(\frac{q^{2k-2}(1+o(1))}{q^{2k-1}}\right)^k|E|^{k+1}(1+o(1))=\frac{|E|^{k+1}(1+o(1))}{q^k},$$

for any $a_{12},\dots,a_{1(k+1)}\in\mathbb{F}_q$.

Let $\mathbb{F}_\square^*$ denote the set of non-zero squares in $\mathbb{F}_q$. For any $a_{12},\dots,a_{1(k+1)}\in\mathbb{F}_\square^*$, then

$$\begin{aligned}|\{(x_1,\dots,x_{k+1})\in E^{k+1}:\|x_1-x_i\|=a_{1i}\}| &= e_{a_{12},\dots,a_{1(k+1)}}(E;\{E,\dots,E\})\\ &= \frac{|E|^{k+1}(1+o(1))}{q^k}.\end{aligned}$$

By the pigeon-hole principle, there exists $x_1\in E$ such that

$$\{(x_2,\dots,x_{k+1})\in E^k:\|x_1-x_i\|=a_{1i}\}|=\frac{|E|^k(1+o(1))}{q^k}.$$

Let $S_t=\{v\in\mathbb{F}_q^{2k-1}:||v||=t\}$ denote the sphere of radius t in $\mathbb{F}_q^{2k-1}$, then

$$|S_t|=q^{2k-2}(1+o(1))$$

for any $t\in\mathbb{F}_q$. Let

$$E_i=\{v\in E:\|x_1-v\|=a_{1i}\}\subset S_{a_i},\ 2\le i\le k+1,$$

then

$$|E_2|\dots|E_{k+1}|=\frac{|E|^k(1+o(1))}{q^k},$$

and

$$|E_2|,\dots,|E_{k+1}|\le O(q^{2k-2}).$$

This implies that

$$|E_i|\ge\Omega\left(\frac{|E|^k(1+o(1))}{q^{k+(2k-2)(k-1)}}\right)\gg q^{2k-\frac{5}{2}}.$$

There are $(q-1)^k/2^k$ possibilities of $a_{12},\dots,a_{1(k+1)}\in\mathbb{F}_\square^*$. From Lemma 4.1, it suffices to show that $T_k(E_2,\dots,E_{k+1})\ge cq^{\binom{k}{2}}$ for some $c>0$. Let

$$E_i'=\{[x]:x\in E_i\}\subset\Omega$$

where Ω is the set of all square type non-isotropic 1-dimensional subspaces of $\mathbb{F}_q^{2k-1}$ with respect to the quadratic form $Q(x) = x_1^2 + \ldots + x_{2k-1}^2$. Since each line through origin in $\mathbb{F}_q^{2k-1}$ intersects the unit sphere S_1 at two points, $|E_i'| \geq |E_i|/2 \gg q^{2k-\frac{5}{2}}$. Suppose that $([U],[V]) \in E_i' \times E_j'$ is an edge of (Ω, R_l), $2 \leq l \leq (q-1)/2$. Then

$$(U+V)\cdot(U+V) = 2 + \alpha_l,$$

where $\alpha_l = 2\nu^{-(l-1)}$. Since $U \cdot U = V \cdot V = 1$, we have $(U-V)\cdot(U-V) = 2-\alpha_l$. The distance between U and V (in $E_i' \times E_j'$) is either $(U+V)\cdot(U+V)$ or $(U-V)\cdot(U-V)$. Hence,

$$||U - V|| \in \{2+\alpha_l, 2-\alpha_l\}. \tag{4.3}$$

Consider the set of colors $L = \{r_1, \ldots, r_{(q+1)/2}\}$ corresponding to classes of the association scheme $\Psi(O_{2k-1}(\mathbb{F}_q), \Omega)$. We color the complete graph P_q with the vertex set Ω, by $(q+1)/2$ colors such that $([U],[V]) \in \Omega \times \Omega$ is colored by r_i whenever $([U],[V]) \in R_i$.

From Theorem 3.2, P_q is a $((1+o(1))q^{2k-2}/2, Kq^{2k-3}, kq^{(2k-3)/2})$-r.c. graph when $k \geq 3$. Since $|E_i'| \gg q^{2k-\frac{5}{2}}$, we have

$$|E_i'| \gg kq^{(2k-3)/2}\left(\frac{(1+o(1))q^{2k-2}/2}{Kq^{2k-3}}\right)^{k-1}.$$

Therefore, applying Theorem 2.4 for colored k-complete subgraphs of P_q then P_q contains all possible colored k-complete subgraphs. From (4.3), $([U],[V])$ is colored by r_l $(2 \leq l \leq (q-1)/2)$ then $||U-V|| \in \{2+\alpha_l, 2-\alpha_l\}$. Hence, $T_k(E_2, \ldots, E_{k+1}) \geq cq^{\binom{k}{2}}$ for some $c > 0$. The theorem follows.

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