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A preorder-free construction of the Kazhdan-Lusztig representations of S_n, with connections to the Clausen representations

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Abstract. We use the polynomial ring $\mathbb{C}[x_{1,1},\ldots,x_{n,n}]$ to modify the Kazhdan-Lusztig construction of irreducible S_n-modules. This modified construction produces exactly the same matrices as the original construction in [*Invent. Math* **53** (1979)], but does not employ the Kazhdan-Lusztig preorders. We also show that our modules are related by unitriangular transition matrices to those constructed by Clausen in [*J. Symbolic Comput.* **11** (1991)]. This provides a $\mathbb{C}[x_{1,1},\ldots,x_{n,n}]$-analog of results of Garsia-McLarnan in [*Adv. Math.* **69** (1988)].

Résumé. Nous utilisons l'anneau $\mathbb{C}[x_{1,1},\ldots,x_{n,n}]$ pour modifier la construction Kazhdan-Lusztig des modules-S_n irreductibles dans $\mathbb{C}[S_n]$. Cette construction modifiée produit exactement les mêmes matrices que la construction originale dans [*Invent. Math* **53** (1979)], mais sans employer les préordres de Kazhdan-Lusztig. Nous montrons aussi que nos modules sont relies par des matrices unitriangulaires aux modules construits par Clausen dans [*J. Symbolic Comput.* **11** (1991)]. Ce résultat donne un $\mathbb{C}[x_{1,1},\ldots,x_{n,n}]$-analogue des résultats de Garsia-McLarnan dans [*Adv. Math.* **69** (1988)].

Keywords: Kazhdan-Lusztig, immanants, irreducible representations, symmetric group

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1 Introduction

In 1979, Kazhdan and Lusztig introduced [8] a family of irreducible modules for Coxeter groups and related Hecke algebras. These modules, which have many fascinating properties, also aid in the understanding of modules for quantum groups and other algebras. Important ingredients in the construction of the Kazhdan-Lusztig modules are the computation of certain polynomials in $\mathbb{Z}[q]$ known as *Kazhdan-Lusztig polynomials*, and the description of preorders on Coxeter group elements known as the *Kazhdan-Lusztig preorders*. These two tasks, which present something of an obstacle to one wishing to construct the modules, have become fascinating research topics in their own right. Even in the simplest case of a Coxeter group, the symmetric group S_n, the Kazhdan-Lusztig polynomials and preorders are somewhat poorly understood.

As an alternative to the "traditional" Kazhdan-Lusztig construction of type-A modules in terms of subspaces of the type-A Hecke algebra $H_n(q)$ (or of its specialization S_n), one may construct modules in terms of subspaces of a noncommutative "quantum polynomial ring" (or of its specialization $\mathbb{C}[x_{1,1},\ldots,x_{n,n}]$). Theoretically, this alternative offers no special advantage over the original construction. On the other hand, a simple modification of this alternative completely eliminates the need for the Kazhdan-Lusztig preorders in a new construction of S_n modules.

In Sections 2-3, we review essential definitions for the symmetric group, Hecke algebra, and Kazhdan-Lusztig modules. In Section 4 we review definitions related to the polynomial ring $\mathbb{C}[x_{1,1},\ldots,x_{n,n}]$ and a particular $n!$-dimensional subspace of $\mathbb{C}[x_{1,1},\ldots,x_{n,n}]$ called the *immanant space*. We recall the definition of the bideterminant basis of the immanant space, which gained notoriety in the work of Désarménien, Kung, and Rota [3], and Clausen's use of this basis to construct irreducible S_n-modules [2]. In Section 5, we use the basis of Kazhdan-Lusztig immanants studied in [12] to transfer the traditional Kazhdan-Lusztig representations to the immanant space.

Aspects of Clausen's work will then motivate us to modify the above representations in Section 6 and to apply vanishing properties of Kazhdan-Lusztig immanants obtained in [13]. This leads to our main result that the resulting representations, which do not rely upon the Kazhdan-Lusztig preorders, have matrices equal to those corresonding to the original Kazhdan-Lusztig representations in [8]. We finish in Section 7 by showing that the relationship between the bideterminant and Kazhdan-Lusztig immanant bases studied in [13] leads to unitriangular transition matrices relating Clausen's irreducible representations of S_n to those of Kazhdan-Lusztig. This provides an analog in $\mathbb{C}[x_{1,1},\ldots,x_{n,n}]$ of the Garsia-McLarnan result [6, Thm. 5.3] relating Young's natural representations to those of Kazhdan-Lusztig in $\mathbb{C}[S_n]$.

2 Tableaux and the symmetric group

We call a weakly decreasing sequence $\lambda = (\lambda_1,\ldots,\lambda_\ell)$ of positive integers with $\sum_{i=1}^{\ell}\lambda_i = r$ an *integer partition* of r, and we denote this by $\lambda \vdash r$ or $|\lambda| = r$. A partial ordering on integer partitions of r called *dominance order* is given by $\lambda \succeq \mu$ if and only if

$$\lambda_1+\cdots+\lambda_i \geq \mu_1+\cdots+\mu_i, \text{ for all } i = 1,\ldots,\ell. \tag{1}$$

From an integer partition λ we can construct a *Ferrers diagram* which has λ_i left justified dots in row i. When we replace the dots in a diagram with $1,\ldots,r$ we have a *Young tableau* where the *shape* of the tableau is λ. An *injective tableau* is merely one in which the replacing is performed injectively, i.e. the $1,\ldots,r$ appear exactly once in the tableau. We call a tableau *column-(semi)strict* if its entries are (weakly) increasing downward in columns. A tableau is *row-(semi)strict* if entries (weakly) increase from left to right in rows. We call a tableau *semistandard* if it is column-strict and row-semistrict, and *standard* if it is semistandard and injective. We define transposition of partitions $\lambda \mapsto \lambda^\top$ (also known as conjugation) and tableaux $T \mapsto T^\top$ in a manner analogous to matrix transposition. We define a *bitableau* to be a pair of tableaux of the same shape, and say that it posesses a certain tableau property if both of its tableaux posess this property.

For each partition λ we define the *superstandard tableau* of shape λ to be the tableau $U(\lambda)$ having entries in reading order. For example,

$$U((4,2,1)) = \begin{matrix} 1 & 2 & 3 & 4 \\ 5 & 6 & & \\ 7 & & & \end{matrix}\,. \tag{2}$$

The standard presentation of S_n is given by generators $s_1,\ldots,s_{n-1}$ and relations

$$\begin{aligned} s_i^2 &= 1, && \text{for } i = 1,\ldots,n-1, \\ s_is_js_i &= s_js_is_j, && \text{if } |i-j| = 1, \\ s_is_j &= s_js_i, && \text{if } |i-j| \geq 2. \end{aligned} \tag{3}$$

Let S_n act on rearrangements of the letters $[n] = \{1,\ldots,n\}$ by

$$s_i \circ v_1\cdots v_n \underset{\text{def}}{=} v_1\cdots v_{i-1}v_{i+1}v_iv_{i+2}\cdots v_n. \tag{4}$$

For each permutation $w = s_{i_1} \cdots s_{i_\ell} \in S_n$ we define the *one-line notation* of w to be the word

$$w_1 \cdots w_n \underset{\text{def}}{=} s_{i_1} \circ (\cdots (s_{i_\ell} \circ (1 \cdots n)) \cdots). \tag{5}$$

For each $w \in S_n$ we define two tableaux, $P(w), Q(w)$ which are obtained from the *Robinson-Schensted correspondence* using row insertion to the one-line notation of w. (See, e.g., [14, Sec. 3.1].) It is well known that these tableaux satisfy $P(w^{-1}) = Q(w)$. Since $\mathrm{sh}(P(w)) = \mathrm{sh}(Q(w))$ we can define the *shape of a permutation* as $\mathrm{sh}(w) = \mathrm{sh}(P(w))$.

Given a permutation $w \in S_n$ expressed in terms of generators $w = s_{i_1} \cdots s_{i_\ell}$ we say this expression is *reduced* if w cannot be expressed as a shorter product of generators of S_n. We call the *length* of a permutation $w \in S_n$ $\ell(w) = \ell$, in the previous equation. We define the *Bruhat order* on S_n by $v \leq w$ if some (equivalently every) reduced expression for w contains a reduced expression for v as a subword (The reader is referred to [1] for more on this topic). Throughout this paper we will use w_0 to denote the unique maximal element in the Bruhat order. Multiplying a permutation on the right by w_0 also changes the bitableau of the Robinson-Schensted correspondence for that permutation. Specifically, this change can be described in terms of transposition and Schützenberger's evacuation algorithm. (See [1, Appendix].)

Lemma 2.1 *If $v \in S_n$, then $P(v) = \mathit{evac}(P(vw_0))^\top$.*

3 Kazhdan-Lusztig representations

Given an indeterminate q we define the *Hecke algebra*, $H_n(q)$, to be the $\mathbb{C}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$-algebra with multiplicative identity $\widetilde{T}_e$ generated by $\{\widetilde{T}_{s_i}\}_{i=1}^{n-1}$ with relations

$$\widetilde{T}_{s_i}^2 = (q^{\frac{1}{2}} - q^{-\frac{1}{2}})\widetilde{T}_{s_i} + \widetilde{T}_e, \qquad \text{for } i = 1, \ldots, n-1, \tag{6}$$

$$\widetilde{T}_{s_i}\widetilde{T}_{s_j}\widetilde{T}_{s_i} = \widetilde{T}_{s_j}\widetilde{T}_{s_i}\widetilde{T}_{s_j}, \qquad \text{if } |i-j| = 1, \tag{7}$$

$$\widetilde{T}_{s_i}\widetilde{T}_{s_j} = \widetilde{T}_{s_j}\widetilde{T}_{s_i}, \qquad \text{if } |i-j| \geq 2. \tag{8}$$

We then can define $\widetilde{T}_w$ for any $w \in S_n$ by $\widetilde{T}_w = \widetilde{T}_{s_{i_1}} \cdots \widetilde{T}_{s_{i_l}}$ where $w = s_{i_1} \cdots s_{i_l}$ is any reduced expression. Inverses of generators are given by

$$\widetilde{T}_{s_i}^{-1} = \widetilde{T}_{s_i} - (q^{\frac{1}{2}} - q^{-\frac{1}{2}})\widetilde{T}_e = \widetilde{T}_{s_i} - q^{-\frac{1}{2}}(q-1)\widetilde{T}_e. \tag{9}$$

When $q = 1$ we see that this presentation is simply that of the group algebra, $\mathbb{C}[S_n]$.

An important involution of the Hecke algebra is the so called *bar involution*. The involution is defined as

$$\sum_w a_w \widetilde{T}_w \mapsto \overline{\sum_w a_w \widetilde{T}_w} = \sum_w \overline{a_w}\, \overline{\widetilde{T}_w} \tag{10}$$

where

$$\overline{q} = q^{-1}, \qquad \overline{\widetilde{T}_w} = \left(\widetilde{T}_{w^{-1}}\right)^{-1}. \tag{11}$$

The *Kazhdan-Lusztig basis*, $\{C_w(q) \,|\, w \in S_n\}$, is the unique basis of $H_n(q)$ such that the basis elements are invariant under the bar involution, $\overline{C}_w = C_w$ for all $w \in S_n$, and that C_w in terms of the $\{\widetilde{T}_v\}$ is given by

$$C_w = \sum_{v \leq w} \epsilon_{v,w} q_{v,w} \overline{P_{v,w}(q)}\, \widetilde{T}_v, \tag{12}$$

where $P_{v,w}(q)$ are polynomials in q of degree at most $\frac{\ell(w)-\ell(v)-1}{2}$ and where we define the convenient notation $\epsilon_{v,w} = (-1)^{\ell(w)-\ell(v)}$, $q_{v,w} = (q^{\frac{1}{2}})^{\ell(w)-\ell(v)}$. These polynomials are known as the *Kazhdan-Lusztig polynomials* and in fact belong to $\mathbb{N}[q]$.

Kazhdan and Lusztig also introduced another basis $\{C'_w(q) \,|\, w \in S_n\}$ with similar properties which we shall call the *signless Kazhdan-Lusztig basis*. $C_w(q)$ and $C'_w(q)$ are related by $C'_w(q) = \psi(C_w(q))$, where ψ is the semilinear map defined by

$$\psi : q^{\frac{1}{2}} \mapsto \bar{q}^{\frac{1}{2}} \text{ and } \widetilde{T}_w \mapsto \epsilon_{e,w}\widetilde{T}_w. \tag{13}$$

Thus $C'_w(q)$ is also bar invariant and its expression in terms of $\{\widetilde{T}_v\}$ is

$$C'_w(q) = \sum_{v \leq w} q_{v,w}^{-1} P_{v,w}(q)\widetilde{T}_v. \tag{14}$$

As a preliminary to the proof of the existence and uniqueness of their bases Kazhdan and Lusztig also defined the following function

$$\mu(u,v) \underset{\text{def}}{=} \begin{cases} \text{coefficient of } q^{(\ell(v)-\ell(u)-1)/2} \text{ in } P_{u,v}(q) & \text{if } u < v, \\ 0 & \text{otherwise.} \end{cases} \tag{15}$$

Note that $\mu(u,v) = 0$ if $\ell(v) - \ell(u)$ is even since $P_{u,v}(q)$ has only integer powers of q. Also, it is well known that $P_{u,v}(q) = P_{w_0uw_0,w_0vw_0}(q)$, and therefore that $\mu(u,v) = \mu(w_0uw_0, w_0vw_0)$. Kazhdan and Lusztig showed further [8, Cor. 3.2] $\mu(u,v) = \mu(w_0v, w_0u)$, even though $P_{u,v}(q)$ and $P_{w_0v,w_0u}(q)$ are not equal in general.

In the existence proof of the Kazhdan-Lusztig basis in [8, Pf. of Thm. 1.1] an expression for the action of $\widetilde{T}_{s_i}$ on the basis element $C_w(q)$ is given by

$$\widetilde{T}_{s_i}C_w(q) = \begin{cases} q^{\frac{1}{2}}C_w(q) + C_{s_iw}(q) + \displaystyle\sum_{\substack{v<w \\ s_iv<v}} \mu(v,w)C_v(q) & \text{if } s_iw > w, \\ -\bar{q}^{\frac{1}{2}}C_w(q) & \text{if } s_iw < w. \end{cases} \tag{16}$$

Along with these bases Kazhdan-Lusztig defined a preorder on S_n in order to construct representations of $H_n(q)$. This preorder, called the *left preorder*, is denoted by $\leq_L$ and is defined as the transitive closure of $\lessdot_L$ where $u \lessdot_L v$ if $C_u(q)$ has nonzero coefficient in the expression of $\widetilde{T}_wC_v(q)$ for some $w \in S_n$. It is known that $w \leq_L v$ implies $\mathrm{sh}(v) \preceq \mathrm{sh}(w)$.

We follow the desription in [7, Appendix] of the Kazhdan-Lusztig construction of an irreducible $H_n(q)$-module (S_n-module) indexed by partition $\lambda \vdash n$. Choosing tableau T of shape λ, we allow $H_n(q)$ to act by left multiplication on

$$K^\lambda \underset{\text{def}}{=} \mathrm{span}\{C_w(q) \,|\, P(w) = T\}, \tag{17}$$

regarded as the quotient $\mathrm{span}\{C_v(q) \,|\, v \leq_L w\}/\mathrm{span}\{C_v(q) \,|\, v \leq_L w, v \not\geq_L w\}$. The quotient is necessary because K^λ is not in general closed under the action of $H_n(q)$. In particular, for $\lambda \neq (n)$ we have the containments $K^\lambda \subset H_n(q)K^\lambda \subseteq K^\lambda \oplus \mathrm{span}\{C_v(q) \,|\, v \leq_L w, v \not\geq_L w\}$.

4 The polynomial ring and Clausen's representations

Let $x = (x_{i,j})$ be an $n \times n$-matrix of variables. The polynomial ring $\mathbb{C}[x]$ has a natural grading $\mathbb{C}[x] = \oplus_{r \geq 0} \mathcal{A}_r$, where $\mathcal{A}_r$ is the span of all monomials of total degree r. Further decomposing each space $\mathcal{A}_r$, we define a multigrading

$$\mathbb{C}[x] = \bigoplus_{r \geq 0} \mathcal{A}_r = \bigoplus_{r \geq 0} \bigoplus_{L,M} \mathcal{A}_{L,M}, \tag{18}$$

where $L = \{\ell(1) \leq \ldots \leq \ell(r)\}$ and $M = \{m(1) \leq \ldots \leq m(r)\}$ are r-element multisets of $[n]$, written as weakly increasing sequences, and where $\mathcal{A}_{L,M}$ is the span of monomials whose row and column indices are given by L and M, respectively. We define the *generalized submatrix* of x with respect to (L, M) by

$$x_{L,M} = \begin{bmatrix} x_{\ell(1),m(1)} & \cdots & x_{\ell(1),m(r)} \\ x_{\ell(2),m(1)} & \cdots & x_{\ell(2),m(r)} \\ \vdots & & \vdots \\ x_{\ell(r),m(1)} & \cdots & x_{\ell(r),m(r)} \end{bmatrix}. \tag{19}$$

We refer to the space

$$\mathcal{A}_{[n],[n]} = \operatorname{span}\{x_{1,w_1} \cdots x_{n,w_n} \,|\, w \in S_n\}, \tag{20}$$

as the *immanant space*, and define the notation $x^{u,v} = x_{u_1,v_1} \cdots x_{u_n,v_n}$ for permutations $u, v \in S_n$.

Given subsets $I, J \subset [n]$ we define the I, J *minor* of x to be the determinant $\Delta_{I,J}(x) = \det(x_{I,J})$, and given a column-strict, semistandard bitableau (S, T) we define the *bideterminant*

$$(S \,|\, T)(x) = \Delta_{I_1,J_1}(x) \cdots \Delta_{I_k,J_k}(x), \tag{21}$$

where $I_1, \ldots, I_k$ are the sets of entries in columns $1, \ldots, k$ of S and $J_1, \ldots, J_k$ are the sets of entries in columns $1, \ldots, k$ of T. For example,

$$\left(\begin{matrix} 1 & 2 & 4 \\ 3 & & \end{matrix} \,\middle|\, \begin{matrix} 1 & 3 & 4 \\ 2 & & \end{matrix}\right)(x) = \Delta_{13,12}(x) x_{2,3} x_{4,4} = x_{1,1}x_{3,2}x_{2,3}x_{4,4} - x_{1,2}x_{3,1}x_{2,3}x_{4,4}. \tag{22}$$

For each permutation w in S_n, define

$$R_w(x) \underset{\text{def}}{=} (Q(w)^\top \,|\, P(w)^\top)(x) \tag{23}$$

where $(P(w), Q(w))$ is the bitableau obtained from the Robinson-Schensted row insertion algorithm. With little effort one can see that each semistandard bideterminant can be viewed as a standard bideterminant of a generalized submatrix. Similarly, standard bideterminants evaluated at generalized submatrices are either zero or a semistandard bideterminant. Therefore, for multisets L, M of $[n]$ with $|L| = |M| = r$ we have that the set $\{R_w(x_{L,M}) \,|\, w \in S_r\}$ is a spanning set for the space $\mathcal{A}_{L,M}$.

A natural S_n-action on $\mathbb{C}[x]$ is given by

$$s_i \circ g(x) \underset{\text{def}}{=} g(s_i x), \tag{24}$$

where $g \in \mathbb{C}[x]$ and $s_i x$ is interpreted as the product of the permutation matrix of s_i and x. Clausen [2, Thm. 8.1] constructed an irreducible S_n-module indexed by $\lambda \vdash n$ by letting $M = 1^{\lambda_1} \cdots n^{\lambda_n}$ and defining

$$B^\lambda \underset{\text{def}}{=} \operatorname{span}\{R_w(x_{[n],M}) \,|\, P(w)^\top = U(\lambda)\}. \tag{25}$$

The matrix representations arising from these modules are the exactly the same as those of *Young's natural representation*. This fact follows from the isomorphism found in [10, Sec. 4.2] between bideterminants and the polytabloids, which are the basis of the irreducible S_n-modules in Young's natural representation.

5 Kazhdan-Lusztig immanants

We now define a generalization of the polynomial ring $\mathbb{C}[x]$ called the *quantum polynomial ring*, $\mathcal{A}(n;q)$. The ring $\mathcal{A}(n;q)$ is a noncommutative $\mathbb{C}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}]$-algebra on n^2 generators $x = (x_{1,1} \dots, x_{n,n})$ with relations (assuming $i < j$ and $k < \ell$),

$$\begin{aligned} x_{i,\ell}x_{i,k} &= q^{\frac{1}{2}} x_{i,k}x_{i,\ell}, \\ x_{j,k}x_{i,k} &= q^{\frac{1}{2}} x_{i,k}x_{j,k}, \\ x_{j,k}x_{i,\ell} &= x_{i,\ell}x_{j,k}, \\ x_{j,\ell}x_{i,k} &= x_{i,k}x_{j,\ell} + (q^{\frac{1}{2}} - q^{\frac{-1}{2}})x_{i,\ell}x_{j,k}. \end{aligned} \tag{26}$$

A natural basis for the quantum polynomial ring consists of the set of monomials in lexicographic order. Analogous to the multigrading of $\mathbb{C}[x]$ is the multigrading

$$\mathcal{A}(n;q) = \bigoplus_{r \geq 0} \mathcal{A}_r(n;q) = \bigoplus_{r \geq 0} \bigoplus_{L,M} \mathcal{A}_{L,M}(n;q), \tag{27}$$

where $\mathcal{A}_r(n;q)$ is the span of all monomials of total degree r, and where $\mathcal{A}_{L,M}(n;q)$ is the span of monomials whose row and column indices are given by r-element multisets L and M of $[n]$. We again call the space $\mathcal{A}_{[n],[n]}(n;q) = \mathrm{span}\{x^{e,w} \mid w \in S_n\}$ the *immanant space* of $\mathcal{A}(n;q)$.

Define a left action of the Hecke algebra on $\mathcal{A}_{[n],[n]}(n;q)$ by

$$\widetilde{T}_{s_i} \circ x^{e,v} = x^{s_i,v} = \begin{cases} x^{e,s_iv} & \text{if } s_iv > v, \\ x^{e,s_iv} + (q^{\frac{1}{2}} - q^{\frac{-1}{2}})x^{e,v} & \text{if } s_iv < v. \end{cases} \tag{28}$$

Related to the bar involution on $H_n(q)$ is another *bar involution* on $\mathcal{A}_{[n],[n]}(n;q)$ defined by

$$\sum_w a_w x^{e,w} \mapsto \overline{\sum_w a_w x^{e,w}} = \sum_w \overline{a_w}\ \overline{x^{e,w}} \tag{29}$$

where

$$\overline{q} = q^{-1}, \qquad \overline{x^{e,w}} = x^{w_0,w_0w} = x_{n,w_n} \cdots x_{1,w_1}. \tag{30}$$

Lemma 5.1 *The bar involutions of (10) and (29) are compatible with the action of $H_n(q)$ on $\mathcal{A}_{[n],[n]}(n;q)$. That is,*

$$\overline{\widetilde{T}_{s_i} \circ x^{e,v}} = \overline{\widetilde{T}_{s_i}} \circ \overline{x^{e,v}} \tag{31}$$

for all $v \in S_n$.

Proof: Omitted. □

It is known that there is a unique, bar-invariant basis of $\mathcal{A}_{[n],[n]}(n;q)$ closely related to the Kazhdan-Lusztig basis of the Hecke algebra. We call the elements of this basis the *Kazhdan-Lusztig immanants* $\{\mathrm{Imm}_v(x;q)\}$. Further description of the Kazhdan-Lusztig immanants can be found in [12, Sec. 2], [4], and also [5]. For the benefit of the reader we provide a proof analogous to that in [8, Thm. 1.1].

Theorem 5.2 *For any $v \in S_n$, there is a unique element $\mathrm{Imm}_v(x;q) \in \mathcal{A}_{[n],[n]}(n;q)$ such that*

$$\overline{\mathrm{Imm}_v(x;q)} = \mathrm{Imm}_v(x;q) \tag{32}$$

$$\mathrm{Imm}_v(x;q) = \sum_{w \geq v} \epsilon_{v,w} q_{v,w}^{-1} Q_{v,w}(q) x^{e,w}, \tag{33}$$

where $Q_{v,w}(q)$ are polynomials in q of degree $\leq \frac{\ell(w)-\ell(v)-1}{2}$ if $v < w$ and $Q_{v,v}(q) = 1$.

Proof: Omitted. □

The polynomials $Q_{u,v}(q)$ in the above proof are actually the *inverse Kazhdan-Lusztig polynomials*, found in [8, Sec. 3]. They are related to the Kazhdan-Lusztig polynomials by

$$Q_{u,v}(q) = P_{w_0 v, w_0 u}(q) = P_{v w_0, u w_0}(q). \tag{34}$$

We can now describe a left action of $H_n(q)$ on the immanant space by its action on the Kazhdan-Lusztig immanants.

Corollary 5.3 *The left action of the Hecke algebra on $\mathcal{A}_{[n],[n]}(n;q)$ is described by*

$$\widetilde{T}_{s_i}\,\mathrm{Imm}_v(x;q) = \begin{cases} q^{\frac{1}{2}}\mathrm{Imm}_v(x;q) + \mathrm{Imm}_{s_i v}(x;q) + \sum\limits_{\substack{w>v \\ s_i w > w}} \mu(v,w)\mathrm{Imm}_w(x;q) & \text{if } s_i v < v, \\ -q^{-\frac{1}{2}}\mathrm{Imm}_v(x;q) & \text{if } s_i v > v. \end{cases} \tag{35}$$

Proof: Omitted. □

A deeper connection between the Kazhdan-Lusztig immanants and the Kazhdan-Lusztig basis is evident in the $\mathbb{C}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$-bilinear form on $\mathcal{A}_{[n],[n]}(n;q) \times H_n(q)$ defined by by $\langle x^{e,v}, \widetilde{T}_w \rangle = \delta_{v,w}$. Specifically, we have $\langle \mathrm{Imm}_v(x;q), C'_w(q) \rangle = \delta_{v,w}$, so the signless Kazhdan-Lusztig basis is dual to the basis of Kazhdan-Lusztig immanants.

In the following lemma we relate the definition of the left preorder in the Hecke algebra with these Kazhdan-Lusztig immanants. The results in the proof will also be essential in describing the relationship of the $H_n(q)$ representations associated with the Kazhdan-Lusztig basis and immanants.

Lemma 5.4 *Let $v, v' \in S_n$. Then $v \lessdot_L v'$ if $\mathrm{Imm}_{v'}(x;q)$ appears with nonzero coefficient in $\widetilde{T}_u \mathrm{Imm}_v(x;q)$ for some $u \in S_n$.*

Proof: Omitted. □

With Lemma 5.4 we can now express the preorder in terms of the Kazhdan-Lusztig immanants. We can now construct $H_n(q)$-modules indexed by $\lambda \vdash n$, like in [7, Appendix], with the Kazhdan-Lusztig immanants. We choose a tableau T of shape λ and allow $H_n(q)$ to act by left multiplication on

$$V^\lambda \underset{\mathrm{def}}{=} \mathrm{span}\{\mathrm{Imm}_w(x;q) \,|\, P(w)^\top = T\}, \tag{36}$$

regarded as the quotient $\mathrm{span}\{\mathrm{Imm}_v(x;q) \,|\, v \geq_L w\} / \mathrm{span}\{\mathrm{Imm}_v(x;q) \,|\, v \geq_L w, v \not\leq_L w\}$. The quotient is necessary because like K^λ, V^λ is not in general closed under the action of $H_n(q)$. In particular, whenever $\lambda \neq (1^n)$ we have the containments

$$V^\lambda \subset H_n(q) V^\lambda \subseteq V^\lambda \oplus \mathrm{span}\{\mathrm{Imm}_v(x;q) \,|\, v \geq_L w, v \not\leq_L w\}. \tag{37}$$

6 Generalized submatrices in Kazhdan-Lusztig immanants

In [13] Rhoades and Skandera found vanishing conditions for immanants and bideterminants of matrices having repeated rows and columns. Using these results we will evaluate the Kazhdan-Lusztig immanants at generalized submatrices, similar to Clausen's construction. It turns out that if we evaluate the Kazhdan-Lusztig immanants at specific generalized submatrices we can eliminate the quotient needed in the construction of the S_n-modules.

To express the vanishing results we need to define the *column repetition partition* of an $n \times n$-matrix A by

$$\nu_{[j]}(A) \underset{\mathrm{def}}{=} (\nu_1, \ldots, \nu_k), \tag{38}$$

where k is the number of distinct columns in the $n \times j$-submatrix $A_{[n],[j]}$, and $\nu_1, \ldots, \nu_k$ are the multiplicities with which distinct columns appear, written in weakly decreasing order.

We will write $\mathrm{Imm}_w(x) = \mathrm{Imm}_w(x; 1)$ for Kazhdan-Lusztig immanants in $\mathcal{A}_{[n],[n]}$. The following vanishing results found in [13, Thms. 4.10-4.11] will be instrumental in later proofs.

Lemma 6.1 *Let $w \in S_n$ and A be an $n \times n$ matrix. If $\mathrm{sh}(w_{[j]}^{-1}) \not\succeq \nu_{[j]}(A)$ for some $1 \leq j \leq n$, then* $\mathrm{Imm}_w(A) = R_w(A) = 0$.

We can now see that the left S_n-action defined in Corollary 5.3 (setting $q = 1$) actually describes an S_n-module if we evaluate the immanants at generalized submatrices.

Theorem 6.2 *Let $\lambda \vdash n$ and set $M = 1^{\lambda_1} \cdots n^{\lambda_n}$. Define*

$$W^\lambda \underset{\mathrm{def}}{=} \mathrm{span}\{\mathrm{Imm}_w(x_{[n],M}) \,|\, P(w)^\top = U(\lambda)\}, \tag{39}$$

where $U(\lambda)$ is the superstandard tableau of shape λ. Then W^λ is an S_n-module.

Proof: By (37) we know that it suffices to show that $\mathrm{Imm}_v(x_{[n],M}) = 0$ for $v >_L w$ where $P(w)^\top = U(\lambda)$. Since $v >_L w$ then we know that $\mathrm{sh}(w) \succ \mathrm{sh}(v)$. The column multiplicity partition of $x_{[n],M}$ is $\nu(x_{[n],M}) = \lambda$. So $\mathrm{sh}(v) \prec \mathrm{sh}(w) = \nu(x_{[n],M})$. Thus $\mathrm{sh}(v) = \mathrm{sh}(v^{-1}) \not\succeq \nu(x_{[n],M})$. Therefore, by Lemma 6.1, $\mathrm{Imm}_v(x_{[n],M}) = 0$ for all $v >_L w$. □

The condition for inclusion in the basis of this module is $P(w)^\top = U(\lambda)$ unlike the condition, $P(w)^\top = T$ where $\mathrm{sh}(T) = \lambda$, used in the definition of V^λ above. The need for the change in conditions will become clear later on in Proposition 7.2.

We would now like to show that these modules, W^λ, are isomorphic to the modules constructed by the action S_n on the Kazhdan-Lusztig basis. We will actually achieve this result by generalizing to the action of the Hecke algebra on the Kazhdan-Lusztig basis and immanants. We shall then show that the action of $\widetilde{T}_{s_i}$ on either basis yields equal matrices, up to ordering of the basis elements. Let $\rho_1 : S_n \to GL(K^\lambda)$ and $\rho_2 : S_n \to GL(W^\lambda)$ be the representations of S_n defined by letting $q = 1$ in the left actions described in (16) and Corollary 5.3, respectively.

Theorem 6.3 *Let $X_1(v)$, $X_2(v)$ be the matrices of $\rho_1(v)$, $\rho_2(v)$ with respect to the Kazhdan-Lusztig basis and the Kazhdan-Lusztig immanant basis. Then $X_1(v) = X_2(v)$.*

Proof: First, we construct K^λ as in (17) with $T = \text{evac}(U(\lambda))$. Let $B = \{v \in S_n \mid P(v) = \text{evac}(U(\lambda))\}$. From Lemma 2.1 we see that if $C_w(1)$ is a basis element of K^λ, i. e. $w \in B$, then $P(ww_0)^\top = \text{evac}(P(w)) = U(\lambda)$. Thus if $w \in Bw_0$, then $\text{Imm}_w(x_{[n],M})$ is a basis element of W^λ, as in (39). Define coefficients $a^{s_i}_{v,w}$ for each generators s_i of S_n and $v, w \in B$ so that

$$\widetilde{T}_{s_i} C_v(1) = \sum_{w \in B} a^{s_i}_{v,w} C_w(1). \tag{40}$$

Then from the proof of Lemma 5.4 we see that for all $v \in B$

$$\widetilde{T}_{s_i} \text{Imm}_{vw_0}(x_{[n],M}) = \sum_{w \in B} a^{s_i}_{v,w} \text{Imm}_{ww_0}(x_{[n],M}). \tag{41}$$

Thus $X_1(s_i) = X_2(s_i)$. Since any element of $v \in S_n$ is a product of generators we have that $X_1(v) = X_2(v)$. $\square$

Corollary 6.4 *The modules W^λ indexed by partitions $\lambda \vdash n$ are the irreducible S_n-modules.*

This result follows immediately from the fact that the modules K^λ are the irreducible S_n-modules.

7 Transition matrices

The goal of this section is to show that the S_n representations constructed with the bideterminant basis and the Kazhdan-Lusztig immanant basis are related by unitriangular matrices. The inspiration for this result comes from the work of Garsia and McLarnan where they showed a similar relationship between the S_n representations constructed with Young's natural basis and the Kazhdan-Lusztig basis [6, Thm. 5.3]. The results of this section are also similar to the work of McDonough and Pallikaros where they found a unitriangular relationship between the $H_n(q)$ representations constructed with Specht modules and the cell modules of Kazhdan and Lusztig [11, Thm. 4.1].

For two standard tableaux S, T with $\text{sh}(S), \text{sh}(T) \vdash n$ we can define *iterated dominance of tableaux* by $S \trianglelefteq_I T$ if for all $j \in [n]$ we have $\text{sh}(T_{[j]}) \preceq \text{sh}(U_{[j]})$, where $T_{[j]}$ is the subtableau of T consisting of all entries less then or equal to j. Also we define the permutation $w_{[j]} \in S_j$ from $w \in S_n$ by arranging $1, \dots, j$ in the same relative order of the first j terms in the one line notation of w. For two standard bitableaux we define *iterated dominance of bitableaux* by componentwise iterated dominance of the tableaux. Thus we have that $(T, U) \trianglelefteq_I (T', U')$ if $T \trianglelefteq_I T'$ and $U \trianglelefteq_I U'$. Using this order on bitableaux and the Robinson-Schensted association we can define *iterated dominance of permutations* by $v \leq_I w$ if and only if $(P(v)^\top, Q(v)^\top) \trianglelefteq_I (P(w)^\top, Q(w)^\top)$ (see [13]).

The following result can be found in [9, Thm. 5.1.4 C] and is usually attributed to Schützenberger.

Lemma 7.1 *If $v \in S_n$ and $1 \leq i \leq n$ then $\text{sh}(w_{[i]}) = \text{sh}(Q(w)^\top_{[i]})$.*

The following proposition was alluded to earlier in the construction of the irreducible S_n-modules with Kazhdan-Lusztig immanants with repeated columns.

Proposition 7.2 *Let $\lambda \vdash n$ and M be defined as above. If $\text{sh}(w) \prec \lambda$ or if $\text{sh}(w) = \lambda$ and $P(w)^\top \neq U(\lambda)$ then $\text{Imm}_w(x_{[n],M}) = 0$.*

Proof: When $\mathrm{sh}(w) \prec \lambda$ then we see that

$$\mathrm{sh}(w^{-1}) = \mathrm{sh}(w) \prec \lambda = \nu(x_{[n],M}). \tag{42}$$

Thus by Lemma 6.1 we see that $\mathrm{Imm}_w(x_{[n],M}) = 0$. Suppose $P(w)^\top \neq U(\lambda)$. It follows that $P(w)^\top \triangleleft_I U(\lambda)$ since $U(\lambda)$ is maximal in iterated dominance of tableaux among all tableaux of shape λ. Since $P(w)^\top \triangleleft_I U(\lambda)$ then there exists an index j such that

$$\mathrm{sh}(P(w)^\top_{[j]}) \succ \mathrm{sh}(U(\lambda)_{[j]}). \tag{43}$$

Let i be the greatest index so that $\lambda_1 + \cdots + \lambda_i \leq j$. By Lemma 7.1 we can see that $\mathrm{sh}(w^{-1}_{[j]}) = \mathrm{sh}(P(w)^\top_{[j]})$. For the generalized submatrix $x_{[n],M}$ we can see that

$$\begin{aligned} \nu_{[j]}(x_{[n],M}) &= (\lambda_1, \ldots, \lambda_i, j - (\lambda_1 + \cdots + \lambda_i)) && (44)\\ &= \mathrm{sh}(U(\lambda)_{[j]}), && (45) \end{aligned}$$

since the entries of $U(\lambda)$ are in reading order. Thus after combining results we have

$$\mathrm{sh}(w^{-1}_{[j]}) = \mathrm{sh}(P(w)^\top_{[j]}) \succ \mathrm{sh}(U(\lambda)_{[j]}) = \nu_{[j]}(x_{[n],M}). \tag{46}$$

Thus $\mathrm{Imm}_w(x_{[n],M}) = 0$ if $P(w)^\top \triangleleft_I U(\lambda)$ by Lemma 6.1. □

An analogous result for the bideterminants has essentially the same proof.

Proposition 7.3 *Let $\lambda \vdash n$ and M be defined as above. If $\mathrm{sh}(w) \prec \lambda$ or if $\mathrm{sh}(w) = \lambda$ and $P(w)^\top \neq U(\lambda)$ then $R_w(x_{[n],M}) = 0$.*

In [13, Sec. 6] Rhoades-Skandera described a filtration of the immanant space. Define for a partition $\lambda \vdash n$ the permutation $w(\lambda)$ to be the unique element of S_n where $P(w(\lambda))^\top = Q(w(\lambda))^\top = U(\lambda)$. We then can define

$$U_\lambda(x) = \mathrm{span}\{R_v(x) \,|\, v \leq_I w(\lambda)\}. \tag{47}$$

Rhoades and Skandera showed that the space $U_\lambda(x)$ also has a spanning set of certain Kazhdan-Lusztig immanants. Specifically their result [13, Thm. 6.4] implies the following.

Lemma 7.4 *Fix a partition $\lambda \vdash n$ then*

$$U_\lambda(x) = \mathrm{span}\{R_v(x) \,|\, \mathrm{sh}(v) \preceq \lambda\} = \mathrm{span}\{\mathrm{Imm}_v(x) \,|\, \mathrm{sh}(v) \preceq \lambda\}. \tag{48}$$

We now can conclude that the modules constructed with the bideterminants and the Kazhdan-Lusztig immanants, both specialized at a generalized submatrix, are the same S_n-module.

Theorem 7.5 *Let $\lambda \vdash n$ then $B^\lambda = W^\lambda$, where B^λ is defined in (25) and W^λ is defined in (39).*

Proof: With $M = 1^{\lambda_1} \cdots n^{\lambda_n}$ we can specialize $U_\lambda(x)$ at the generalized submatrix $x_{[n],M}$ to get

$$U_\lambda(x_{[n],M}) = \mathrm{span}\{R_v(x_{[n],M}) \,|\, \mathrm{sh}(v) \preceq \lambda\} = \mathrm{span}\{\mathrm{Imm}_v(x_{[n],M}) \,|\, \mathrm{sh}(v) \preceq \lambda\}. \tag{49}$$

Then by Proposition 7.3

$$\mathrm{span}\{R_v(x_{[n],M}) \,|\, \mathrm{sh}(v) \preceq \lambda\} = \mathrm{span}\{R_v(x_{[n],M}) \,|\, P(v)^\top = U(\lambda)\} = B^\lambda. \tag{50}$$

Similarly, by Proposition 7.2

$$\mathrm{span}\{\mathrm{Imm}_v(x_{[n],M}) \,|\, \mathrm{sh}(v) \preceq \lambda\} = \mathrm{span}\{\mathrm{Imm}_v(x_{[n],M}) \,|\, P(v)^\top = U(\lambda)\} = W^\lambda. \tag{51}$$

Thus

$$U_\lambda(x_{[n],M}) = B^\lambda = W^\lambda. \tag{52}$$

□

By [13, Cor. 5.11], the Kazhdan-Lusztig immanant basis, evaluated at generalized submatrices, is related to the bideterminant basis by a unitriangular matrix. Specifically we have the following result.

Proposition 7.6 *Fix a permutation $v \in S_n$ and n-element multiset M of $[n]$. Define coefficients $\{d_{u,v}^{[n],M} \,|\, u \in S_n\}$ by*

$$R_v(x_{[n],M}) = \sum_{u \in S_n} d_{u,v}^{[n],M} \mathrm{Imm}_u(x_{[n],M}). \tag{53}$$

Then we have $d_{u,v}^{[n],M} = 0$ if $u \not\leq_I v$ and $d_{v,v}^{[n],M} = 1$ for all $v \in S_n$.

Proposition 7.6 describes the change of basis matrix between the bideterminants and the Kazhdan-Lusztig immanant basis of $\mathcal{A}_{[n],M}(n;q)$. The change-of-basis matrix is given by

$$\mathcal{Z} = \left[d_{u,v}^{[n],M} \right], \tag{54}$$

where the permutations are in any linear extension of the iterated dominance order. These change-of-basis matrices imply a close relationship between the S_n representations generated by the Kazhdan Lusztig immanants, evaluated at generalized submatrices, and the bideterminant representations. Let $\rho_3 : S_n \to GL(B^\lambda)$ be the representation of S_n defined in (25) and let $X_3(v)$ be the matrix of this representation with respect to the Clausen basis. By the above argument, the matrix $X_2(v)$ defined before Theorem 6.3 is the matrix of $\rho_3(v)$ with respect to the Kazhdan-Lusztig immanant basis.

Theorem 7.7 *For all $v \in S_n$, $X_3(v) = Z^{-1} X_2(v) Z$, where Z is a unitriangular matrix.*

Proof: Let Z be the principal submatrix of the matrix $\mathcal{Z}$ (54) corresponding to rows and columns indexed by permutations u satisfying $P(u) = U(\lambda)^\top$. □

Since the matrix representation arising from the bideterminants is the equivalent to that of Young's natural representation then the previous theorem gives a new interpretation of Garsia and McLarnan's result [6, Thm. 5.3], in the setting of the polynomial ring.

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