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# *Words and polynomial invariants of finite groups in non-commutative variables*

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**Abstract.** Let $V$ be a complex vector space with basis $\{x_1, x_2, \dots, x_n\}$ and $G$ be a finite subgroup of $GL(V)$. The tensor algebra $T(V)$ over the complex is isomorphic to the polynomials in the non-commutative variables $x_1, x_2, \dots, x_n$ with complex coefficients. We want to give a combinatorial interpretation for the decomposition of $T(V)$ into simple $G$-modules. In particular, we want to study the graded space of invariants in $T(V)$ with respect to the action of $G$. We give a general method for decomposing the space $T(V)$ into simple $G$-module in terms of words in a particular Cayley graph of $G$. To apply the method to a particular group, we require a surjective homomorphism from a subalgebra of the group algebra into the character algebra. In the case of the symmetric group, we give an example of this homomorphism from the descent algebra. When $G$ is the dihedral group, we have a realization of the character algebra as a subalgebra of the group algebra. In those two cases, we have an interpretation for the graded dimensions of the invariant space in term of those words.

**Résumé.** Soit $V$ un espace vectoriel complexe de base $\{x_1, x_2, \dots, x_n\}$ et $G$ un sous-groupe fini de $GL(V)$. L'algèbre $T(V)$ des tenseurs de $V$ sur les complexes est isomorphe aux polynômes à coefficients complexes en variables non-commutatives $x_1, x_2, \dots, x_n$. Nous voulons donner une décomposition de $T(V)$ en $G$-modules simples de manière combinatoire. Plus particulièrement, nous étudions l'espace gradué des invariants de $T(V)$ sous l'action de $G$. Nous présentons une méthode générale donnant la décomposition de $T(V)$ en modules simples via certains mots dans un graphe de Cayley donné. Pour appliquer la méthode à un groupe particulier, nous avons besoin d'un homomorphisme surjectif entre une sous-algèbre de l'algèbre de groupe et l'algèbre des charactères. Pour le cas du groupe symétrique, nous donnons un example de cet homomorphisme qui provient de la théorie de l'algèbre des descentes. Pour le groupe diédral, nous avons une réalisation de l'algèbre des charactères comme une sous-algèbre de l'algèbre de groupe. Dans ces deux cas, nous avons une interprétation des dimensions graduées de l'espace des invariants en terme de ces mots.

**Keywords:** Invariant theory, Non-commutative variables, Symmetric group, Dihedral group, Cayley Graph, Words

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# 1 Introduction

Let $V$ be a vector space over $\mathbb{C}$ with basis $\{x_1, x_2, \dots, x_n\}$ and $G$ a finite subgroup of $GL(V)$, then

$$T(V) = \mathbb{C} \oplus V \oplus V^{\otimes 2} \oplus V^{\otimes 3} \oplus \cdots \simeq \mathbb{C}\langle x_1, x_2, \dots, x_n \rangle$$

is the ring of non-commutative polynomials in the basis elements where we use the notation $V^{\otimes d} = V \otimes V \otimes \cdots \otimes V$. We will consider the subalgebra $T(V)^G \simeq \mathbb{C}\left\langle x_1, x_2, \dots, x_n \right\rangle^G$ as the graded space of invariants with respect to the action of $G$. It is convenient to conserve the information on the dimension of each homogeneous component $\mathbb{C}\langle x_1, x_2, \dots, x_n \rangle_d^G \simeq (V^{\otimes d})^G$ of degree $d$ in the *Hilbert-Poincaré series*

$$P(T(V)^G) = \sum_{d \geq 0} \dim(V^{\otimes d})^G q^d.$$

Several algebraic tools allow us to study the invariants for $T(V)$ with respect to the group $G$. The graded character of $T(V)$ can be found in terms by what we might identify as a 'master theorem' for the tensor space,

$$\chi^{(V^{\otimes d})}(g) = tr(M(g))^d = \left[q^d\right] \frac{1}{1 - tr(M(g))q} ,$$

where $\left[q^d\right]$ represents taking the coefficient of $q^d$ in the expression to the right and $M(g)$ is a matrix which represents the action of the group element $g$ on a basis of $V$. The analogue of Molien's theorem [3] for the tensor algebra says that

$$dim\ (V^{\otimes d})^G = \left[q^d\right] \frac{1}{|G|} \sum_{g \in G} \frac{1}{1 - tr(M(g))q} .$$

In general, we can say that the invariants $T(V)^G$ are freely generated [4] by an infinite set of generators (except when $G$ is scalar, *i*.e. when $G$ is generated by a nonzero scalar multiple of the identity matrix) [3]. No simple general description of the invariants or the generators is known for large classes of groups and these algebraic tools do not clearly show the underlying combinatorial structure of these invariant algebras.

Our goal is to find a combinatorial method for computing the graded dimensions of $T(V)^G$. The main idea of a general theorem would be the following. To a $G$-module $V$, we associate a subalgebra of the group algebra together with a homomorphism of algebras into the ring of characters. Then we get as a consequence a combinatorial description of the invariants of $T(V)$ as words generated by a particular Cayley graph of $G$. To compute the coefficient of $q^d$ in the Hilbert-Poincaré series of $T(V)^G$, it then suffices to look at the multiplicity of the trivial in $(V^{\otimes d})$. At this point, since there is not a general relation between the group algebra and the character ring, we are only able to treat some examples that we decided to present here and the method used gives rise to objects that are a priori not natural in that context. In particular, we compute the graded dimensions of $T(V)^G$ for $V$ being the geometric module (see below) of the symmetric group and for $V$ being any module of the dihedral group in term of words generated by a Cayley graph of $G$ in some specific generators. The subalgebra we use in the case of the symmetric group is the Solomon's descent algebra, that will make the bridge between words in a particular Cayley graph in those generators and the decomposition of $T(V)$ into simple $S_n$-module. In the case of the dihedral group, we present a new non-commutative realization of the character ring as a subalgebra of the group algebra.

When the group $G$ is generated by pseudo-reflections acting on a vector space $V$, then if $V$ is simple, $V$ is called the geometric G-module. When $G$ is the symmetric group $S_n$ on $n$ letters and acts on the vector space $V$ spanned by the vectors $\{x_1, x_2, \ldots, x_n\}$ by the permutation action then $G$ is generated by pseudo-reflections, but is not a simple $S_n$-module. The space $\mathbb{C}\langle x_1, x_2, \ldots, x_n\rangle^{S_n}$ is known as the symmetric functions in non-commutative variables which was first studied by Wolf [8] and more recently by Rosas-Sagan [6]. The dimension of $(V^{\otimes d})^{S_n}$ is the number of set partitions of the numbers $\{1, 2, \ldots, d\}$ into at most $n$ parts. If $G$ is the symmetric group but acting on the vector space spanned by the vectors $\{x_1 - x_2, x_2 - x_3, \ldots, x_{n-1} - x_n\}$ (again with the permutation action on the $x_i$) then this is also a group generated by pseudo-reflections but the invariant space $T(V)^{S_n}$ is not as well understood. The graded dimensions of the invariant space are given by the number of oscillating tableaux studied by Chauve-Goupil [1]. This interpretation for the graded dimensions has a very different nature to that of set partitions. By applying the results in this paper we find a combinatorial interpretation for the graded dimensions of these spaces, and many others, which unifies the interpretations of their graded dimensions.

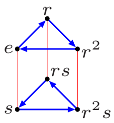
The paper is organized as follows. In section 2 we recall the definition of a Cayley graph and present a technical lemma that we will need to link the number of words of length $d$ in a particular Cayley graph of $G$ to some coefficients in the $d$-th power of a particular element of the group algebra. We will then present in section 3 the particular case of the symmetric group $S_n$ and make explicit the result for $V$ being the geometric $S_n$-module. Since the bridge between the words in the Cayley graph of $S_n$ and the decomposition of $T(V)$ is the descent algebra, we will recall in section 3.3 some results about the Solomon's descent algebra of $S_n$. Section 3.6 contains some results about the invariant algebra $T(V)^{S_n}$ where we present a conjecture for a closed formula for the Hilbert-Poincaré series of $T(V)^{S_n}$, where $V$ is the geometric $S_n$-module. Finally in section 4, we apply our general method in the case of the dihedral group $D_m$ and then study in section 4.3 the particular case of the invariant algebra $T(V)^{D_m}$ when $V$ is the geometric module and give a closed formula for the Hilbert-Poincaré series of $T(V)^{D_m}$.

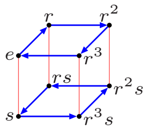
## 2 Cayley graph of a group $G$

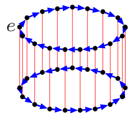
Let us recall the definition of a Cayley graph given in Coxeter [2]. A presentation of a finite group $G$ with generating set $S$ can be encoded by its Cayley graph. A *Cayley graph* is an oriented graph $\Gamma = \Gamma(G, S)$, having one vertex for each element of the group $G$ and the edges associated with generators in $S$. Two vertices $g_1$ and $g_2$ are joined by a directed edge associated to $s \in S$ if $g_2 = g_1 s$. Then a path along the edges corresponds to a word in the generators in $S$. A word which *reduces to* $g \in G$ in $\Gamma$ will be a path along the edges from the vertex corresponding to the identity to the one corresponding to the element $g$. We will denote by $w(g; d; \Gamma)$ the set of words of length $d$ which reduce to $g$ in $\Gamma$. We will say that a word *does not cross the identity* if it has no proper prefix which reduces to the identity.

More generally, we will consider *weighted* Cayley graphs $\Gamma(G, S)$. In other words, we will associate a weight $\omega(s)$ to each generator $s \in S$. Then we will define the *weight of a word* $w = s_1 s_2 \cdots s_r$ in the generators to be the product of the weights of the generators, $\omega(w) = \omega(s_1)\omega(s_2)\cdots\omega(s_r)$. To simplify the image, undirected edges will represent bidirectional edges and non-labelled edges will represent edges of weight one .

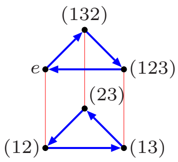
**Example 2.1** *Consider the dihedral group* $D_m$ *with presentation* $\langle s, r \mid s^2 = r^m = srsr = e\rangle$. *The Cayley graphs* $\Gamma(D_3, \{s, r\})$, $\Gamma(D_4, \{s, r\})$ *and more generally* $\Gamma(D_m, \{s, r\})$ *will look like*




**Example 2.2** *The symmetric group $S_n$ on $n$ letters is generated by the permutations $(12)$ and $(1\,n\,\cdots 432)$ (see [2]), hence also by the permutations $(12)$, $(132)$, $(1432), \dots, (1\,n\,\cdots 432)$, written in cyclic notation. The Cayley graph $\Gamma(S_3, \{(12), (132)\})$ is*




**Lemma 2.3** *Let $\Gamma = \Gamma(G, \{s_1, s_2, \dots, s_r\})$ be a Cayley graph of $G$ with associated weights $\omega(s_i) = \omega_i$. Then the coefficient of $\sigma \in G$ in the element $(\omega_1 s_1 + \omega_2 s_2 + \cdots + \omega_r s_r)^d$ of the group algebra $\mathbb{C}G$ is equal to*

$$\sum_{w \in w(\sigma, d; \Gamma)} \omega(w),$$

*where $w(\sigma, d; \Gamma)$ is the set of words of length $d$ which reduce to $\sigma$ in $\Gamma$.*

**Example 2.4** *Let us consider the Cayley graph $\Gamma = (S_3, \{(12), (132)\})$ of Example 2.2. Set $a = (12)$ and $b = (132)$ to simplify. Then the table below shows that the coefficient of a specific element in $(a+b)^4$ coincides with the number of words of length three which reduce to that specific element in $\Gamma$.*

$$(a+b)^4 = \mathbf{3}\, e + \mathbf{2}\,(12) + \mathbf{3}\,(23) + \mathbf{3}\,(123) + \mathbf{2}\,(132) + \mathbf{3}\,(13)$$

| $e$ | $(12)$ | $(23)$ | $(123)$ | $(132)$ | $(13)$ |
|---|---|---|---|---|---|
| $aaaa$ | $abbb$ | $aaba$ | $aabb$ | $abba$ | $aaab$ |
| $abab$ | $bbba$ | $baaa$ | $baab$ | $bbbb$ | $abaa$ |
| $baba$ | | $bbab$ | $bbaa$ | | $babb$ |

# 3 Symmetric group $S_n$

We will give in that section a combinatorial way to decompose the tensor algebra on $V$ into simple $S_n$-modules, for $V$ being the geometric $S_n$-module, by means of words in a particular Cayley graph of $S_n$. We will also give a combinatorial way to compute the graded dimensions of the invariant space $T(V)^{S_n}$, which is the multiplicity of the trivial in the decomposition of $T(V)$. But first let us recall some definition and the theory of the descent algebra.

## 3.1 Partitions and tableaux

To fix the notation, recall the definition of a partition. A *partition* $\lambda$ of a positive integer $n$ is a decreasing sequence $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_\ell > 0$ of positive integers such that $n = |\lambda| = \lambda_1 + \lambda_2 + \ldots + \lambda_\ell$. We will write $\lambda = (\lambda_1, \lambda_2, \ldots \lambda_\ell) \vdash n$. For example, the partitions of 3 are

$$(1,1,1) \qquad (2,1) \qquad (3).$$

It is natural to represent a partition by a diagram. The *Ferrers diagram* of a partition $\lambda = (\lambda_1, \lambda_2, \ldots \lambda_\ell)$ is the finite subset $\lambda = \{(a, b) \,|\, 0 \le a \le \ell - 1$ and $0 \le b \le \lambda_{a+1} - 1\}$ of $\mathbb{N} \times \mathbb{N}$. Visually, each element of $\lambda$ corresponds to the bottom left corner of a square of dimension $1 \times 1$ in $\mathbb{N} \times \mathbb{N}$. A ***tableau*** of shape $\lambda \vdash n$, denoted $sh(t) = \lambda$, with values in $T = \{1, 2, \ldots, n\}$ is a function $t : \lambda \longrightarrow T$. We can visualize it with filling each square $c$ of a Ferrers diagram $\lambda$ with the value $t(c)$. A tableau is said to be ***standard*** if its entries form an increasing sequence along each line and along each column. We will denote by $STab_n$ the set of standard tableau with $n$ squares. For example, $STab_3$ contains the four standard tableaux

$$\begin{array}{|c|}\hline 3\\ \hline 2\\ \hline 1\\ \hline\end{array} \qquad \begin{array}{|c|c}\cline{1-1} 2 & \multicolumn{1}{c}{}\\ \hline 1 & 3 \\ \hline\end{array} \qquad \begin{array}{|c|c}\cline{1-1} 3 & \multicolumn{1}{c}{}\\ \hline 1 & 2 \\ \hline\end{array} \qquad \begin{array}{|c|c|c|}\hline 1 & 2 & 3\\ \hline\end{array}\,.$$

The *Robinson-Schensted correspondence* is a bijection between the elements $\sigma$ of the symmetric group $S_n$ and pairs $(P(\sigma), Q(\sigma))$ of standard tableaux of the same shape, where $P(\sigma)$ is the insertion tableau and $Q(\sigma)$ the recording tableau.

## 3.2 Simple $S_n$-modules

Since the conjugacy classes in $S_n$ are in bijection with the partitions of $n$, it is natural to index the simple $S_n$-modules by the partitions $\lambda$ of $n$ and we will denote them by $V^\lambda$. In particular, the simple $S_n$-module $V^{(n)}$ indexed by the partition $(n)$ is the the trivial one. Let us consider the linear span $V = \mathcal{L}\{x_1, x_2, \ldots, x_n\}$ on which $S_n$ acts by permuting the coordinates. Then we have

$$V = \mathcal{L}\{x_1 + x_2 + x_3 + \ldots + x_n\} \oplus \mathcal{L}\{x_1 - x_2, x_2 - x_3, \ldots, x_{n-1} - x_n\},$$

so the decomposition of $V$ into simple $S_n$-modules is $V = V^{(n)} \oplus V^{(n-1,1)}$. Note that the $S_n$-module $V^{(n-1,1)}$ corresponds to the geometric $S_n$-module. Let $X_n$ denote the set of variables $x_1, x_2, \ldots, x_n$ and $Y_{n-1}$ denote the set of variables $y_1, y_2, \ldots, y_{n-1}$. If we identify $T(V)$ with $\mathbb{R}\langle X_n\rangle$, then $T(V^{(n-1,1)}) \simeq \mathbb{R}\langle X_n\rangle / \langle x_1 + x_2 + \cdots + x_n\rangle$ can be identified with $\mathbb{R}\langle Y_{n-1}\rangle$, where $y_i = x_i - x_{i+1}$ for $1 \le i \le n - 1$.

## 3.3 Solomon's descent algebra of $S_n$

Surprisingly, the key to prove the general result is the theory of descent algebra of the symmetric group. Let us recall some of that theory here. Let $I = \{1, 2, \ldots, n - 1\}$. The descent set of $\sigma \in S_n$ is the set $Des(\sigma) = \{i \in I \,|\sigma(i) > \sigma(i + 1)\}$. For $K \subseteq I$, set

$$d_K = \sum_{\substack{\sigma \in S_n \\ Des(\sigma) = K}} \sigma.$$

The Solomon's descent algebra $\Sigma(S_n)$ is a subalgebra of the group algebra $\mathbb{Z}S_n$ with basis $\{d_K | K \subseteq I\}$ [7]. For a standard tableau $t$ of shape $\lambda \vdash n$ define

$$z_t = \sum_{\substack{\sigma \in S_n \\ Q(\sigma) = t}} \sigma,$$

where $Q(\sigma)$ corresponds to the recording tableau in the Robinson-Schenstead correspondence. Then consider the linear span $\mathcal{Q}_n = \mathcal{L}\{z_t |\ t \in STab_n\}$. Note in general that $\mathcal{Q}_n$ is not a subalgebra of $\mathbb{Z}S_n$, for $n \geq 4$. Define the descent set of a standard tableau $t$ by $Des(t) = \{i | i+1 \text{ is above i in } t\}$. Then

$$d_K = \sum_{\substack{t \in STab_n \\ Des(t)=K}} z_t.$$

and $\Sigma(S_n) \subseteq \mathcal{Q}_n$. There is an algebra morphism $\theta : \Sigma(S_n) \to \mathbb{Z}\mathrm{Irr}(S_n)$ due to Solomon [7]. Moreover, there is a linear map [5] $\tilde{\theta} : \mathcal{Q}_n \to \mathbb{Z}\mathrm{Irr}(S_n)$ defined by $\tilde{\theta}(z_t) = \chi^{\mathrm{sh}(t)}$, and $\tilde{\theta}$ restricted to $\Sigma(S_n)$ corresponds to $\theta$. We can observe that

$$z_{\begin{smallmatrix} \boxed{2} & & & & \\ \boxed{1} & \boxed{3} & \boxed{4} & \boxed{\cdots} & \boxed{n} \end{smallmatrix}} = (12) + (132) + (1432) + \cdots + (1\,n\,\cdots 432) = d_{\{1\}}$$

hence $\theta(d_{\{1\}}) = \chi^{(n-1,1)}$.

## 3.4 General method for $S_n$

We are developing a general combinatorial method for determining the multiplicity of $V^\lambda$ in $V^{\otimes d}$, when $V$ is any $S_n$-module. To this end, we will consider the algebra morphism $\theta : \Sigma(S_n) \to \mathbb{Z}\mathrm{Irr}(S_n)$ of section 3.3. The next proposition says that this multiplicity is given as the sum of some coefficients in $f^d$, when $f$ is an element of $\Sigma(S_n)$ such that $\theta(f) = \chi^V$.

**Proposition 3.1** *Let $V$ be an $S_n$-module such that $\theta(f) = \chi^V$, for some $f \in \Sigma(S_n)$. For $\lambda \vdash n$, the multiplicity of $V^\lambda$ in $V^{\otimes d}$ is equal to*

$$\sum_{\substack{t \in STab_n \\ sh(t)=\lambda}} [z_t] f^d,$$

*where $[z_t]f^d$ is the coefficient of $z_t$ in $f^d$.*

Although the next theorem is an easy consequence of the Lemma 2.3 and Proposition 3.1, it provides us with an interesting interpretation for the multiplicity of $V^\lambda$ in the $d$-fold Kronecker product of a $S_n$-module. This multiplicity is the weighted sum of words in a particular Cayley graph of $S_n$ which reduce to the element $\sigma_t$, where $\sigma_t$ has recording tableau $t$ of shape $\lambda$ in the Robinson-Schensted correspondance. Recall that the *support* of an element $f$ of the group algebra is defined by $\mathrm{supp}(f) = \{g \in G | [g]f \neq 0\}$.

**Theorem 3.2** *Let $V$ be an $S_n$-module such that $\theta(f) = \chi^V$, for some $f \in \Sigma(S_n)$. For $\lambda \vdash n$, the multiplicity of $V^\lambda$ in $V^{\otimes d}$ is*

$$\sum_{\substack{t \in STab_n \\ sh(t)=\lambda}} \sum_{w \in w(\sigma_t, d; \Gamma)} \omega(w),$$

*where $\sigma_t$ is such that $Q(\sigma_t) = t$, $\Gamma = \Gamma(S_n, supp(f))$ with $\omega(\sigma) = [\sigma](f)$ for each $\sigma \in supp(f)$ and $w(\sigma_t, d; \Gamma)$ is the set of words of length $d$ which reduce to $\sigma_t$ in $\Gamma$.*

### 3.5 Decomposition of $T(V^{(n-1,1)})$ and words in a Cayley graph of $S_n$

Since we are particularly interested in the geometric $S_n$-module, we make explicit the following two corollaries respectively of Proposition 3.1 and Theorem 3.2 needed to draw a connection between the multiplicity of $V^\lambda$ in $(V^{(n-1,1)})^{\otimes d}$ and words of length $d$ in a particular Cayley graph of $S_n$. To this end, we use the fact that the element $d_{\{1\}}$ of the descent algebra, which is the sum of elements of $S_n$ having descent set $\{1\}$, is sent to $\chi^{(n-1,1)}$ under the $\theta$ morphism.

**Corollary 3.3** *Let $\lambda \vdash n$. The multiplicity of $V^\lambda$ in $(V^{(n-1,1)})^{\otimes d}$ is*

$$\sum_{\substack{t \in STab_n \\ sh(t)=\lambda}} [z_t] {d_{\{1\}}}^d.$$

**Corollary 3.4** *Let $\lambda \vdash n$. The multiplicity of $V^\lambda$ in $(V^{(n-1,1)})^{\otimes d}$ is equal to*

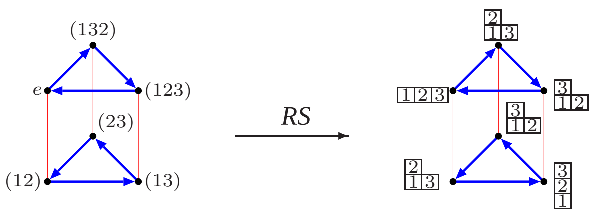
$$\sum_{\substack{t \in STab_n \\ sh(t)=\lambda}} |w(\sigma_t, d; \Gamma)|,$$

*where $\sigma_t \in S_n$ is such that $Q(\sigma_t) = t$ and $\Gamma = \Gamma(S_n, \{(12), (132), \ldots, (1\, n \cdots 432)\})$. In particular, the multiplicity of the trivial is $|w(e, d; \Gamma)|$.*

**Example 3.5** *The $S_3$-module $(V^{(2,1)})^{\otimes 4}$ decomposes as $3\,V^{(3)} \oplus 5\,V^{(2,1)} \oplus 3\,V^{(1,1,1)}$ since*

$$\begin{aligned} {d_{\{1\}}}^4 &= 3\,d_\emptyset + 3\,d_{\{2\}} + 2\,d_{\{1\}} + 3\,d_{\{1,2\}} \\ &= 3\,z_{\begin{smallmatrix}1&2&3\end{smallmatrix}} + 3\,z_{\begin{smallmatrix}3\\1&2\end{smallmatrix}} + 2\,z_{\begin{smallmatrix}2\\1&3\end{smallmatrix}} + 3\,z_{\begin{smallmatrix}3\\2\\1\end{smallmatrix}}. \end{aligned}$$

*These multiplicities can also be computed using Corollary 3.4 in the following way. The Cayley graph $\Gamma = \Gamma(S_3, \{(12), (132)\})$ looks like*




*and if we write $a$ for $(12)$ and $b$ for $(132)$ to simplify, and choose the representatives*

$$\sigma_{\begin{smallmatrix}1&2&3\end{smallmatrix}} = e \qquad \sigma_{\begin{smallmatrix}3\\1&2\end{smallmatrix}} = (23) \qquad \sigma_{\begin{smallmatrix}2\\1&3\end{smallmatrix}} = (12) \qquad \sigma_{\begin{smallmatrix}1\\2\\3\end{smallmatrix}} = (13)$$

*the multiplicities are respectively given by the cardinalities of the sets of words (see Example 2.4)*

$$\begin{array}{lll} V^{(3)}: & |w(e\,,4;\Gamma)| & = |\{aaaa, abab, baba\}| = 3, \\ V^{(2,1)}: & |w((23)\,,4;\Gamma)| + |w((12)\,,4;\Gamma)| & = |\{aaba, baaa, bbab\}| + |\{abbb, bbba\}| = 5, \\ V^{(1,1,1)}: & |w((13)\,,4;\Gamma)| & = |\{aaab, abaa, babb\}| = 3. \end{array}$$

### 3.6 Invariant algebra $T(V^{(n-1,1)})^{S_n} \simeq \mathbb{R}\langle Y_{n-1}\rangle^{S_n}$

We have an interpretation of the invariant algebra $T(V^{(n-1,1)})^{S_n}$ in terms of words which reduce to the identity in the Cayley graph $\Gamma(S_n, \{(12), (132), \ldots, (1\,n \cdots 432)\})$. As a corollary of Corollary 3.4, we can now show that the dimension of $T(V^{(n-1,1)})^{S_n}$ in each degree $d$, which is also the multiplicity of the trivial representation in $(V^{(n-1,1)})^{\otimes d}$, can be indexed by those precise words of length $d$.

**Corollary 3.6** *The dimension of $((V^{(n-1,1)})^{\otimes d})^{S_n} \simeq \mathbb{R}\langle Y_{n-1}\rangle_d^{S_n}$ is equal to the number of words of length $d$ which reduce to the identity in the Cayley graph $\Gamma(S_n, \{(12), (132), \ldots, (1\,n \cdots 432)\})$.*

**Example 3.7** *Consider the symmetric group $S_3$. Using the Reynold's operator $\sum_{\sigma\in S_n}\sigma$ acting on the monomials, a basis for the invariant space $\mathbb{R}\langle y_1, y_2\rangle_4^{S_3}$ is given by the three following polynomials*

$$y_1^2y_2^2 - y_1y_2^2y_1 - y_2y_1^2y_2 + y_2^2y_1^2,$$
$$y_1y_2y_1y_2 - y_1y_2^2y_1 - y_2y_1^2y_2 + y_2y_1y_2y_1,$$
$$2y_1^4 + y_1^3y_2 + y_1^2y_2y_1 + y_1y_2y_1^2 + 3y_1y_2^2y_1 + y_1y_2^3 + y_2y_1^3 + 3y_2y_1^2y_2 + y_2y_1y_2^2 + y_2^2y_1y_2 + y_2^3y_1 + 2y_2^4.$$

*which agree with the number of words $\{aaaa, abab, baba\}$ in the letters $a = (12)$ and $b = (132)$ which reduce to the identity in the Cayley graph $\Gamma(S_3, \{(12), (132)\})$ (see Example 2.4).*

**Proposition 3.8** *The number of free generators of $T(V^{(n-1,1)})^{S_n}$ as an algebra are counted by the words which reduce to the identity without crossing the identity in $\Gamma(S_n, \{(12), (132), \ldots, (1\,n \cdots 432)\})$.*

**Example 3.9** *The number of free generators of $T(V^{(2,1)})^{S_3}$ are counted by the number of words in the following subsets of words which reduce to the identity without crossing the identity in $\Gamma(S_3, \{(12), (132)\})$*

$$\{aa\}, \{bbb\}, \{abab, baba\}, \{abbba, baabb, bbaab\}, \{abaaab, abbabb, baaaba, babbab, bbabba\}, \ldots$$

*with cardinalities corresponding to the Fibonacci numbers.*

We present next a conjecture for a closed formula giving the Hilbert-Poincaré series of $T(V^{(n-1,1)})^{S_n}$ which does not seem to obviously follow from our combinatorial interpretations for the dimensions.

**Conjecture 3.10** *The Hilbert-Poincaré series of $T(V^{(n-1,1)})^{S_n}$ is*

$$P(T(V^{(n-1,1)})^{S_n}) = \frac{1}{1+q} + \frac{q}{1+q}\sum_{k=0}^{n-1}\frac{q^k}{(1-q)\,(1-2\,q)\cdots(1-k\,q)}.$$

## 4 Dihedral group $D_m$

The same kind of results can be observed for other finite groups, for example in the case of cyclic and dihedral groups. We will present in this section the case of the dihedral group $D_m$ with presentation $D_m = \langle s, r \mid s^2 = r^m = srsr = e\rangle$. We will give a combinatorial way to decompose the tensor algebra on any $D_m$-module into simple modules by looking to words in a particular Cayley graph of $D_m$. The bridge between those words and the decomposition of the tensor algebra into simple modules is made possible via a subalgebra of the group algebra $\mathbb{R}D_m$ and a surjective algebra morphism from this subalgebra into the algebra of characters that we will present in next section.

## 4.1 Simple $D_m$-modules

For our purpose, let us first compute the irreducible characters of the dihedral group $D_m$. For $m = 2\,k$ even, there are $k+3$ simple $D_m$-modules (up to isomorphisms) $V^{id}$, $V^{\gamma}$, $V^{\epsilon}$, $V^{\gamma\epsilon}$ and $V^{i}$, for $1 \le i \le k-1$ with associated irreducible characters

$$\begin{array}{rcl} id : D_m & \to & \mathbb{C} \\ r^{\eta} & \mapsto & 1 \\ s & \mapsto & 1 \\ rs & \mapsto & 1 \end{array} \qquad \begin{array}{rcl} \gamma : D_m & \to & \mathbb{C} \\ r^{\eta} & \mapsto & (-1)^{\eta} \\ s & \mapsto & -1 \\ rs & \mapsto & 1 \end{array} \qquad \begin{array}{rcl} \chi_i : D_m & \to & \mathbb{C} \\ r^{\eta} & \mapsto & 2\cos(\frac{2\pi\eta i}{m}) \\ s & \mapsto & 0 \\ rs & \mapsto & 0 \end{array}$$

$$\begin{array}{rcl} \epsilon : D_m & \to & \mathbb{C} \\ r^{\eta} & \mapsto & 1 \\ s & \mapsto & -1 \\ rs & \mapsto & -1 \end{array} \qquad \begin{array}{rcl} \gamma\epsilon : D_m & \to & \mathbb{C} \\ r^{\eta} & \mapsto & (-1)^{\eta} \\ s & \mapsto & 1 \\ rs & \mapsto & -1 \end{array}$$

For $m = 2\,k+1$ odd, the $k+2$ simple $D_m$-modules (up to isomorphisms) are $V^{id}$, $V^{\epsilon}$ and $V^{i}$, for $1 \le i \le k$ and the associated irreducible characters are respectively $id$, $\epsilon$ and $\chi_i$. The next two propositions define the surjective algebra morphism needed to link the decomposition of $T(V)$ to words in a Cayley graph of $D_m$.

**Proposition 4.1** *Let $y_i = r^{1-i}s + r^{i}$. For $m = 2k$ even, $\mathcal{Q} = \mathcal{L}\{e, r^k, rs, r^{k+1}s, y_i, y_i rs\}_{1\le i\le k-1}$ is a subalgebra of $\mathbb{Z}D_m$, and there is a surjective algebra morphism $\theta : \mathcal{Q} \to \mathbb{Z}\mathrm{Irr}(D_m)$ defined by $\theta(e) = id$, $\theta(rs) = \epsilon$, $\theta(r^k) = \gamma$, $\theta(r^{k+1}s) = \gamma\epsilon$ and $\theta(y_i) = \theta(y_i rs) = \chi_i$.*

**Proposition 4.2** *Let $y_i = r^{1-i}s + r^{i}$. For $m = 2k+1$ odd, the linear span $\mathcal{Q} = \mathcal{L}\{e, rs, y_i, y_i rs\}_{1\le i\le k}$ is a subalgebra of $\mathbb{Z}D_m$, and there is a surjective algebra morphism $\theta : \mathcal{Q} \to \mathbb{Z}\mathrm{Irr}(D_m)$ defined by $\theta(e) = id$, $\theta(rs) = \epsilon$ and $\theta(y_i) = \theta(y_i rs) = \chi_i$.*

## 4.2 Decomposition of $T(V)$ and words in a Cayley graph of $D_m$

To simplify the notation, we will denote the subalgebras of Proposition 4.1 and 4.2 by $\mathcal{Q} = \mathcal{L}\{b_i\}_{i\in I}$, where each element $b_i$ of the basis is sent to an irreducible character by $\theta$ and $V^{(i)}$ will denote a simple $D_m$-module with irreducible character $\chi^{(i)}$. As for the symmetric group, we have the following two results. Recall that $\mathrm{supp}(f) = \{g \in G | [g]f \neq 0\}$.

**Proposition 4.3** *Let $V$ be a $D_m$-module. If $f \in \mathcal{Q}$ is such that $\theta(f) = \chi^V$, then the multiplicity of $V^{(k)}$ in $V^{\otimes d}$ is equal to*

$$\sum_{\substack{b_i \\ \theta(b_i)=\chi^{(k)}}} [b_i] f^d.$$

**Theorem 4.4** *Let $V$ be a $D_m$-module. If $f \in \mathcal{Q}$ is such that $\theta(f) = \chi^V$, then the multiplicity of $V^{(k)}$ in $V^{\otimes d}$ is equal to*

$$\sum_{\substack{b_i \\ \theta(b_i)=\chi^{(k)}}} \sum_{w \in w(\sigma_i, d; \Gamma)} \omega(w),$$

*where $\sigma_i \in supp(b_i)$, $\Gamma = \Gamma(D_m, supp(f))$ with $\omega(g) = [g](f)$ for each $g \in supp(f)$.*

**Example 4.5** *Consider the $D_4$-module $(2\,V^1 \oplus V^{\gamma\epsilon})^{\otimes 2}$. By Theorem 4.1 , there is a subalgebra $\mathcal{Q} = \mathcal{L}\{e, r^2, rs, r^3s, s+r, r^3+r^2s\}$ of the group algebra and $\theta : \mathcal{Q} \to \mathbb{Z}\mathrm{Irr}(D_4)$ defined by*

$$\theta(e) = id, \quad \theta(rs) = \epsilon, \quad \theta(r^2) = \gamma, \quad \theta(r^3s) = \gamma\epsilon, \quad \theta(s+r) = \theta(r^3+r^2s) = \chi_1.$$

*Let $f = 2\,(r^3+r^2s)+r^3s$. Applying $\theta$, $f^2 = 5\,e+4\,rs+4\,r^2+4\,r^3s+2\,(s+r)+2\,(r^3+r^2s)$ is sent to $(2\,\chi_1+\gamma\epsilon)^2 = 5\,id+4\,\epsilon+4\,\gamma+4\,\gamma\epsilon+2\,\chi_1+2\,\chi_1$ so the decomposition into simple modules is*

$$(2\,V^1 \oplus V^{\gamma\epsilon})^{\otimes 2} = 5\,V^{id} \oplus 4\,V^{\epsilon} \oplus 4\,V^{\gamma} \oplus 4\,V^{\gamma\epsilon} \oplus 4V^1.$$

*These multiplicities can also be computed using words in the Cayley graph $\Gamma = \Gamma(D_4, \{r^3, r^2s, r^3s\})$ with weights $\omega(r^3) = \omega(r^2s) = 2$ and $\omega(r^3s) = 1$. Applying Theorem 4.4, the multiplicities are*

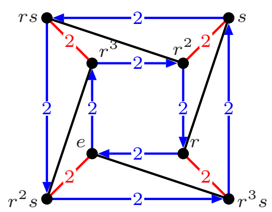
$$V^{id}: \sum_{w\in w(e,2;\Gamma)} \omega(w) = \omega(aa)+\omega(cc) = 2\cdot 2+1\cdot 1 = 5$$

$$V^{\epsilon}: \sum_{w\in w(rs,2;\Gamma)} \omega(w) = \omega(ba) = 2\cdot 2 = 4$$

$$V^{\gamma}: \sum_{w\in w(r^2,2;\Gamma)} \omega(w) = \omega(bb) = 2\cdot 2 = 4$$

$$V^{\gamma\epsilon}: \sum_{w\in w(r^3s,2;\Gamma)} \omega(w) = \omega(ab) = 2\cdot 2 = 4$$

$$V^{1}: \sum_{w\in w(r,2;\Gamma)} \omega(w) + \sum_{w\in w(r^3,2;\Gamma)} \omega(w) = \omega(ca)+\omega(ac) = 1\cdot 2+2\cdot 1 = 4.$$




## 4.3 Invariant algebra $T(V^1)^{D_m} \simeq \mathbb{R}\langle x_1, x_2\rangle^{D_m}$

We were particularly interested in studying the invariant space of the tensor algebra on the geometric representation $V^1$ and we have the following results. Since the dimension of $((V^1)^{\otimes d})^{D_m} \simeq \mathbb{R}\langle x_1, x_2\rangle_d^{D_m}$ is equal to the multiplicity of the trivial in $(V^1)^{\otimes d} \simeq \mathbb{R}\langle x_1, x_2\rangle_d$, the following Corollary follows from Theorem 4.4 and the fact that $\theta(s+r) = \chi_1$.

**Corollary 4.6** *The dimension of $((V^1)^{\otimes d})^{D_m} \simeq \mathbb{R}\langle x_1, x_2\rangle_d^{D_m}$ is equal to the number of words of length $d$ which reduce to the identity in the Cayley graph $\Gamma(D_m, \{r, s\})$.*

**Proposition 4.7** *The number of free generators of $T(V^1)^{D_m}$ as an algebra are counted by the words in the Cayley graph $\Gamma(D_m, \{r, s\})$ which reduce to the identity without crossing the identity.*

**Proposition 4.8** *The Hilbert-Poincaré series of $T(V^1)^{D_m} \simeq \mathbb{R}\langle x_1, x_2\rangle^{D_m}$ is*

$$P(T(V^1)^{D_m}) = 1 + \frac{1}{2}\left(\frac{(2q)^m + \sum_{i=0}^{\lfloor m/2\rfloor}\left(\binom{m+1}{2i+1} - 2\binom{m}{2i}\right)(1-4q^2)^i}{\sum_{i=0}^{\lfloor m/2\rfloor}\binom{m}{2i}(1-4q^2)^i - (2q)^m}\right).$$

# 5 Appendix

| $S_n\backslash d$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $S_3$ | 1 | 0 | 1 | 1 | 3 | 5 | 11 | 21 | 43 | 85 | 171 | 341 | 683 | 1365 |
| $S_4$ | 1 | 0 | 1 | 1 | 4 | 10 | 31 | 91 | 274 | 820 | 2461 | 7381 | 22144 | 66430 |
| $S_5$ | 1 | 0 | 1 | 1 | 4 | 11 | 40 | 147 | 568 | 2227 | 8824 | 35123 | 140152 | 559923 |
| $S_6$ | 1 | 0 | 1 | 1 | 4 | 11 | 41 | 161 | 694 | 3151 | 14851 | 71621 | 350384 | 1729091 |

**Tab. 1:** Dimension of $((V^{(n-1,1)})^{\otimes d})^{S_n} \simeq \mathbb{R}\langle Y_{n-1}\rangle_d^{S_n}$. Number of words of length $d$ which reduce to the identity in $\Gamma(S_n, \{(12), (132), (1432), \ldots, (1\, n\, \cdots\, 432)\})$.

| $d\backslash S_n$ | $S_3$ | $S_4$ | $S_5$ | $S_6$ |
|---|---|---|---|---|
| 2 | $aa$ | $aa$ | $aa$ | $aa$ |
| 3 | $bbb$ | $bbb$ | $bbb$ | $bbb$ |
| 4 | $aaaa$<br>$abab$<br>$baba$ | $aaaa$<br>$abab$ $cccc$<br>$baba$ | $aaaa$<br>$abab$ $cccc$<br>$baba$ | $aaaa$<br>$abab$ $cccc$<br>$baba$ |
| 5 | $aabbb$<br>$abbba$<br>$baabb$<br>$bbaab$<br>$bbbaa$ | $aabbb$ $accbc$<br>$abbba$ $bcacc$<br>$baabb$ $caccb$<br>$bbaab$ $cbcac$<br>$bbbaa$ $ccbca$ | $aabbb$ $accbc$<br>$abbba$ $bcacc$<br>$baabb$ $caccb$ $ddddd$<br>$bbaab$ $cbcac$<br>$bbbaa$ $ccbca$ | $aabbb$ $accbc$<br>$abbba$ $bcacc$<br>$baabb$ $caccb$ $ddddd$<br>$bbaab$ $cbcac$<br>$bbbaa$ $ccbca$ |

**Tab. 2:** Words of length $d$ in the letters $a = (12), b = (132), c = (1432), d = (15432)$ which reduce to the identity in $\Gamma(S_n, \{(12), (132), (1432), \ldots, (1\, n\, \cdots\, 432)\})$.

| $D_m\backslash d$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $D_3$ | 1 | 0 | 1 | 1 | 3 | 5 | 11 | 21 | 43 | 85 | 171 | 341 | 683 | 1365 |
| $D_4$ | 1 | 0 | 1 | 0 | 4 | 0 | 16 | 0 | 64 | 0 | 256 | 0 | 1024 | 0 |
| $D_5$ | 1 | 0 | 1 | 0 | 3 | 1 | 10 | 7 | 35 | 36 | 127 | 165 | 474 | 715 |
| $D_6$ | 1 | 0 | 1 | 0 | 3 | 0 | 11 | 0 | 43 | 0 | 171 | 0 | 683 | 0 |

**Tab. 3:** Dimension of $((V^1)^{\otimes d})^{D_m} \simeq \mathbb{R}\langle x_1, x_2\rangle_d^{D_m}$. Number of words in the letters $r$ and $s$ of length $d$ which reduce to the identity in $\Gamma(D_m, \{r, s\})$.

| $d\backslash D_m$ | $D_3$ | $D_4$ | $D_5$ | $D_6$ |
|---|---|---|---|---|
| 2 | $ss$ | $ss$ | $ss$ | $ss$ |
| 3 | $rrr$ | | | |
| 4 | $ssss$ $rsrs$<br>$srsr$ | $ssss$ $rsrs$<br>$srsr$ $rrrr$ | $ssss$ $rsrs$<br>$srsr$ | $ssss$ $rsrs$<br>$srsr$ |
| 5 | $ssrrr$ $rrssr$<br>$srrrs$ $rrrss$<br>$rssrr$ | | $rrrrr$ | |

**Tab. 4:** Words of length $d$ in the letters $r$ and $s$ which reduce to the identity in $\Gamma(D_m, \{r, s\})$.

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## References

[1] C. Chauve and A. Goupil. Combinatorial operators for Kronecker powers of representations $S_n$. *Séminaire Lotharingien de Combinatoire*, 54(Article B54j), 2006.

[2] H. S. M. Coxeter and W. O. J. Moser. *Generators and Relations for Discrete Groups*. Ergebnisse der Mathematik und Ihrer Grenzgebrete, New Series, no. 14. Berlin-Gottingen-Heidelberg, Springer, 1957.

[3] W. Dicks and E. Formanek. Poincaré series and a problem of S. Montgomery. *Linear and Multilinear Algebra*, 12(1):21–30, 1982/83.

[4] V.K. Kharchenko. Algebras of invariants of free algebras. *Algebra i logika*, 17:478–487, 1978.

[5] S. Poirier and C. Reutenauer. Algèbres de Hopf de tableaux. *Ann. Sci. Math. Québec*, 19:79–90, 1995.

[6] M. H. Rosas and B. E. Sagan. Symmetric functions in noncommuting variables. *Trans. Amer. Math. Soc.*, 358(1):215–232 (electronic), 2006.

[7] L. Solomon. A Mackey formula in the group ring of a Coxeter group. *J. Algebra*, 41:255–268, 1976.

[8] M. C. Wolf. Symmetric functions of non-commutative elements. *Duke Math. J.*, 2(4):626–637, 1936.