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Brauer-Schur functions

Kazuya Aokage[1]

[1] *Department of Mathematics, Okayama University, Okayama 700-8530, Japan*

A new class of functions is studied. We define the Brauer-Schur functions $B_\lambda^{(p)}$ for a prime number p, and investigate their properties. We construct a basis for the space of symmetric functions, which consists of products of p-Brauer-Schur functions and Schur functions. We will see that the transition matrix from the natural Schur function basis has some interesting numerical properties.

Keywords: Schur function, compound basis, transition matrix

1 Introduction

Let V denote the space of polynomials with infinitely many variables:

$$V = \mathbb{Q}[t_j; j \geq 1] = \bigoplus_{n=0}^{\infty} V_n,$$

where V_n is the subspace of homogeneous polynomials of degree n, with $\deg t_j = j$. The Schur functions form a basis for V. For a partition λ of n, the Schur function $S_\lambda(t)$ indexed by λ is defined by

$$S_\lambda(t) = \sum_\rho \chi_\rho^\lambda \frac{t_1^{m_1} t_2^{m_2} \cdots}{m_1! m_2! \cdots} \quad \in V_n.$$

Here the summation runs over all partitions $\rho = (1^{m_1} 2^{m_2} \cdots)$ of n, and the integer χ_ρ^λ is the irreducible character of λ of the symmetric group S_n, evaluated at the conjugacy class ρ. The "original" (symmetric) Schur function is recovered by putting

$$t_j = \frac{1}{j}(x_1^j + x_2^j + \cdots).$$

It is known that these Schur functions are ortho-normal with respect to the inner product

$$\langle F, G \rangle = F(\partial) G(t)|_{t=0},$$

where $\partial = (\frac{\partial}{\partial t_1}, \frac{1}{2}\frac{\partial}{\partial t_2}, \frac{1}{3}\frac{\partial}{\partial t_3}, \cdots)$.

In this paper we will consider yet another basis for V, which we call the compound basis. Our new basis comes from modular representations of the symmetric group at characteristic p. We will simply replace the character χ_ρ^λ by the p-Brauer character φ_ρ^λ. It is natural that the decomposition matrices play an essential role in the argument. The aim of this note is to investigate the transition matrices between Schur function basis and our compound basis.

For $p = 2$ the compound basis was introduced in (1) in connection with the basic representation of the affine Lie algebra of type $A_1^{(1)}$. In this case Schur's Q-functions are used. However that basis cannot be defined for odd primes p. Instead we consider here the functions $B_\lambda^{(p)}(t)$, which we call the "Brauer-Schur functions".

Throughout the note, $P(n)$ always denotes the set of partitions of n, and P denotes the set of all partitions.

2 The Symmetric Functions $B_\lambda^{(p)}$

We introduce a new family of symmetric functions. It has an origin in the modular representations of the symmetric groups (6). Let p be a fixed prime number. A partition $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_\ell)$ is said to be p-regular if there are no parts satisfying $\lambda_i = \lambda_{i+1} = \cdots = \lambda_{i+p-1}$. The set of p-regular partitions of n is denoted by $P^p(n)$. A partition $\rho = (1^{m_1}2^{m_2}\cdots)$ is said to be p-class regular if $m_p = m_{2p} = \cdots = 0$. The set of p-class regular partitions of n is denoted by $P_p(n)$. For example, a partition is 2-regular if it is strict, and 2-class regular if it is odd. If $p = 3$ and $n = 4$, then $P^3(4) = \{4, 31, 2^2, 21^2\}$ and $P_3(4) = \{4, 1^4, 2^2, 21^2\}$.
For $\lambda \in P^p(n)$, we define the Brauer-Schur function $B_\lambda^{(p)}(t)$ indexed by λ as follows.

$$B_\lambda^{(p)}(t) = \sum_{\rho \in P_p(n)} \varphi_\rho^\lambda \frac{t_1^{m_1} t_2^{m_2} \cdots}{m_1! m_2! \cdots} \quad \in V_n,$$

where φ_ρ^λ is the irreducible Brauer character for the symmetric group S_n of characteristic p corresponding to λ, evaluated at the conjugacy class ρ. These functions form a basis for the space $V_n^{(p)} = V^{(p)} \cap V_n$, where

$$V^{(p)} = \mathbb{Q}[t_j; j \geq 1, j \not\equiv 0 \,(\mathrm{mod}\, p)].$$

The p-decomposition matrix records the relation between ordinary irreducible characters and Brauer characters. Given a Schur function $S_\lambda(t)$, define the p-reduced Schur function $S_\lambda^{(p)}(t)$ by "killing" all variables $t_p, t_{2p}, \cdots$;

$$S_\lambda^{(p)}(t) = S_\lambda(t)|_{t_{jp}=0}.$$

By definition, the p-decomposition matrix $D_n^{(p)} = D_n = (d_{\lambda\mu})$ is given by

$$S_\lambda^{(p)}(t) = \sum_{\mu \in P^p(n)} d_{\lambda\mu} B_\mu^{(p)}(t)$$

for $\lambda \in P(n)$. It is known that the entries $d_{\lambda\mu}$ satisfies the properties; $d_{\lambda\mu} \in \mathbb{Z}_{\geq 0}$, $d_{\lambda\mu} = 0$ unless $\mu \geq \lambda$ and $d_{\lambda\lambda} = 1$. Here " $\geq$ " denotes the dominance order.
We define an inner product $\langle\, , \,\rangle$ on V_n by $\langle F(t), G(t)\rangle := F(\partial)G(t)|_{t=0}$, where $\partial = (\frac{\partial}{\partial t_1}, \frac{1}{2}\frac{\partial}{\partial t_2}, \frac{1}{3}\frac{\partial}{\partial t_3} \cdots)$. In contarst with the Schur functions which are ortho-normal with respect to this inner product, our

p-Brauer-Schur functions are not orthogonal in general. Therefore we need the dual basis for the p-Brauer-Schur functions $B_\lambda^{(p)}(t)$. To this end we introduce another symmetric functions $\widetilde{B_\lambda^{(p)}(t)}$ indexed by $\lambda \in P^p(n)$ as follows.

$$\widetilde{B_\lambda^{(p)}}(t) = \sum_{\rho \in P_p(n)} \widetilde{\varphi_\rho^\lambda} \frac{t_1^{m_1} t_2^{m_2} \cdots}{m_1! m_2! \cdots} \quad \in V_n,$$

where $\widetilde{\varphi_\rho^\lambda}$ are the entries of the matrix

$$\widetilde{\Psi}_n := {}^t D_n D_n \Psi_n$$

with $\Psi_n = (\varphi_\rho^\lambda)_{\lambda \in P^p(n), \rho \in P_p(n)}$. Then the orthogonality of the Brauer characters implies

$$\langle B_\lambda^{(p)}(t), \widetilde{B_\mu^{(p)}}(t) \rangle = \langle \widetilde{B_\lambda^{(p)}(t)}, B_\mu^{(p)}(t) \rangle = \delta_{\lambda\mu}.$$

It is known that

$$B_\lambda^{(p)}(t) = S_\lambda^{(p)}(t) = S_\lambda(t)$$

for a p-core λ, and hence, it is a homogeneous τ-function for the p-reduction of KP hierarchy (9; 4).

Here these functions are being expressed in terms of the "Sato variables" $t = (t_1, t_2, \ldots)$ appearing in the theory of soliton equations. However, for the description and the proof of our formula, it is sometimes more convenient to use the "original" variables of the symmetric functions, i.e., the "eigenvalues" $x = (x_1, x_2, \ldots)$. The variables are connected by the formula

$$t_j = \frac{1}{j}(x_1^j + x_2^j + \cdots).$$

We will denote by $B_\lambda^{(p)}(x)$ etc. when the functions are expressed in terms of variables x.

First we notice the following Cauchy identity.

Proposition 2.1

$$\sum_{\lambda \in P^p} B_\lambda^{(p)}(px) \widetilde{B_\lambda^{(p)}}(y) = \prod_{i,j} \frac{1 - x_i^p y_j^p}{(1 - x_i y_j)^p},$$

where $(px) := (\underbrace{x_1, \ldots x_1}_{p}, \underbrace{x_2, \ldots x_2}_{p}, \ldots)$.

Proof: It is known that the Schur functions form a selfdual basis for the space V with respect to the inner product $\langle\ ,\ \rangle$. Hence we have the well-known Cauchy identity

$$\sum_{\lambda \in P} S_\lambda(x) S_\lambda(y) = \prod_{i,j} \frac{1}{1 - x_i y_j} = \exp(\sum_{i,j} t_i {y_j}^i).$$

Our functions $B_\lambda^{(p)}(t)$ and $\widetilde{B_\lambda^{(p)}}(t)$ are dual bases for the p-reduced space $V^{(p)}$. Therefore we have

$$\sum_{\lambda \in P^p} B_\lambda^{(p)}(px) \widetilde{B_\lambda^{(p)}}(y) = \exp(\sum_{j \geq 1} \sum_{n \not\equiv 0 (mod\ p)} p t_n y_j^n).$$

The right-hand side equals

$$\begin{aligned}\exp(\sum_{j\geq 1}\sum_{n\not\equiv 0 (mod\ p)} pt_n y_j^n) &= \exp(\sum_{j\geq 1}\sum_{n\geq 1} pt_n y_j^n - \sum_{j\geq 1}\sum_{n\geq 1} pt_{pn} y_j^{pn}) \\ &= \prod_{i,j} \frac{1-x_i^p y_j^p}{(1-x_i y_j)^p}.\end{aligned}$$

□

3 Compound Basis

We begin with three bijections among sets of partitions. We here remark that these bijections can be defined for any nutural number p. The first bijection is

$$\psi^{(p)} : P(n) \longrightarrow \bigcup_{n_1+pn_2=n} P^p(n_1) \times P(n_2),$$

defined by $\lambda \longmapsto (\lambda^{r(p)}, \lambda^{d(p)})$. Here the multiplicities $m_i(\lambda^{r(p)})$ and $m_i(\lambda^{d(p)})$ of $i \geq 1$ are given respectively by

$$m_i(\lambda^{r(p)}) = \begin{cases} k & \text{if} \quad m_i(\lambda) \equiv k_{\neq 0} \pmod p \\ 0 & \text{if} \quad m_i(\lambda) \equiv 0 \pmod p, \end{cases}$$

and

$$m_i(\lambda^{d(p)}) = \begin{cases} \frac{m_i(\lambda)-k}{p} & \text{if} \quad m_i(\lambda) \equiv k_{\neq 0} \pmod p \\ \frac{m_i(\lambda)}{p} & \text{if} \quad m_i(\lambda) \equiv 0 \pmod p. \end{cases}$$

For example, if $p = 3$ and $\lambda = (5^4 4^6 2^{11} 1^2)$, then $\lambda^{r(3)} = (52^2 1^2)$, $\lambda^{d(3)} = (54^2 2^3)$.
In view of this bijection, we can define the function, for a prime p and $\lambda \in P(n)$,

$$B^{(p)}_{\lambda^{r(p)}}(t) S_{\lambda^{d(p)}}(t_{(p)}), \qquad t_{(p)} = (t_p, t_{2p}, t_{3p}, \cdots).$$

These functions are linearly independent, and therefore, form a basis for the space V_n. We call $\{B^{(p)}_{\lambda^{r(p)}}(t) S_{\lambda^{d(p)}}(t_{(p)}); \lambda \in P(n)\}$ the "p-compound basis" for V_n.
The second bijection reads

$$\pi^{(p)} : P(n) \longrightarrow \bigcup_{n_1+pn_2=n} P_p(n_1) \times P(n_2), \ \ \lambda \longmapsto (\lambda^{o(p)}, \lambda^{e(p)}),$$

where $\lambda^{o(p)}$ is obtained by removing all parts from λ which are multiples of p, and $\lambda^{e(p)} := (1^{m_p} 2^{m_{2p}} 3^{m_{3p}} \ldots)$ if $\lambda = (1^{m_1} 2^{m_2} 3^{m_3} \ldots)$. For example, if $p = 3$ and $\lambda = (7^4 6^3 4^5 3\, 2^6 1^2)$, then $\lambda^{o(p)} = (7^4 4^5 2^6 1^2)$, $\lambda^{e(p)} = (2^3 1)$.

The last bijection is called the Glaisher map. Let $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ be a p-reguler partition. Write each part as $\lambda_i = p^{a_i} q_i$ with $(p, q_i) = 1$. Let $\mu(i)$ be the rectanguler partition of λ_i given by $\mu(i) = (q_i, \ldots, q_i)$

with length p^{a_i}. Suppose that $q_{j_1} \geq \ldots \geq q_{j_l}$. Let $\tilde{\lambda}$ be the vertical concatenation $(\mu(j_1), \ldots, \mu(j_l))$, which is p-class reguler. Then the bijection $\gamma : P^p(n) \longrightarrow P_p(n)$ is defined by $\lambda \longmapsto \tilde{\lambda}$. For example, if $p = 3$ and $\lambda = (6^2 5^3 431)$, then $\tilde{\lambda} = (5^3 42^6 1^4)$.
By composing these three bijections, we can define the map

$$\Phi^{(p)} : P(n) \longrightarrow P(n), \;\; \Phi^{(p)}(\lambda) := \pi^{(p)^{-1}}(\gamma \otimes id)(\psi^{(p)}(\lambda)).$$

For example, here is a table of the case $p = 3$ and $n = 6$.

$P(n)$	$\longrightarrow$	$P^p(n_1) \times P(n_2)$	$\longrightarrow$	$P_p(n_1) \times P(n_2)$	$\longrightarrow$	$P(n)$
6	$\longmapsto$	$6, \emptyset$	$\longmapsto$	$2^3, \emptyset$	$\longmapsto$	2^3
51	$\longmapsto$	$51, \emptyset$	$\longmapsto$	$51, \emptyset$	$\longmapsto$	51
42	$\longmapsto$	$42, \emptyset$	$\longmapsto$	$42, \emptyset$	$\longmapsto$	42
41^2	$\longmapsto$	$41^2, \emptyset$	$\longmapsto$	$41^2, \emptyset$	$\longmapsto$	41^2
3^2	$\longmapsto$	$3^2, \emptyset$	$\longmapsto$	$1^6, \emptyset$	$\longmapsto$	1^6
321	$\longmapsto$	$321, \emptyset$	$\longmapsto$	$21^4, \emptyset$	$\longmapsto$	21^4
$2^2 1^2$	$\longmapsto$	$2^2 1^2, \emptyset$	$\longmapsto$	$2^2 1^2, \emptyset$	$\longmapsto$	$2^2 1^2$
31^3	$\longmapsto$	$3, 1$	$\longmapsto$	$1^3, 1$	$\longmapsto$	31^3
21^4	$\longmapsto$	$21, 1$	$\longmapsto$	$21, 1$	$\longmapsto$	321
2^3	$\longmapsto$	$\emptyset, 2$	$\longmapsto$	$\emptyset, 2$	$\longmapsto$	6
1^6	$\longmapsto$	$\emptyset, 1^2$	$\longmapsto$	$\emptyset, 1^2$	$\longmapsto$	3^2

For a pair (n_1, n_2) with $n_1 + pn_2 = n$, we call the set $\psi^{(p)^{-1}}(P^p(n_1) \times P(n_2))$ the "(n_1, n_2)-block". Looking at the table above, we notice the following relations for lengths.

Proposition 3.1

$$(i) \quad \sum_{\lambda \in P(n)} \ell(\lambda) = \sum_{\lambda \in P(n)} (\ell(\lambda^{r(p)}) + p\ell(\lambda^{d(p)})) = \sum_{\lambda \in P(n)} (\ell(\lambda^{o(p)}) + \ell(\lambda^{e(p)}))$$
$$= \sum_{\lambda \in P(n)} (\ell(\widetilde{\lambda^{r(p)}}) + \ell(\lambda^{e(p)})),$$

$$(ii) \quad \frac{\ell(\widetilde{\lambda^{r(p)}}) - \ell(\lambda^{r(p)})}{p-1} = \ell((\Phi^{(p)}(\lambda))^{d(p)}).$$

4 Transittion Matrices

We investigate the transition matrix between two bases. Let $A_n^{(p)} := (a_{\lambda\mu})_{\lambda,\mu \in P(n)}$ be defined by

$$S_\lambda(t) = \sum_{\mu \in P(n)} a_{\lambda\mu} B^{(p)}_{\mu^{r(p)}}(t) S_{\mu^{d(p)}}(t_{(p)}), \;\; \lambda \in P(n).$$

We see that the transition matrix $A_n^{(p)}$ is an integral matrix, and that the determinant of $A_n^{(p)}$ has a combinatorial interpretation. The definition of $a_{\lambda\mu}$ is rewritten as

$$S_\lambda(px) = \sum_{\mu \in P(n)} a_{\lambda\mu} B^{(p)}_{\mu^{r(p)}}(px) S_{\mu^{d(p)}}(x^p),$$

where $(x^p) := (x_1^p, x_2^p, \ldots)$.

Proposition 4.1

$$\sum_{\lambda\in P} S_\lambda(px)S_\lambda(y) = \sum_{\lambda\in P} B^{(p)}_{\lambda^{r(p)}}(px)S_{\lambda^{d(p)}}(x^p)\widetilde{B^{(p)}_{\lambda^{r(p)}}}(y)S_{\lambda^{d(p)}}(y^p).$$

Proof: By looking at the Cauchy identity for the Schur functions, we see that

$$\sum_{\lambda\in P} S_\lambda(x^p)S_\lambda(y^p) = \prod_{i,j}\frac{1}{1-x_i^p y_j^p}.$$

Hence, from Proposition 2.1, we have

$$\begin{aligned}
&\sum_{\lambda\in P} B^{(p)}_{\lambda^{r(p)}}(px)S_{\lambda^{d(p)}}(x^p)\widetilde{B^{(p)}_{\lambda^{r(p)}}}(y)S_{\lambda^{d(p)}}(y^p)\\
=\ &\sum_{\mu\in P^p} B^{(p)}_{\mu}(px)\widetilde{B^{(p)}_{\mu}}(y)\sum_{\nu\in P} S_\nu(x^p)S_\nu(y^p)\\
=\ &\prod_{i,j}\frac{1-x_i^p y_j^p}{(1-x_iy_j)^p}\times\prod_{i,j}\frac{1}{1-x_i^p y_j^p}\\
=\ &\prod_{i,j}\frac{1}{(1-x_iy_j)^p}.
\end{aligned}$$

$\square$

Theorem 4.2 *The entries $a_{\lambda\mu}$ are integers given by*

$$a_{\lambda\mu} = \langle\widetilde{B^{(p)}_{\lambda^{r(p)}}}(y)S_{\lambda^{d(p)}}(y^p),\ S_\mu(y)\rangle.$$

Proof: We have

$$\sum_{\lambda\in P} S_\lambda(px)S_\lambda(y) = \sum_{\lambda\in P} B^{(p)}_{\lambda^{r(p)}}(px)S_{\lambda^{d(p)}}(x^p)\widetilde{B^{(p)}_{\lambda^{r(p)}}}(y)S_{\lambda^{d(p)}}(y^p).$$

Taking the inner product $\langle\ ,\ \rangle$ with $S_\mu(y)$, we obtain

$$S_\mu(px) = \sum_{\lambda\in P}\langle\widetilde{B^{(p)}_{\lambda^{r(p)}}}(y)S_{\lambda^{d(p)}}(y^p), S_\mu(y)\rangle\ B^{(p)}_{\lambda^{r(p)}}(px)S_{\lambda^{d(p)}}(x^p).$$

Thus we see that

$$a_{\lambda\mu} = \langle\widetilde{B^{(p)}_{\lambda^{r(p)}}}(y)S_{\lambda^{d(p)}}(y^p), S_\mu(y)\rangle.$$

Here we use the following formula of plethysm, which is found in (3).

$$S_{\lambda^{d(p)}}(x^p) = \sum_{\mu \in P} c^{\mu}_{\lambda^{d(p)} p} S_\mu(x), \quad c^{\mu}_{\lambda p} \in \mathbb{Z}.$$

Also, by definition, we have

$$\widetilde{B^{(p)}_{\lambda^{r(p)}}}(y) = \sum_{\mu \in P(n)} d_{\mu \lambda^{r(p)}} S^{(p)}_\mu(y).$$

This shows that

$$\widetilde{B^{(p)}_{\lambda^{r(p)}}}(y) = \sum_{\mu \in P(n)} d_{\mu \lambda^{r(p)}} S_\mu(y).$$

From the orthonormality of the Schur functions, the assertion holds. $\square$

Here we give the matrix $A^{(p)}_n$ for the case $p = 3$ and $n = 5$. Columns are labeled by $(\mu^{r(3)}, \mu^{d(3)})$.

$$A^{(3)}_5 = \begin{array}{c|ccccccc} & (5,\emptyset) & (2^2 1,\emptyset) & (41,\emptyset) & (32,\emptyset) & (31^2,\emptyset) & (2,1) & (1^2,1) \\ \hline (5) & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ (41) & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ (32) & 0 & 0 & 1 & 1 & 0 & 0 & -1 \\ (31^2) & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ (2^2 1) & 1 & 1 & 0 & 0 & 0 & -1 & 0 \\ (21^3) & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ (1^5) & 0 & 0 & 0 & 1 & 0 & 0 & 1 \end{array}.$$

If we expand the Schur function $S_\lambda(x)$ and the p-Brauer-Schur function $B^{(p)}_\lambda(x)$ in terms of the power sum symmetric functions, then we have

$$S_\lambda(x) = \sum_{\rho \in P(n)} Y_{\lambda,\rho} p_\rho(x), \;\; B^{(p)}_\lambda(x) = \sum_{\rho \in P_p(n)} B^{(p)}_{\lambda,\rho} p_\rho(x).$$

Put

$$Y_n := (Y_{\lambda,\rho})_{\lambda,\rho \in P(n)}, \quad B^{(p)}_n := (B^{(p)}_{\lambda,\rho})_{\lambda \in P^p(n), \rho \in P_p(n)}.$$

We will remark that

$$B^{(p)}_n = \Psi_n Z_n^{-1},$$

where $Z_n := \operatorname{diag}(z_\rho; \rho \in P_p(n))$ with $z_\rho := 1^{m_1} 2^{m_2} \cdots m_1! m_2! \cdots$ for $\rho = (1^{m_1} 2^{m_2} \cdots)$.

Now we are going to discuss determinants of the transition matrices. Let the symbol "det" mean the absolute value of the determinant. By a standard argument, we have

$$1 = (\det Y_n)^2 \prod_{\rho \in P(n)} z_\rho.$$

Here $M(W, U)$ denotes the transition matrix from the basis W to the basis U for V_n. We compute

$$\begin{aligned} \det M(S(px), B^{(p)}(px) S(x^p)) &= \det M(S(px), p(px)) \det M(p(px), B^{(p)}(px) S(x^p)) \\ &= \det M(S(x), p(x)) \det M(p(px), B^{(p)}(px) S(x^p)) \\ &= \det M(S(x), p(x)) \det M(p(px), p(x)) \det M(p(x), B^{(p)}(px) S(x^p)). \end{aligned}$$

Also, for $\lambda \in P^p(n_1), \mu \in P(n_2)$, we write

$$\begin{aligned} B_\lambda^{(p)}(px)S_\mu(x^p) &= \sum_{\rho \in P_p(n_1), \sigma \in P(n_2)} B_{\lambda,\rho}^{(p)} Y_{\mu,\sigma} p_\rho(px) p_\sigma(x^p) \\ &= \sum_{\rho \in P_p(n_1), \sigma \in P(n_2)} B_{\lambda,\rho}^{(p)} Y_{\mu,\sigma} p^{\ell(\rho)} p_\rho(x) p_{p\sigma}(x), \end{aligned}$$

where $p\sigma := (p\sigma_1, p\sigma_2, \ldots)$. This shows that the matrix

$$M(B^{(p)}(px)S(x^p), p(x))$$

is block diagonal and each block is indexed by the pair (n_1, n_2) with $n_1 + pn_2 = n$.

Proposition 4.3

$$\det M(B^{(p)}(px)S(x^p), p(x)) = \prod_{n_1+pn_2=n} (\det B_{n_1}^{(p)})(\det Y_{n_2})(\det L_{n_1}).$$

where $L_n = \operatorname{diag}(p^{\ell(\rho)}; \rho \in P_p(n))$.

There is a compact formula for the elementary divisors of the Cartan matrix $C_n = {}^t D_n D_n$ (10):

$$\{p^{\frac{\ell(\tilde{\lambda})-\ell(\lambda)}{p-1}}; \lambda \in P^p(n)\}.$$

Clearly,

$$\det C_n = \prod_{\lambda \in P^p(n)} p^{\frac{\ell(\tilde{\lambda})-\ell(\lambda)}{p-1}}.$$

Proposition 4.4

$$\prod_{\rho \in P_p(n)} z_\rho = (\det \Psi_n)^2 \times \prod_{\lambda \in P^p(n)} p^{\frac{\ell(\tilde{\lambda})-\ell(\lambda)}{p-1}}.$$

Our main theorem involves an interesting combinatorial fact.

Theorem 4.5

$$\det A_n^{(p)} = p^T.$$

where

$$T = \sum_{\lambda \in P^p(n)} \frac{\ell(\tilde{\lambda}) - \ell(\lambda)}{p-1} = \sum_{\lambda \in P(n)} \ell(\lambda^{d(p)}),$$

which is the sum of the number of parts of multiples of p *in the partitions of* n.

Proof: We recall

$$\begin{aligned} \det A_n^{(p)} &= \det M(S(px), B^{(p)}(px)S(x^p)) \\ &= \det M(S(x), p(x)) \det M(p(px), p(x)) \det M(p(x), B^{(p)}(px)S(x^p)). \end{aligned}$$

By Propositions 4.3 and 4.4, we see that

$$
\begin{aligned}
& \det M(p(x), B^{(p)}(px)S(x^p)) \\
= & \left(\prod_{n_1+pn_2=n} (\det B_{n_1})(\det Y_{n_2})(\det L_{n_1}) \right)^{-1} \\
= & \prod_{n_1+pn_2=n} \left((\det \Psi_{n_1}) \left(\prod_{\sigma\in P_p(n_1)} z_\sigma^{-1} \right) (\det Y_{n_2}) \left(\prod_{\rho\in P(n_1)} p^{\ell(\rho)} \right) \right)^{-1} \\
= & \prod_{n_1+pn_2=n} \left(\left(\prod_{\rho\in P_p(n_1)} p^{\ell(\rho)} \right) \left(\prod_{\sigma\in P_p(n_1)} z_\sigma^{-1} \right) \right)^{-1} \\
\times & \quad \left(\left(\prod_{\rho\in P_p(n_1)} z_\rho^{-1} \right) \left(\prod_{\lambda\in P^p(n_1)} p^{\frac{\ell(\tilde\lambda)-\ell(\lambda)}{p-1}} \right) \right)^{1/2} \left(\prod_{\rho\in P(n_2)} z_\rho \right)^{1/2} .
\end{aligned}
$$

Hence we have

$$
\begin{aligned}
\det A_n^{(p)} = & \prod_{\rho\in P(n)} z_\rho^{-1/2} \times \prod_{\rho\in P(n)} p^{\ell(\rho)} \\
& \times \prod_{n_1+pn_2=n} \left(\left(\prod_{\rho\in P_p(n_1)} p^{\ell(\rho)} \right) \left(\prod_{\sigma\in P_p(n_1)} z_\sigma^{-1} \right) \right)^{-1} \\
& \quad \times \left(\left(\prod_{\rho\in P_p(n_1)} z_\rho^{-1} \right) \left(\prod_{\lambda\in P^p(n_1)} p^{\frac{\ell(\tilde\lambda)-\ell(\lambda)}{p-1}} \right) \right)^{1/2} \left(\prod_{\rho\in P(n_2)} z_\rho \right)^{1/2} .
\end{aligned}
$$

Paying attention to the bijection $\pi^{(p)}$ and the relation

$$z_\lambda = p^{\ell_p(\lambda)} z_{\lambda^{o(p)}} z_{\lambda^{e(p)}},$$

where $\ell_p(\lambda)$ denotes the number of parts of multiples of p in the partition λ, we notice that

$$\left(\prod_{\rho\in P(n)} z_\rho^{-1/2} \right) \times \left(\prod_{n_1+pn_2=n} \prod_{\sigma\in P_p(n_1)} z_\sigma \right) \times \left(\prod_{n_1+pn_2=n} \prod_{\rho\in P_p(n_1)} z_\rho^{-1/2} \right) \times \left(\prod_{n_1+pn_2=n} \prod_{\rho\in P(n_2)} z_\rho^{1/2} \right)$$

is equal to $(p^T)^{-1/2}$. Next, we look at

$$\prod_{\rho\in P(n)} p^{\ell(\rho)} \times \prod_{n_1+pn_2=n} \left(\prod_{\sigma\in P_p(n_1)} p^{-\ell(\sigma)} \right) \times \prod_{n_1+pn_2=n} \left(\prod_{\lambda\in P^p(n_1)} p^{\frac{\ell(\tilde\lambda)-\ell(\lambda)}{p-1}} \right)^{1/2} .$$

Through the bijection $\pi^{(p)}$, we have

$$\prod_{\rho\in P(n)} p^{\ell(\rho)} \times \prod_{n_1+pn_2=n}\left(\prod_{\sigma\in P_p(n_1)} p^{-\ell(\sigma)}\right) = p^T.$$

Also, from Proposition 3.1 (ii), we obtain

$$\prod_{n_1+pn_2=n}\left(\prod_{\lambda\in P^p(n_1)} p^{\frac{\ell(\tilde{\lambda})-\ell(\lambda)}{p-1}}\right)^{1/2} = \left(\prod_{\tau\in P(n)} p^{\ell(\tau^{d(p)})}\right)^{1/2} = (p^T)^{1/2}.$$

Hence, $\det A_n^{(p)} = (p^T)^{-1/2}(p^T)(p^T)^{1/2} = p^T$. $\square$

For example, we have $\det A_5^{(3)} = 9 = 3^2$. Here is a small list of T for $p = 3$.

n	1	2	3	4	5	6	7	$8\cdots$
T	0	0	1	1	2	5	7	$11\cdots$

We also see that the elementary divisors of $A_n^{(p)}$ coincide with

$$\{p^{\frac{\ell(\tilde{\mu})-\ell(\mu)}{p-1}}; \mu = \lambda^{r(p)}, \lambda\in P(n)\}.$$

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